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GAUß-MANIN CONNECTION ARISING FROM ARRANGEMENTS OF HYPERPLANES

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ABSTRACT. We study local systems arising from flat line bundles over topologically trivial families $U \to S$ of hyperplane complements in $\mathbb{P}^n$. Imposing some genericity condition on the monodromy, one knows that fiberwise the cohomology of the local system is concentrated in the middle dimension and is computed by the Aomoto complex, a subcomplex of global differential forms on a good compactification $\pi\colon X \to S$ with logarithmic poles along $D' = X \setminus U$.

The families $\mathcal{A}'$ considered are obtained by fixing a configuration $\mathcal{A}$ of hyperplanes and moving one additional hyperplane. The line bundle is the structure sheaf, endowed with the connection $d_{\text{rel}} + \omega$, for a logarithmic relative differential form ω. In this situation we construct the Gauß-Manin connection ∇ on $R^n\pi_*(\Omega^\bullet_{X/S}(\log D'), d_{\text{rel}} + \omega)$. We show that these sheaves are free. Using the combinatorics of $\mathcal{A}'$ we give a basis for these sheaves and an algorithm to express the connection ∇ in this basis. The corresponding matrix depends too much on the combinatorics of the arrangement to be given in a closed form, but we illustrate the method with some examples.

These results can be seen as a generalization of the hypergeometric functions.

Introduction

Let $\{H_i\}_{i\in I}$ be a collection of different hyperplanes in $\mathbb{P}^n$, let $U = \mathbb{P}^n \setminus \bigcup_{i\in I} H_i$ and let X be a smooth compactification of U such that the divisor $D = X \setminus U$ has normal crossings. Let ω be a global holomorphic one-form on U with logarithmic poles along the divisor D. As $d\omega = 0$, this form induces an integrable connection $\nabla = d + \omega$ on $\mathcal{O}_X$. The flat sections of ∇ define a rank 1 local system V. Deligne proved in [D1] that if ω has no positive integers as residues the cohomology of the local system V is given by the cohomology of the de Rham complex induced by ∇. Moreover, Esnault, Schechtman and Viehweg showed in [ESV] that if ω has no integers as residues, the cohomology of the local system is given by the Aomoto complex which is the subcomplex of global sections of the de Rham complex with logarithmic poles. Under the same genericity conditions, Esnault and Viehweg proved in [EV] that the cohomology of V is concentrated in the n-th term.

In this article we consider a topologically trivial deformation of the arrangement $\mathcal{A}' = \cup_{i\in I} H_i \subset \mathbb{P}^n$ and study how the cohomology of the local system varies. We take $\mathcal{A}'$ in such a way that there exists $i_0 \in I$ such that the hyperplane H_{i_0} is in general position with respect to the arrangement $\mathcal{A} = \cup_{i\in I\setminus\{i_0\}} H_i \subset \mathbb{P}^n$. We

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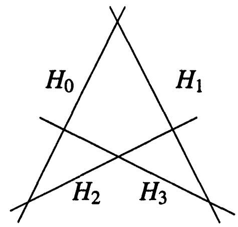
assume that $\mathcal{A}$ contains a system of coordinate hyperplanes. The deformation we consider is given by moving the hyperplane H_{i_0} and leaving $\mathcal{A}$ fixed in such a way that we do not obtain new bad crossings. Given an arrangement of hyperplanes in $\mathbb{P}^n$ the discriminant Discr($\mathcal{A}$) is defined as the locus in $\mathbb{P}^{n^\vee}$ given by the hyperplanes $H \subset \mathbb{P}^n$ for which the arrangement $H \cup \mathcal{A}$ gets new bad crossings. In fact, it is itself an arrangement in the dual projective space. Our deformation is a family X of arrangements over $S = \mathbb{P}^{n^\vee} \backslash \mathrm{Discr}(\mathcal{A})$. When talking about the family, we will denote the hyperplane H_{i_0} by H_s with $s \in S$. For the fiber of our family over $s \in S$, we consider a local system given as before by the kernel of a differential ∇_s with logarithmic poles and constant residues along the arrangement $H_s \cup \mathcal{A}$ such that the residues remain constant when we move $s \in S$. This gives us a relative connection on our family of arrangements. We show that this relative connection can be lifted to an absolute connection $\bar{\nabla}$ on X. This implies the existence of the Gauß-Manin connection. This connection is defined on the Gauß-Manin bundles $H^i_{DR}(X/S, D, \nabla)$ which are defined as the relative de Rham cohomology sheaves. This connection has logarithmic poles along Discr($\mathcal{A}$), and under the genericity conditions has as flat sections the direct images of the absolute local system $V = \ker(\bar{\nabla})$. Proposition 3.2 together with Theorem 3.3 give a generalization of the results obtained by Esnault, Schechtman and Viehweg concerning the cohomology of the family. They imply that the Gauß-Manin bundle is free and generated by global sections. In a standard way, one can calculate a representation of the Gauß-Manin connection. For the classical case of hypergeometric functions on $\mathbb{P}^1$ with poles along three different points, it is well known, from Riemann's integral representation formula, that one can express the hypergeometric system of differential equations as a direct image (as a variation of cohomology) of a rank one system; see [S, Theorem 2] and [M, Proposition, p. 373]. Hence the representation of the Gauß-Manin connection induced by Theorem 3.3 gives a generalization of hypergeometric differential equations. For some cases one can apply results from the theory of differential equations [D1, II.5.6] to calculate the local monodromies.

Section 1 of our article is devoted to show the existence of the Gauß-Manin connection.

Section 2 contains several results concerning aspects of the combinatorics for the theory of hyperplane arrangements. We present two important results. The first is due to Björner [Bj] and gives a basis for the cohomology with constant coefficients on the complement of an arrangement. The second is a description of a basis for the ideal of relations $\mathcal{J}$ for the Orlik-Solomon's Algebra. We will need these two results in Section 3 to obtain a representation of the Gauß-Manin connection.

In Section 3 we give a representation of the Gauß-Manin connection. Using Proposition 3.2 and Theorem 3.3, we show that under the genericity conditions, the cohomology of the relative local system is given by the relative Aomoto complex, even for the non-normal crossings case. This allows us to describe an algorithm for the Gauß-Manin connection. In Sections 4 and 5 we illustrate the method with some examples. We construct the Gauß-Manin connection for the following arrangement:



This example is of particular interest because its discriminant is the "Ceva" configuration which has been intensively studied in [BHH] for the construction of ball quotient surfaces. The second example illustrates the method in the case where we have a non-normal crossings arrangement.

The appendix contains some geometric constructions and vanishing theorems needed throughout the article.

This article is a short version of my PhD. thesis of the University of Essen which I defended in May 1996. After finishing this article we learned that Kaneko [Kj] independently studied the Gauß-Manin connection of arrangements of hyperplanes, obtaining similar results for some special cases.

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1. Existence of the Gauß-Manin connection

Let $\mathcal{A}$ be an arrangement of $m = n + r + 1$ hyperplanes in $\mathbb{P}^n$. We do not assume $\mathcal{A}$ to have normal crossings but that it contains a complete system of coordinate hyperplanes. We fix once and for all an order "$<$" on the set of hyperplanes such that $\{H_0, \ldots, H_n\}$, the first $n+1$ hyperplanes, are linearly independent. Let $(z_0, \ldots, z_n)$ be homogeneous coordinates for $\mathbb{P}^n$. We choose the coordinates so that z_i is a homogeneous defining equation of H_i for $i = 0, \ldots, n$. We have

$$H_j := \sum_{i=0}^{n} \lambda_{j,i} z_i = 0 \qquad (j = n+1, \ldots, m).$$

We denote by $\mathbb{P}^{n^\vee}$ the projective space dual to $\mathbb{P}^n$. As every point $p \in \mathbb{P}^{n^\vee}$ represents a hyperplane $H_p \subset \mathbb{P}^n$ we can now consider the locus in $\mathbb{P}^{n^\vee}$ defined as the set $p \in \mathbb{P}^{n^\vee}$ such that the configuration $\mathcal{A} \cup H_p \subset \mathbb{P}^n$ has more non-normal crossings than $\mathcal{A}$. This set, is known as the discriminant of $\mathcal{A}$, forms a divisor in $\mathbb{P}^{n^\vee}$ and will be denoted by $\mathrm{Discr}(\mathcal{A})$. Even if $\mathcal{A}$ is a normal crossing divisor, $\mathrm{Discr}(\mathcal{A})$ need not be.

Let $\{h_0, \dots, h_n\}$ be homogeneous coordinates of $\mathbb{P}^{n^\vee}$ dual to $(z_0, \dots, z_n)$ and let $\mathcal{S}^\vee$ be the homogeneous coordinate ring of $\mathbb{P}^{n^\vee}$. We can identify $\mathcal{S}^\vee$ with the set of homogeneous polynomials in the h_i's. Let $J \in M_{m+1,n+1}(\mathcal{S}^\vee)$ be given by

$$J = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \lambda_{0,0} & \cdots & \lambda_{0,n} \\ \vdots & \ddots & \vdots \\ \lambda_{r,0} & \cdots & \lambda_{r,n} \\ h_0 & \cdots & h_n \end{pmatrix}. \tag{1}$$

The discriminant of $\mathcal{A}$ is defined as the union of the zero set of all non-trivial $(n+1)$-minors of J that contain the row $\{h_0, \dots, h_n\}$.

Let us consider now a family of arrangements in $\mathbb{P}^n$ given by the projection

$$\pi\colon \mathbb{P}^{n^\vee} \times \mathbb{P}^n \longrightarrow \mathbb{P}^{n^\vee}.$$

The fiber of π over $p = (h_0, \dots, h_n) \in \mathbb{P}^{n^\vee}$ represents the arrangement $\mathcal{A}$ plus a hyperplane H_p that moves with $p \in \mathbb{P}^{n^\vee}$. The extra hyperplane is defined by the fibers of

$$\Delta = \left\{ (h_0, \dots h_n) \times (z_0, \dots, z_n) \in \mathbb{P}^{n^\vee} \times \mathbb{P}^n \mid z_h := \sum_{i=0}^{n} h_i z_i = 0 \right\}.$$

The divisor $D = [\mathbb{P}^{n^\vee} \times \mathcal{A}] \cup \Delta \cup [\mathrm{Discr}(\mathcal{A}) \times \mathbb{P}^n] \subset \mathbb{P}^{n^\vee} \times \mathbb{P}^n$ does not have normal crossings.

Let $\mathcal{L}(\mathrm{Discr}(\mathcal{A}))$ be as in Definition 5 in the appendix. Let

$$\varrho\colon \tilde{S} \longrightarrow \mathbb{P}^{n^\vee}$$

be a blow up along the elements of $\mathcal{L}(\mathrm{Discr}(\mathcal{A}))$ as constructed in the appendix. Let $X = \tilde{S} \times \mathbb{P}^n$ and $D' = (\varrho \times \mathrm{id}_{\mathbb{P}^n})^{-1}(D)$. We will denote the pullback of the projection π under the morphism ϱ again by π. Let $\rho\colon \tilde{X} \longrightarrow X$ be the blow up, constructed in the appendix, along the elements of $\mathcal{L}(D')$ such that $\rho^* D'$ has normal crossings. We will denote $\rho^* D'$ by $\tilde{D}$. We have the diagram

$$\begin{tikzcd} \tilde{X} \arrow[d, "\rho"'] \arrow[dr, "\sigma"] & & \\ X \arrow[r] \arrow[d] & \mathbb{P}^{n^\vee} \times \mathbb{P}^n \arrow[r, "\pi'"] \arrow[d, "\pi"] & \mathbb{P}^n \\ \tilde{S} \arrow[r, "\varrho"] & \mathbb{P}^{n^\vee}. & \end{tikzcd} \tag{2}$$

Let $\Omega^i_{\tilde{X}/\tilde{S}}(\log \tilde{D})$ be the coherent sheaf of $\mathcal{O}_{\tilde{X}}$-modules of relative i-forms of $\tilde{X}$ over $\tilde{S}$ with logarithmic poles along $\tilde{D}$.

We fix once and for all the hyperplane H_0 as the hyperplane at infinity. For $1 \leq i \leq m$ let $\omega_i = d_{\mathrm{rel}} x_i / x_i$, with $x_i = z_i/z_0$, be the differential form holomorphic on $U = \mathbb{P}^n \setminus \mathcal{A}$ with a logarithmic pole along H_i with residue 1, and a logarithmic pole along H_0 with residue -1. Let $\tilde{\omega}_i = (\sigma \circ \pi')^* \omega_i$. Let $W = \mathbb{P}^{n^\vee} \times \mathbb{P}^n \setminus D$ and let $\omega_s = d_{\mathrm{rel}} x_s / x_s$ be the differential form holomorphic on W where $x_s := 1 + \sum_{i=1}^n l_i x_i$ with $l_i = h_i/h_0$, $x_i = z_i/z_0$ and where the differential d_{rel} is the relative differential, i.e., $d_{\mathrm{rel}}|_{\pi^{-1}\mathcal{O}_S} \equiv 0$. The differential form ω_s has then a logarithmic pole along Δ with residue 1, and a logarithmic pole with residue -1 along $z_0 = 0$.

Remark 1.1. The absolute differential form $\omega_s^{\mathrm{abs}} = dx_s/x_s$ has also a logarithmic pole along $h_0 = 0$ with residue -1.

Notation 1.2. Let H_0 and H' two hyperplanes in $\mathbb{P}^n$ defined by the equations z_0 and z'. We take H_0 as the hyperplane at infinity. Let $x' = z'/z_0$ be the affine equation of the hyperplane induced by H' on the affine space $\mathbb{P}^n \setminus H_0$. We denote by $[\frac{dx'}{x'}]$ or by $[\frac{dz'}{z'} - \frac{dz_0}{z_0}]$ the global differential form, holomorphic on $W = \mathbb{P}^n \setminus H' \cup H_0$, with a logarithmic pole along H' with residue 1 and a logarithmic pole along H_0 with residue -1. Sometimes we omit the brackets.

Let $\omega \in H^0(W, \Omega^1_{W/S})$ be given by

$$\omega = \sum_{i=1}^m a_i \omega_i + a_h \omega_s$$

with $a_l \in \mathbb{C}$ for $l = 1, \ldots, m, h$ with ω_i and ω_s as before. This form induces the differential form $\tilde{\omega} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}/\tilde{S}}(\log \tilde{D}))$. This form has residue a_i along the pole H_i, a_h as residue along Δ and $\sum_{i \in I_L} a_i$ as residue along the exceptional divisor $e_L = \sigma^{-1}(L)$ for $L \in \mathcal{L}(D)$.

We consider the operator $\nabla = d_{\mathrm{rel}} + \tilde{\omega}$. As $d_{\mathrm{rel}} \tilde{\omega} = 0$ we have a logarithmic de Rham complex

$$0 \longrightarrow \mathcal{O}_{\tilde{X}} \xrightarrow{\nabla} \Omega^1_{\tilde{X}/\tilde{S}}(\log \tilde{D}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n_{\tilde{X}/\tilde{S}}(\log \tilde{D}) \longrightarrow 0. \tag{3}$$

$\ker(\nabla \colon \mathcal{O}_{\tilde{X}} \longrightarrow \Omega^1_{\tilde{X}/\tilde{S}}(\log \tilde{D}))$ defines a relative local constant system V_{rel} over the complement of $\tilde{D}$ in $\tilde{X}$ relative to $\tilde{S}$.

Definition 1.3. The i-th de Rham cohomology on $\tilde{X}$ relative to $\tilde{S}$ with respect to ∇ as the sheaf of $\mathcal{O}_{\tilde{S}}$-modules $\mathbf{R}^i\pi_*(\Omega^\bullet_{\tilde{X}/\tilde{S}}(\log \tilde{D}), \nabla)$ where $\mathbf{R}^i\pi_*\Omega^\bullet_{\tilde{X}/\tilde{S}}(\log \tilde{D})$ is the hyperderived functor of the functor $\mathbf{R}^0\pi_*$. We will denote it by $H^i_{DR}(\tilde{X}/\tilde{S}, \tilde{D}, \nabla)$.

To show the existence of the Gauß-Manin connection we follow the algebraic presentation given by Katz in [Ka, 3.0].

Procedure 1.4. Let $(\Omega^{\bullet}_{\tilde{X}/\tilde{S}}(\log \tilde{D}), \nabla)$ be the relative de Rham complex as above.

Step 1. We extend the differential ∇ on the relative complex to an absolute one by taking

$$\Omega = \sum_{i=1}^{m} a_i \frac{dx_i}{x_i} + a_h \frac{dx_s}{x_s} \quad \in \quad H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log \tilde{D})) \tag{4}$$

where x_i and x_s are as before and where d is the absolute differential on $\mathcal{O}_{\tilde{X}}$. We define $\overline{\nabla} = d + \Omega$. As $d\Omega = 0$ we have $\overline{\nabla}^2 = 0$. This defines the complex $(\Omega^{\bullet}_{\tilde{X}}(\log \tilde{D}), \overline{\nabla})$. Let $V = \ker(\overline{\nabla}\colon \mathcal{O}_{\tilde{X}} \longrightarrow \Omega^1_{\tilde{X}}(\log \tilde{D}))$ be the absolute local system on $\tilde{X} \setminus \tilde{D}$.

Step 2. We filter the complex $(\Omega^{\bullet}_{\tilde{X}}(\log \tilde{D}), \overline{\nabla})$ by

$$\cdots \mathbf{F}^{i+1} \subset \mathbf{F}^{i} \subset \cdots \subset \mathbf{F}^{0} = \Omega^{\bullet}_{\tilde{X}}(\log \tilde{D})$$

where

$$\mathbf{F}^{i} = \pi^{*}\Omega^{i}_{\tilde{S}}(\log(\varrho^{*}\mathrm{Discr}(\mathcal{A}))) \wedge \Omega^{\bullet-i}_{\tilde{X}}(\log \tilde{D}).$$

Step 3. We construct a spectral sequence abutting $\mathbf{R}^{\bullet}\pi_{*}(\Omega^{\cdot}_{\tilde{X}}(\log \tilde{D}))$. The $E_1^{a,b}$ terms are equal to $\Omega^{a}_{\tilde{S}}(\log \varrho^{*}(\mathrm{Discr}(\mathcal{A}))) \otimes_{\mathcal{O}_{\tilde{S}}} \mathbf{R}^{b}\pi_{*}(\Omega^{\bullet}_{\tilde{X}/\tilde{S}}(\log \tilde{D}))$ and the differential

$$d_1\colon E_1^{a,b} \longrightarrow E_1^{a+1,b} \tag{5}$$

has bidegree $(1, 0)$.

Step 4. From the filtration in Step 2 we have

$$0 \longrightarrow \frac{\mathbf{F}^1}{\mathbf{F}^2} \longrightarrow \frac{\mathbf{F}^0}{\mathbf{F}^2} \longrightarrow \frac{\mathbf{F}^0}{\mathbf{F}^1} \longrightarrow 0.$$

This is just the exact sequence of complexes

$$\begin{aligned} 0 &\longrightarrow \pi^{*}\Omega^{1}_{\tilde{S}}(\log \varrho^{*}(\mathrm{Discr}(\mathcal{A})) \otimes \Omega^{\bullet-1}_{\tilde{X}/\tilde{S}}(\log \tilde{D}) \\ &\longrightarrow \frac{\Omega^{\bullet}_{\tilde{X}}(\log \tilde{D})}{\pi^{*}\Omega^{2}_{\tilde{S}}(\log \varrho^{*}\mathrm{Discr}(\mathcal{A})) \wedge \Omega^{\bullet-2}_{\tilde{X}}(\log \tilde{D})} \longrightarrow \Omega^{\bullet}_{\tilde{X}/\tilde{S}}(\log \tilde{D}) \longrightarrow 0. \end{aligned} \tag{6}$$

The differential (5), for the case $a = 0$, is the connecting morphism for the long exact sequence of cohomology obtained from (6). Using the projection formula, with local calculation one can show that it satisfies the Leibniz property. This connection is called the Gauß-Manin connection; see [Ka,4.6]. We still denote it by $\overline{\nabla}$.

Step 5. To show the integrability of the Gauß-Manin connection we have the following diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{F}^1/\mathbf{F}^2 & \longrightarrow & \mathbf{F}^0/\mathbf{F}^2 & \longrightarrow & \mathbf{F}^0/\mathbf{F}^1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathbf{F}^1/\mathbf{F}^3 & \longrightarrow & \mathbf{F}^0/\mathbf{F}^3 & \longrightarrow & \mathbf{F}^0/\mathbf{F}^1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathbf{F}^2/\mathbf{F}^3 & = & \mathbf{F}^2/\mathbf{F}^3 & & & & \\
 & & \uparrow & & \uparrow & & . & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{7}
$$

The curvature is then given by the map

$$\overline{\nabla}^2\colon \mathbf{R}^b\pi_*(\mathbf{F}^0/\mathbf{F}^1) \longrightarrow \mathbf{R}^{b+2}\pi_*(\mathbf{F}^2/\mathbf{F}^3).$$

For an element $\alpha \in \mathbf{R}^b\pi_*(\mathbf{F}^0/\mathbf{F}^1)$, the connecting morphism of the middle horizontal exact sequence in (7) gives us an element in $\mathbf{R}^{b+1}\pi_*(\mathbf{F}^1/\mathbf{F}^3)$. From the left vertical exact sequence in (7) one has the integrability of the Gauß-Manin connection.

This proves the following key proposition.

PROPOSITION 1.5. *For π and ω as above, the form Ω defines an integrable connection $\overline{\nabla}$ on $H^i_{DR}(\tilde{X}/\tilde{S}, \tilde{D}, \nabla)$ which is called the Gauß-Manin connection.*

Remark 1.6. As we will see in Section 3 the condition that none of the residues of ω is an integer will imply that $\mathbf{R}^i\pi_*V$ is zero, for $i \neq n$ and is equal to the local system of flat sections of $\overline{\nabla}$ on $\mathbf{R}^n\pi_*(\Omega\, \bullet_{\tilde{X}/\tilde{S}}\,(\log \tilde{D})$. Moreover the latter sheaf will turn out to be free. To this aim, and to give an explicit representation of the connection we use some combinatorics to obtain a basis of the Gauß-Manin bundle.

2. Some combinatorics

Let $\mathcal{A}$ be an arrangement of $m = n + r$ hyperplanes in the affine space $\mathbb{C}^n$.

Definition 2.1. Let $L = L(\mathcal{A})$ be the semi-lattice of all non-empty intersections $\bigcap_{i \in J} H_j \neq \emptyset$ of elements of $\mathcal{A}$. We assume $\mathbb{C}^n \in L(\mathcal{A})$ as the intersection over the empty set, i.e., $J = \emptyset$.

One has a partial order "$\preceq$" on the elements of L given by reverse inclusion: for $X, Y \in L$ we say that $X \preceq Y$ if and only if $Y \subseteq X$. Let $X, Y \in L$ be such that $X \prec Y$. A *chain* from X to Y is a set $\{Z_0, \dots, Z_n\} \subset L$ such that $X = Z_0 \prec Z_1 \prec \cdots \prec Z_n = Y$. We then say that this chain has *length* n. A chain is *maximal* if for every $i \in \{0, \dots, n-1\}$ there does not exist $W \in L$ such that $Z_i \prec W \prec Z_{i+1}$.

Definition 2.2. For $X \in L$ we define $\operatorname{rank}(X)$ as the codimension of $X \subset \mathbb{C}^n$. Let $L_{n-r} = L^r = \{Z \in L \mid \operatorname{rank}(Z) = r\}$. We have $L^0 = \mathbb{C}^n$, $L_{n-1} = L^1$ is the set of hyperplanes of our arrangement and L_0 is a set of points in $\mathbb{C}^n$.

LEMMA 2.3. *Maximal elements of $L(\mathcal{A})$ have the same rank.*

Proof. See [OT, Lemma 2.4]. □

Definition 2.4. We define *rank* of $L = L(\mathcal{A})$ as the rank of any maximal element.

Given a subset $S \subset L^1$ of *points* in a semi-lattice we have $\operatorname{rank}(\cap_{H \in S} H) \leq |S|$.

Definition 2.5 [OT, Definition 3.2]. For $S \subset L^1$ we say that S is *independent* if $\cap_{H \in S} H$ is non-empty and if $\operatorname{rank}(\cap_{H \in S} H) = |S|$; otherwise we say that S is *dependent*.

Definition 2.6. Maximal independent subsets will be called *bases* and the minimal dependent sets will be called *circuits* and will be denoted by $\mathcal{C}(\mathcal{A})$.

Remark 2.7. In our case, one can easily see that, for $\mathcal{L}$ as in Definition A.1, we have $l \in \mathcal{L}$ if and only if the set $\{H_i \mid i \in I_l\}$ contains a circuit.

Recall that we fixed a linear order for the elements of $\mathcal{A}$.

Definition 2.8. Let $C \subset L^1$ be a circuit and $p \in L^1$ the least element of C. Then we say that $C - \{p\}$ is a *broken circuit*. The family of sets which do not contain a broken circuit will be called *non broken circuits* and will be denoted as nbc-elements. Maximal nbc-elements will be called nbc-bases.

Definition 2.9. For a broken circuit $C \subset L^1$ let $\operatorname{princ}(C) \in L^1$ be such that $C \cup \{\operatorname{princ}(C)\}$ is a circuit and is the smallest element with this property. In a similar way, for $B \subset L^1$ independent but not nbc we define $\operatorname{princ}(B) \in L^1$ as the least element of L^1 with the property that there exists a broken circuit $C \subset B$ such that $\operatorname{princ}(B) \cup C$ is a circuit.

Let $\mathcal{E}_1$ be the complex vector space freely generated by elements $\{e_H\colon H \in \mathcal{A}\}$ and let $\mathcal{E}$ be its exterior algebra. For $S \subset \mathcal{A}$ we denote the element $\wedge_{H\in S} e_H$ by e_S respecting the order chosen. A subset $S \subset \mathcal{A}$ is said to be *dependent* if there exists $H' \in \mathcal{A}$ such that $\cap_{H\in S\setminus\{H'\}} H = \cap_{H\in S} H$. We have the morphism

$$\partial\colon \mathcal{E} \longrightarrow \mathcal{E} \tag{8}$$

given by $\partial e_{(H_1,\dots,H_p)} = \sum_{i=1}^{p} (-1)^i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_p$ for $e_{(H_1,\dots,H_p)} \in \mathcal{E}_p$ where as usual $\widehat{\ }$ means that this element does not appear. It is a morphism of algebras and it is easy to see that $\partial^2 = 0$. Let $\mathcal{J}$ be the graded ideal of $\mathcal{E}$ generated by ∂e_S with $S \subset \mathcal{A}$ dependent. The quotient $\mathcal{E}/\mathcal{J}$ is a graded algebra known as the Orlik-Solomon algebra and appeared for the first time in [OS]; see [OT] also. From [OS, Theorem 5.2] we have the isomorphism

$$\mathcal{E}/\mathcal{J} \longrightarrow H^*(U, \mathbb{C}) \tag{9}$$

with $U = \mathbb{C}^n \setminus \mathcal{A}$.

We have the following theorem due to A. Björner; see [Bj, Theorem 4.2] and [SV].

THEOREM 2.10 (Björner). *The set of r-*nbc*-elements forms a basis for A^r.*

Remark 2.11. This last theorem together with formula (9) is a generalization of Brieskorn's theorem [B, Lemma 5].

Let $\mathcal{J}_r = \mathcal{J} \cap \mathcal{E}_r$ where $\mathcal{E}_r$ denotes the degree r subgroup of the graded algebra $\mathcal{E}$ and let $\mathcal{J}$ be as before. We would like to give a basis for the ideal of relations $\mathcal{J}$ in the Orlik-Solomon algebra. Actually in Section 3 we will need a basis for $\ker(\phi\colon A^n \longrightarrow H(U, \mathbb{C}^n))$.

Consider the map $\phi'\colon \mathcal{E}_r \longrightarrow A^r$ induced in the natural way by the map $\mathcal{E}_1 \longrightarrow A^1$ where $\phi'(e_H) = d\log(H)$ for $H \in \mathcal{A}$. The map $\phi''\colon \mathcal{E}_r \longrightarrow H^r(U, \mathbb{C})$ factors through A^r as $\phi'' = \phi \circ \phi'$.

We have the following proposition obtained jointly with V. Welker.

PROPOSITION 2.12. *Let $\mathcal{A} = \cup_{i=0}^{m=n+r} H_i \subset \mathbb{C}^n$ be an arrangement and $\mathcal{E}$ the Orlik-Solomon algebra. Then a basis for $\mathcal{J}_r$ is given by elements of $\mathcal{E}_r$ of the following form:*

(i) *e_B for $B \subset \mathcal{A}$ dependent and $|B| = r$;*
(ii) *$\partial e_{\hat{B}}$ for B independent of rank r but not an r-nbc.*

Proof. We have

$$|\mathcal{E}_r| = |r\text{-nbc}| + \{|r - \text{circuits}| + |r - \text{broken circuits}|\}.$$

One can see that the sets (i) and (ii) are disjoint and that (i) generates $\ker\phi'$. As the number of elements in (i) together with (ii) equals $|\mathcal{E}_r| - |r\text{-nbc}|$, to show that they form a basis we only have to show that they generate $\ker(\phi)$. We will prove this by induction by showing that, with the help of the elements in (i) and (ii), one can write any element of A^r as a linear combination of non broken circuits. The induction will be taken on the lexicographic order on the elements of A^r induced by the order chosen for the set of hyperplanes.

As the first element of A^r is already a non broken circuit, the statement is true for the base of induction. Let $C = (H_1, \ldots, H_r)$ be independent but not nbc. By the induction hypothesis we can assume that the statement is true for any base $B \leq C$. Let $\hat{C} = C \cup \operatorname{princ}(C)$ as in Definition 2.8; then

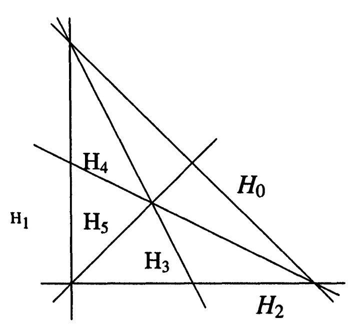
$$\partial e_{\hat{C}} = \sum_{i=1}^{\operatorname{princ}(C)} e_{\hat{C}\setminus\{H_i\}} + \sum_{i=\operatorname{princ}(C)}^{r} e_{\hat{C}\setminus\{H_i\}}. \tag{10}$$

Every summand in the first sum in (10) contains a circuit so they all are elements of (i). For the second sum, the first element is C and the rest contain $\operatorname{princ}(C)$ so they have lexicographic order smaller than C. Applying our induction hypothesis we can write C as linear combination of circuits, which vanish under ϕ', and r-nbc. $\square$

Remark 2.13. A basis for $\ker(\phi\colon A^n \longrightarrow H^n(U, \mathbb{C}))$ is given by the image of the elmements in (ii) under ϕ'.

Example 2.14. Let $\mathcal{A} = \cup_{i=0}^{5} H_i$ be an arrangement in $\mathbb{P}^2$ given by

$$\begin{aligned}
H_0 &:= z_0 = 0\\
H_1 &:= z_1 = 0\\
H_2 &:= z_1 = 0\\
H_3 &:= z_3 := z_0 - z_1 = 0\\
H_4 &:= z_4 := z_0 - z_2 = 0\\
H_5 &:= z_5 := z_1 - z_2 = 0.
\end{aligned} \tag{11}$$



(12)

Figure 1

Take H_0 to be the hyperplane at infinity. On the affine complement of H_0 we have the following arrangement:

$$\begin{aligned} L_1 &:= x_1 = 0 \\ L_2 &:= x_2 = 0 \\ L_3 &:= x_3 := x_1 - 1 = 0 \\ L_4 &:= x_4 := x_2 - 1 = 0 \\ L_5 &:= x_5 := x_1 - x_2 = 0. \end{aligned} \tag{13}$$

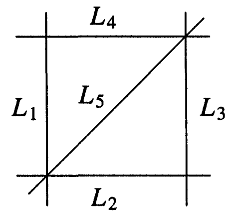


Figure 2

The set of circuits is

$$\mathcal{C}(\mathcal{A}) = \{(L_1, L_3), (L_2, L_4), (L_1, L_2, L_5), (L_3, L_4, L_5)\}.$$

These are the only *dependent* subsets of $\mathcal{A}$. The nbc's are

$$\mathrm{nbc}(\mathcal{A}) = \{(L_1, L_2), (L_1, L_4), (L_1, L_5), (L_2, L_3), (L_3, L_4), (L_3, L_5)\}.$$

Clearly the only *broken circuits* are $\{(L_2, L_5), (L_4, L_5)\}$ for which $(L_1) = \mathrm{princ}(L_2, L_5)$ and $(L_3) = \mathrm{princ}(L_4, L_5)$.

Let $\mathcal{E}_1$ be freely generated by $\{e_i \mid L_i \in \mathcal{A}\}$ and let $\mathcal{E}$ be its exterior algebra. Let $\mathcal{J}$ be the ideal of $\mathcal{E}$ generated by ∂e_S for $S \subset \mathcal{A}$ dependent. By Proposition 2.12, $\mathcal{J}_n = \mathcal{J} \cap \mathcal{E}_n$ is generated by

$$\mathcal{J}_n = \langle e_{13}; e_{24}; \partial e_{125}; \partial e_{345}\rangle = \langle e_{13}; e_{24}; e_{12} - e_{15} + e_{25}; e_{34} - e_{35} + e_{45}\rangle.$$

Under the natural identification of $\mathcal{E}_1$ with $H^0(U, \Omega^1_U$, where $U = \mathbb{P}^2 \backslash \mathcal{A}$, given by $e_i \mapsto \frac{dx_i}{x_i}$, these relations lead to the following relations of 2-forms:

$$\begin{aligned} \frac{dx_1 dx_3}{x_1\, x_3} &= 0 \\ \frac{dx_2 dx_4}{x_2\, x_4} &= 0 \\ \frac{dx_1 dx_2}{x_1 x_2} - \frac{dx_1 dx_5}{x_1 x_5} + \frac{dx_2 dx_5}{x_2 x_5} &= 0 \\ \frac{dx_3 dx_4}{x_3 x_4} - \frac{dx_3 dx_5}{x_3 x_5} + \frac{dx_4 dx_5}{x_4 x_5} &= 0. \end{aligned} \tag{14}$$

By Proposition 2.12 these relations are linearly independent.

3. The Gauß-Manin matrix

Let $\mathcal{A}$ be an arrangement of $m = n + r + 1$ hyperplanes in $\mathbb{P}^n$ as in Section 1. Contrary to Section 1 we don't compactify the space of parameters. We have a family of arrangements in $\mathbb{P}^n$ given by the projection

$$\pi\colon\ S \times \mathbb{P}^n \longrightarrow S$$

where $S = \mathbb{P}^{n^\vee} \setminus \mathrm{Discr}(\mathcal{A})$. Let D, $W = S \times \mathbb{P}^n \setminus D$ and $\omega \in H^0(W, \Omega^1_W)$ be as in Section 1 but restricted to $S \times \mathbb{P}^n$. Hence $D = S \times \mathcal{A} + \Delta \cap [S \times \mathbb{P}^n]$ where $\mathcal{A}$ is the constant arrangement and Δ the additional hyperplane.

Let $\rho\colon\ \tilde{X} \longrightarrow S \times \mathbb{P}^n$ be the blow up along $\mathcal{L}(D)$, as in (2). As our space of parameters is taken as the non compactified space, we have $\mathcal{L}(D) = \mathcal{L}(S \times \mathcal{A})$.

Remark 3.1. Under the assumptions made above, $\mathcal{L}(D) = S \times \mathcal{L}(\mathcal{A}) \subset D$, i.e., the bad loci have at most codimension n. Letting $\pi'\colon\ \tilde{Y} \longrightarrow \mathbb{P}^n$ be the standard resolution along elements of $\mathcal{L}(\mathcal{A})$ as described in (32) in the appendix, one has $\tilde{X} = S \times \tilde{Y}$.

We denote $\rho^*(D)$ by $\tilde{D}$. Let $\Omega^i_{\tilde{X}/S}(\log \tilde{D})$ be the coherent sheaf of $\mathcal{O}_{\tilde{X}}$-modules of relative i-forms of $\tilde{X}$ relative to S with logarithmic poles along $\tilde{D}$. Let $\tilde{\omega} = \rho^*\omega \in H^0(\tilde{X}, \Omega^1_{\tilde{X}/S}(\log \tilde{D}))$. Then $\tilde{\omega}$ is the differential form with residues a_h along $\sum_{i=0}^n h_i z_i = 0$, a_i along H_i with $a_0 = \sum_{i=1}^m -a_i - a_h$ and such that for every $L \in \mathcal{L}(\mathcal{A})$ the form $\tilde{\omega}$ has residue $\sum_{i \in I_L} a_i$ along the exceptional divisor $e_L = \rho^{-1}(S \times L)$.

We consider the operator $\nabla = d_{\mathrm{rel}} + \tilde{\omega}$. As $d_{\mathrm{rel}}\tilde{\omega} = 0$, it gives a logarithmic de Rham complex $(\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}), \nabla)$.

Let $A^p \subset \pi_*\Omega^p_{W/S}$ be generated over $\mathcal{O}_S$ by

$$\left\{ \bigwedge_{j=1}^{p} \frac{dx_{i_j}}{x_{i_j}};\ \bigwedge_{j=1}^{p-1} \frac{dx_{i_j}}{x_{i_j}} \wedge \frac{d_{\mathrm{rel}}x_s}{x_s} \mid i_j \in \{1, \dots, m\} \right\}.$$

We have the subcomplex $A^\bullet \subset \pi_*\Omega^\bullet_{W/S}$ given by

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\nabla} A^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} A^n \longrightarrow 0. \tag{15}$$

PROPOSITION 3.2. *Let D be the family of arrangements on $\pi\colon\ S \times \mathbb{P}^n \longrightarrow S$ considered above. Let $\rho\colon\ \tilde{X} \longrightarrow S \times \mathbb{P}^n$ be the standard resolution along the elements of $\mathcal{L}(D)$ such that the divisor $\tilde{D} = \rho^*(D)$ has normal crossings. We have $A^p = (\pi \circ \rho)_*\Omega^p_{\tilde{X}/S}(\log(\tilde{D}))$.*

Proof. For $\Omega^q_{\tilde{X}/S}(\log \tilde{D})$ we define the function h^p on S as

$$h^p(s, \Omega^q_{\tilde{X}/S}(\log \tilde{D})) = \dim H^p(\tilde{X}|_s, \Omega^q_{\tilde{X}/S}(\log \tilde{D})|_s)$$

where $\Omega^q_{\tilde{X}/S}(\log \tilde{D})|_s$ is just the restriction to the fiber $\pi^{-1}(s) = s \times \mathbb{P}^n$. By the lemma in [ESV, Section 2] (see Lemma A.4 in the appendix), for $p > 0$ this function is constant zero. Applying base change [Ha, III, 12.11] implies the result. □

Let V be the kernel of the absolute connection $\overline{\nabla} = d + \Omega$ considered in (4).

THEOREM 3.3. *Under the hypothesis of Proposition* 3.2, *assuming further that* $a_i \notin \mathbb{Z}$ *for* $i \in \{0, \dots, m, h\}$ *with* $a_0 = -\sum_{i=1}^m a_i - a_h$ *and* $\sum_{i\in I_L} a_i \notin \mathbb{Z}$ *for every* $L \in \mathcal{L}$, *it follows that* $\mathbf{R}^p(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) = 0$ *for* $p \neq n$ *and* $\mathbf{R}^n V$ *is the kernel of the Gauß-Manin connection* $\mathbf{R}^n(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \longrightarrow \mathbf{R}^n(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \otimes \Omega^1_S$ *constructed in Section* 1. *Moreover, for* $s \in S$, *the fibre* $R^n\pi_*V|_s$ *is the* n*-th cohomology of the restriction of* V_{rel} *to the fibre* U_s *of* $\pi\colon W \longrightarrow S$ *over* s.

Proof. From Proposition 1.5 the sheaf of $\mathcal{O}_S$-modules $\mathbf{R}^p(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D})$ carries an integrable connection. As is well known this implies that this bundle is locally free. As the sheaf $\Omega^i_{\tilde{X}/S}(\log \tilde{D})$ is locally free it is flat over S. Applying [Ka, Theorem 8.0] we have base change. For $s \in S$,

$$\mathbf{R}^p(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \otimes k(s) = H^p(\tilde{X}_s, \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D})\,|_s) \tag{16}$$

where $k(s)$ is the residue field over $s \in S$. Applying [EV1, Corollary 1.5] (see Theorem A.6 in the appendix), completes the proof for $p \neq n$.

Since $\pi\colon W \longrightarrow S$ is topologically trivial, $R^n\pi_*V|_s = H^n(U_s, V_{\mathrm{rel}}|_s)$. By the construction of the sequence (6) one has a natural map

$$R^n\pi_*V \to \mathrm{Ker}\,\mathbf{R}^n(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \longrightarrow \mathbf{R}^n(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \otimes \Omega^1_S.$$

By (16) this map is an isomorphism on all fibers, hence it is an isomorphism. □

Under the hypothesis of Theorem 3.3 the Gauß-Manin connection is given as

$$\overline{\nabla}\colon \mathbf{R}^n(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \longrightarrow \Omega^1_S \otimes \mathbf{R}^n(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}). \tag{17}$$

Definition 3.4. Let $A^p_c \subset A^p$ be the subalgebra generated by

$$\left\{ \bigwedge_{j=1}^{p} \frac{dx_{i_j}}{x_{i_j}} \mid i_j \in \{1, \dots, m\} \text{ and } x_{i_j} < x_{i_k} \text{ if } j < k \right\}.$$

As we will see in the next lemma, $A^\bullet_c$ is the Aomoto complex for the constant part of part for the family of arrangements.

LEMMA 3.5. *The complex $A_c^\bullet$ is obtained as the tensor product with $\mathcal{O}_S$ of the Aomoto complex of the constant arrangement $\mathcal{A}$ on $\mathbb{P}^n$. One has the decomposition*

$$A^p = A_c^p \oplus \frac{dx_s}{x_s} \wedge A_c^{p-1}. \tag{18}$$

Proof. We have the exact sequence

$$\begin{aligned} 0 &\longrightarrow R^0\pi_*(X, \Omega_X^p(\log \gamma^*(\rho_c^*(S \times \mathcal{A}))) \longrightarrow R^0\pi_*(X, \Omega_X^p(\log(\rho_c^*(D)))) \\ &\longrightarrow R^0\pi_*(\Delta, \Omega_\Delta^{p-1}(\log(\rho_c^*(\mathcal{A}))|_\Delta)) \longrightarrow 0 \end{aligned} \tag{19}$$

where ρ and ρ_c are the standard resolution of $S \times \mathbb{P}^n$ along D and $S \times \mathcal{A}$ respectively and where γ is as in Lemma A.3. This sequence gives an injection of the Aomoto complex on $\mathcal{A}$ into $A^\bullet$. Since the generators are the right ones, one obtains the first part of the lemma.

For the second half, one can see that on the fiber over $s \in S$ we have the arrangement $\mathcal{A} \cup H_s$, where H_s is defined by $x_s := 1 + \sum_{i=1}^n l_i x_i = 0$ with $s = (1, l_1, \ldots, l_n) \in S \subset \mathbb{P}^{n^\vee}$. For every $l \in \mathcal{L}(\mathcal{A} \cup H_s)$ we have $l \not\subset H_s$ which from Remark A.9 implies that

$$A_s^p = A_c^p\big|_s \oplus \frac{dx_s}{x_s} \wedge A_c^{p-1}\big|_s. \tag{20}$$

By Brieskorn's Lemma, A_s^p generates $H^p(\tilde{X}|_s, \Omega^q_{\tilde{X}/S}(\log \tilde{D})|_s)$; see Lemma A.7. There exists a non empty Zariski open set $U \subset S$ where the kernel of the natural morphism

$$A_c^p \oplus \frac{dx_s}{x_s} \wedge A_c^{p-1} \to A^p \tag{21}$$

is locally free. As a consequence we can extend the decomposition (20) to global sections as

$$A^p = A_c^p \oplus \frac{dx_s}{x_s} \wedge A_c^{p-1}. \tag{22}$$

□

THEOREM 3.6. *Let* $\operatorname{nbc}(\mathcal{A})$ *be an nbc-basis for the arrangement* $\mathcal{A}$. *Then* $\mathbf{R}^n(\pi \circ \rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D})$ *is a free* $\mathcal{O}_S$ *module, generated by* $\operatorname{nbc}(\mathcal{A})$.

Proof. From the exact sequence

$$0 \longrightarrow A^0 \xrightarrow{\tilde{\omega}} A^1 \xrightarrow{\tilde{\omega}} \cdots \xrightarrow{\tilde{\omega}} A^n \xrightarrow{\tilde{\omega}} \mathbf{R}^n(\pi \circ \rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \longrightarrow 0$$

where as always, $\tilde{\omega} = \rho^*\omega$ with $\omega = \sum_{i=1}^m a_i\omega_i + a_h\omega_s$ and $a_h \neq 0$, calculating the homology one has a telescoping series which together with the exactness of the complex one can write any element of $\mathbf{R}^n(\pi \circ \rho)_*\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D})$ in terms of the $\operatorname{nbc}(\mathcal{A})$ which implies the result. □

In [FT] the authors show that the set of β-nbc's form a basis for the local system; see [Z, Section 1]. In our case we prefer to use Theorem 3.6. Actually, when taking an order for the set of hyperplanes such that the moving hyperplane is the first one, this basis is composed of β-nbc's.

Remark 3.7. If the arrangement $\mathcal{A}$ has normal crossings, the sheaf of $\mathcal{O}_S$-modules $\mathbf{R}^n\pi_*\Omega^\bullet_{S\times\mathbb{P}^n/S}(\log D)$ is free of rank $\binom{m}{n}$ over $\mathcal{O}_S$ with basis

$$\left\{\bigwedge_{j=1}^{n}\frac{dx_{i_j}}{x_{i_j}} \mid i_j\in\{1,\dots,n+r\} \quad \text{and} \quad i_j<i_k \quad \text{when} \quad j<k\right\}.$$

Remark 3.8. The order on the set of hyperplanes induces an order on the basis of Theorem 3.6 for $\mathbf{R}^n(\pi\circ\rho)_*\Omega^\bullet_{\tilde{X}/S}(\log\tilde{D})$, where we say that $\bigwedge_{k=1}^{n}\frac{dx_{i_k}}{x_{i_k}}\le\bigwedge_{k=1}^{n}\frac{dx_{j_k}}{x_{j_k}}$ when there exists $k\in\{1,\dots,n\}$ such that $\frac{dx_{i_k}}{x_{i_k}}\le\frac{dx_{j_k}}{x_{j_k}}$ and $\frac{dx_{i_l}}{x_{i_l}}=\frac{dx_{j_l}}{x_{j_l}}$ for $l<k$.

Procedure 3.9. The procedure to write the matrix of the Gauß-Manin connection with respect to the basis given in Corollary 3.6 is as follows: As before, we take affine coordinates for the complement of $z_0=0$ in $\mathbb{P}^n$ as $x_i=z_i/z_0$. We do the same for the complement of $h_0=0$ in $\mathbb{P}^{n^\vee}$ by taking $l_i=h_i/h_0$. We extend the relative differential form ω to a global form Ω, as in Section 1. In affine coordinates $\Omega=\sum_{i=1}^{n}a_i\frac{dx_i}{x_i}+a_h\frac{dx_l}{x_l}$ where $x_l=1+l_1x_1+\cdots+l_nx_n$ and where the differential is the absolute one. The procedure is the standard one. We take an element of the basis given in Theorem 3.6, apply to it the connection and write its image again in terms of this basis. To do this, in the non-normal crossing case, we need to apply the basis of relations given in Proposition 2.12. These relations allow one to write the image under the connection of an element of the basis canonically, in terms of the former basis. In the source of the computation, we have basically two cases. The first is when we apply the connection to the first element of the basis in Corollary 3.6. The hyperplanes involved with this element are given by the set of affine coordinates chosen. We use the standard procedure, applying the connection and writing the image in terms of the basis. In the second case, we apply the connection to any other elements of the basis. We then reduce the problem to the first case by making a change of the affine coordinates.

As the basis for the Gauß-Manin bundle given in Theorem 3.6 depends on the combinatorics of our arrangement, we cannot give an explicit form for the matrix. Nevertheless, the basis nbc($\mathcal{A}$) and the basis of relations in Proposition 2.12 are given in such a precise way that, for any explicit example, we are able to compute the matrix of the Gauß-Manin connection. Moreover for the normal crossings case there is an explicit form of the Gauß-Manin matrix; see [K, section 4].

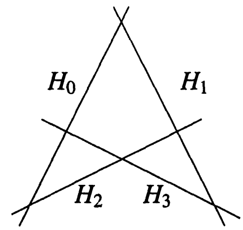


Figure 3

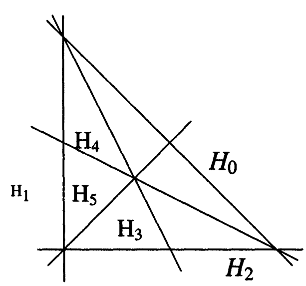
4. Example I

In this section we give an example for the method given in the previous section. We take an arrangement of six lines in $\mathbb{P}^2$ in general position. The discriminant in this case is Ceva's arrangement, (see [BHH] for an intensive study of this configuration).

Let $\mathcal{A} = \cup_{i=0}^3 H_i$ be the arrangement in $\mathbb{P}^2$ given by

$$\begin{aligned} H_0 &:= z_0 = 0 \\ H_1 &:= z_1 = 0 \\ H_2 &:= z_2 = 0 \\ H_3 &:= z_3 := z_0 + z_1 + z_2 = 0 \end{aligned} \tag{23}$$

where we can take z_0, z_1, z_2 as a local frame for $\mathbb{P}^2$. In this case, the discriminant is given as $\mathrm{Discr}(\mathcal{A}) = \{h_0 = 0, h_1 = 0, h_2 = 0, h_0 - h_1 = 0, h_0 - h_2 = 0, h_2 - h_1 = 0\}$.



(24)

Figure 4

Let $X = S \times \mathbb{P}^2 \setminus \{\Delta := h_0 z_0 + h_1 z_1 + h_2 z_2 = 0\} \cup \{S \times \mathcal{A}\}$ and $\pi\colon X \longrightarrow S$ be a family of arrangements parameterized by $S = \mathbb{P}^{2^\vee} \setminus \mathrm{Discr}(\mathcal{A})$. We denote the divisor $S \times \mathcal{A} \cup \{\Delta \cap S \times \mathbb{P}^2\}$ by D.

We fix H_0 as the hyperplane at infinity of the arrangement (23). Let

$$\omega \in H^0(S \times \mathbb{P}^2, \Omega^1_{S \times \mathbb{P}^2}(\log D))$$

be given as

$$\omega = \sum_{i=1}^{3} a_i \frac{d_{\mathrm{rel}} x_i}{x_i} + a_h \frac{d_{\mathrm{rel}} x_l}{x_l}$$

where $x_i = z_i/z_0$, $l_i = h_i/h_0$ and $x_l = l_1 x_1 + l_2 x_2 + 1$, where $\frac{dx_i}{x_i}$ is taken as in Remark 1.2 and where the differential is taken as the relative differential along S. We assume that $a_i \notin \mathbb{Z}$ for $i \in \{0, \dots, 3, h\}$, and that $\sum_{i=0}^{3} a_i + a_h = 0$.

The operator $\nabla = d_{\mathrm{rel}} + \omega$ defines the complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\nabla} \Omega^1_{S \times \mathbb{P}^2/S}(\log D) \xrightarrow{\nabla} \Omega^2_{S \times \mathbb{P}^2/S}(\log D) \longrightarrow 0.$$

Let V be the relative local system defined as the sheaf of flat sections of ∇. Theorem 3.3 implies that for $i = 0, 1$,

$$\mathbf{R}^i \pi_* \Omega^\bullet_{S \times \mathbb{P}^2/S}(\log D) = 0. \tag{25}$$

For $i = 2$ we have

$$\mathbf{R}^2 \pi_* \Omega^\cdot_{S \times \mathbb{P}^2/S}(\log D) = \mathcal{O}_S \frac{dx_1 \wedge dx_2}{x_1 x_2} \oplus \mathcal{O}_S \frac{dx_1 \wedge dx_3}{x_1 x_3} \oplus \mathcal{O}_S \frac{dx_2 \wedge dx_3}{x_2 x_3} \tag{26}$$

We can now extend ω to

$$\Omega = \sum_{i=0}^{3} a_i \frac{dz_i}{z_i} + a_h \frac{dz_h}{z_h}$$

where the differential is no longer the relative differential but the absolute one over S. The operator $\overline{\nabla} = d + \Omega$ which, when using affine coordinates in particular on the complement of $z_0 = 0$ and $h_0 = 0$, takes the form

$$\overline{\nabla} = d + \sum_{i=1}^{3} a_i \frac{dx_i}{x_i} + a_h \frac{dx_l}{x_l}.$$

With respect to the basis (26) the Gauß-Manin connection

$$\overline{\nabla}\colon H^2(X/S, V) \longrightarrow H^2(X/S, V) \otimes \Omega^1_S(\log(\mathrm{Discr}(\mathcal{A}))).$$

is represented by the matrix

$$\begin{pmatrix} \begin{matrix} -a_1[\frac{dh_1}{h_1} - \frac{dh_0}{h_0}] \\ -a_2[\frac{dh_2}{h_2} - \frac{dh_0}{h_0}] \end{matrix} & -a_2[\frac{dh_2}{h_2} - \frac{d(h_0-h_2)}{h_0-h_2}] & a_1[\frac{dh_1}{h_1} - \frac{d(h_0-h_1)}{h_0-h_1}] \\ -a_3[\frac{dh_2}{h_2} - \frac{dh_0}{h_0}] & \begin{matrix} -a_1[\frac{d(h_1-h_2)}{h_1-h_2} - \frac{d(h_0-h_2)}{h_0-h_2}] \\ -a_3[\frac{dh_2}{h_2} - \frac{d(h_0-h_2)}{h_0-h_2}] \end{matrix} & -a_1[\frac{d(h_1-h_2)}{h_1-h_2} - \frac{d(h_0-h_1)}{h_0-h_1}] \\ a_3[\frac{dh_1}{h_1} - \frac{dh_0}{h_0}] & -a_2[\frac{d(h_1-h_2)}{h_1-h_2} - \frac{d(h_0-h_2)}{h_0-h_2}] & \begin{matrix} -a_2[\frac{d(h_1-h_2)}{h_1-h_2} - \frac{d(h_0-h_1)}{h_0-h_1}] \\ -a_3[\frac{dh_1}{h_1} - \frac{d(h_0-h_1)}{h_0-h_1}] \end{matrix} \end{pmatrix}. \tag{27}$$

We would now like to calculate the monodromy of the Gauß-Manin connection along different elements of the fundamental group of S.

For $H_i \in \mathrm{Discr}(\mathcal{A})$ we have the residue map along H_i,

$$\begin{aligned}\mathrm{Res}_{H_i}(\overline{\nabla})\colon\ H^2(X/S, V) &\longrightarrow H^2(X/S, V) \otimes \Omega^1_S(\log \mathrm{Discr}(\mathcal{A}))\\ &\longrightarrow H^2(X/S, V) \otimes \mathcal{O}_{H_i},\end{aligned}$$

defined in the usual way; see [D1, II.3.7].

Fix a base point $p \in S$ and let $\gamma_i \in \pi(S, p)$ be a loop around $H_i \in \mathrm{Discr}(\mathcal{A})$ with base point p.

Let

$$T_i = \exp(-2\pi i \cdot \mathrm{Res}_{H_i}(\overline{\nabla})). \tag{28}$$

If we suppose that the difference of pairs of different eigenvalues of $\mathrm{Res}_{H_i}(\overline{\nabla})$ are not in $\mathbb{Z} \setminus \{0\}$ then by [D1, II.5.6] the local monodromy around H_i is given by T_i. The global monodromy is then given as a conjugacy class of T_i.

In our case, from the residue matrices along the hyperplanes of the discriminant one can see that almost all the eigenvalues are zero. Assume that $a_i + a_j \notin \mathbb{Z} \setminus \{0\}$ for $1 \le i < j \le 3$. By [D1, II.5.6], the local monodromy is given by T_i. The image of $\gamma_i \in \pi(S, p)$ under the global monodromy is conjugate to T_i.

To illustrate we compute the local monodromy around $H_1\colon h_1 = 0$ which is given as follows.

Let A_{H_1} be the residue matrix of the connection along H_1. From (27) we have

$$A_{H_1} = \mathrm{Res}_{H_1}(\overline{\nabla}) = \begin{pmatrix} -a_1 & 0 & a_1 \\ 0 & 0 & 0 \\ a_3 & 0 & -a_3 \end{pmatrix}.$$

For $n \ge 1$, we have

$$A^n_{H_1} = (-a_1 - a_3)^{n-1} A_{H_1}.$$

One can see that $(-a_1 - a_3)$ is the trace of the matrix A_{H_1} which is an eigenvalue. We have $A^n_{H_1} = (\mathrm{tr}(A_{H_1}))^{n-1} A_{H_1}$ where tr is the trace of the matrix. If $a_1 + a_3 \notin \mathbb{Z} \setminus \{0\}$ then, from (28), the monodromy transform is given by a conjugacy class of

$$T_1 = I + (\exp(-2\pi i \cdot (-a_1 - a_3)) - 1) \cdot \begin{pmatrix} \frac{-a_1}{(-a_1-a_3)} & 0 & \frac{a_1}{(-a_1-a_3)} \\ 0 & 0 & 0 \\ \frac{a_3}{(-a_1-a_3)} & 0 & \frac{-a_3}{(-a_1-a_3)} \end{pmatrix}.$$

For any other hyperplane $H \in \mathcal{A}$ the residue matrix A_H has the same property, namely

$$A^n_H = \alpha^{n-1} A_H$$

where α is the trace of the residue matrix which at the same time is an eigenvalue of A_H. This implies that if $\alpha \notin \mathbb{Z} \setminus \{0\}$, the local monodromy is given as

$$T_H = I + (\exp(-2\pi i \cdot \mathrm{tr}(A_{H_1})) - 1)\mathrm{tr}(A_{H_1})^{-1} \cdot \alpha A_{H_1} \alpha^{-1}.$$

5. Ceva's configuration

Let $\mathcal{A} = \cup_{i=0}^{5} H_i$ be Ceva's arrangement in $\mathbb{P}^2$ given by

$$\begin{aligned} H_0 &:= z_0 = 0 \\ H_1 &:= z_1 = 0 \\ H_2 &:= z_1 = 0 \\ H_3 &:= z_3 := z_0 - z_1 = 0 \\ H_4 &:= z_4 := z_0 - z_2 = 0 \\ H_5 &:= z_5 := z_1 - z_2 = 0; \end{aligned} \tag{29}$$

see (12).

The discriminant is

$$\begin{aligned} \mathrm{Discr}(\mathcal{A}) = \{ & h_0 = 0, h_1 = 0, h_2 = 0, h_0 + h_1 = 0, h_1 + h_2 = 0, h_0 + h_2 = 0, \\ & h_0 + h_1 + h_2 = 0\}. \end{aligned}$$

Let $X = S \times \mathbb{P}^2 \setminus \{\Delta := h_0 z_0 + h_1 z_1 + h_2 z_2 = 0\} \cup \{S \times \mathcal{A}\}$ and $\pi \colon X \longrightarrow S$ be a family of arrangements parameterized by $S = \mathbb{P}^{2^\vee} \setminus \mathrm{Discr}(\mathcal{A})$. We denote the divisor $(S \times \mathcal{A}) \cup \{\Delta \cap (S \times \mathbb{P}^2)\}$ by D.

Let $\rho \colon \tilde{X} \longrightarrow S \times \mathbb{P}^2$ be the blow up along the elements of $\mathcal{L}(D)$ as in (2); see Remark 3.1. Let $\tilde{D} = \rho^*(D)$.

Let H_0 be the hyperplane at infinity of the projective arrangement (29). Let $W = \tilde{X} \setminus \tilde{D}$. Let $\omega \in H^0(W, \Omega^1_W)$ be given by

$$\omega = \sum_{i=1}^{5} a_i \frac{d_{\mathrm{rel}} x_i}{x_i} + a_h \frac{d_{\mathrm{rel}} x_l}{x_l}$$

where $x_i = z_i/z_0, l_i = h_i/h_0, x_l = l_1 x_1 + l_2 x_2 + 1$ and $\frac{d_{\mathrm{rel}} x_i}{x_i}$ is taken as in Remark 1.2 with the relative differential along S. We assume that $\sum_{i=0}^{5} a_i + a_h = 0$, $a_i \notin \mathbb{Z}$ for $i \in \{0, \dots, 5, h\}$ and $\sum_{i \in I_L} a_i \notin \mathbb{Z}$ for $L \in \mathcal{L}(\mathcal{A})$. Let $\tilde{\omega} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log \tilde{D}))$ be given as $\tilde{\omega} = \rho^* \omega$.

As in Section 4, the operator $\bar{\nabla} = d_{\mathrm{rel}} + \tilde{\omega}$ defines the complex

$$0 \longrightarrow \mathcal{O}_{\tilde{X}} \xrightarrow{\bar{\nabla}} \Omega^1_{\tilde{X}}(\log \tilde{D}) \xrightarrow{\bar{\nabla}} \Omega^2_{\tilde{X}/S}(\log \tilde{D}) \longrightarrow 0.$$

From Theorem 3.3 and [D1, II.6], the cohomology of the local system V obtained as the flat sections of $\bar{\nabla}$ is

$$H^i(\tilde{X}/S, V) = \mathbf{R}^i \pi_* \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) = 0 \tag{30}$$

for $i = 0, 1$.

For $i = 2$, from Theorem 3.6 we have

$$
\begin{aligned}
H^2(\tilde{X}/S, V) &= \mathbf{R}^2\pi_*\Omega^{\cdot}_{\tilde{X}/S}(\log \tilde{D}) \\
&= \mathcal{O}_S\frac{dx_1 \wedge dx_2}{x_1x_2} \oplus \mathcal{O}_S\frac{dx_1 \wedge dx_4}{x_1x_4} \oplus \mathcal{O}_S\frac{dx_1 \wedge dx_5}{x_1x_5} \\
&\quad \oplus \mathcal{O}_S\frac{dx_2 \wedge dx_3}{x_2x_3} \oplus \mathcal{O}_S\frac{dx_3 \wedge dx_4}{x_3x_4} \oplus \mathcal{O}_S\frac{dx_3 \wedge dx_5}{x_3x_5}.
\end{aligned}
\tag{31}
$$

We lift ω to $\Omega \in H^0(W, \Omega^1_W)$ given by

$$\Omega = \sum_{i=1}^{5} a_i\frac{dx_i}{x_i} + a_h\frac{dx_l}{x_l}$$

where the differential is no longer the relative differential and where $x_i = z_i/z_0$, $l_i = h_i/h_0$, $x_l = 1 + l_1x_1 + l_2x_2$ and dx_i/x_i are taken as in Remark 1.2. We extend $\tilde{\omega}$ to $\tilde{X}$ to an element $\tilde{\Omega} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log \tilde{D}))$ as $\tilde{\Omega} = \rho^*\Omega$. We have the operator $\overline{\nabla} = d + \tilde{\Omega}$. We have the Gauß-Manin connection

$$\overline{\nabla}\colon H^2(\tilde{X}/S, V) \longrightarrow H^2(\tilde{X}/S, V) \otimes \mathcal{O}_S.$$

To write the matrix of the Gauß-Manin connection with respect to the basis (31) we apply the relations (14). The matrix is given as follows.

The first column is

$$
\begin{pmatrix}
(-a_1 - a_5)[\frac{dh_1}{h_1} - \frac{dh_0}{h_0}] - a_2[\frac{dh_2}{h_2} - \frac{dh_0}{h_0}] \\
-a_4[\frac{dh_2}{h_2} - \frac{dh_0}{h_0}] \\
a_5[\frac{dh_1}{h_1} - \frac{dh_2}{h_2}] \\
a_3[\frac{dh_1}{h_1} - \frac{dh_0}{h_0}] \\
0 \\
0
\end{pmatrix}.
$$

The second column is

$$
\begin{pmatrix}
-a_2[\frac{dh_2}{h_2} - \frac{d(h_0+h_2)}{(h_0+h_2)}] \\
-a_1[\frac{dh_1}{h_1} - \frac{d(h_0+h_2)}{(h_0+h_2)}] - a_4[\frac{dh_2}{h_2} - \frac{d(h_0+h_2)}{(h_0+h_2)}] \\
-a_5[\frac{dh_2}{h_2} - \frac{d(h_0+h_2)}{(h_0+h_2)}] \\
0 \\
(-a_3 - a_5)[\frac{dh_1}{h_1} - \frac{d(h_0+h_2)}{(h_0+h_2)}] \\
a_5[\frac{dh_1}{h_1} - \frac{d(h_0+h_2)}{(h_0+h_2)}]
\end{pmatrix}.
$$

The third column is

$$\begin{pmatrix} a_2[\frac{d(h_1+h_2)}{h_1+h_2} - \frac{dh_2}{h_2}] \\ -a_4[\frac{dh_2}{h_2} - \frac{dh_0}{h_0}] \\ (-a_1 - a_2)[\frac{d(h_1+h_2)}{h_1+h_2} - \frac{dh_0}{h_0}] - a_5[\frac{dh_2}{h_2} - \frac{dh_0}{h_0}] \\ 0 \\ a_4[\frac{d(h_1+h_2)}{h_1+h_2} - \frac{dh_0}{h_0}] \\ (-a_3 - a_4)[\frac{d(h_1+h_2)}{h_1+h_2} - \frac{dh_0}{h_0}] \end{pmatrix}.$$

The fourth column is

$$\begin{pmatrix} (a_1 + a_5)[\frac{dh_1}{h_1} - \frac{d(h_0+h_1)}{h_0+h_1}] \\ 0 \\ (-a_5[\frac{dh_1}{h_1} - \frac{d(h_0+h_1)}{h_0+h_1}] \\ -a_3[\frac{dh_1}{h_1} - \frac{d(h_0+h_1)}{h_0+h_1}] - a_2[\frac{dh_2}{h_2} - \frac{d(h_0+h_1)}{h_0+h_1}] \\ a_4[\frac{dh_2}{h_2} - \frac{d(h_0+h_1)}{h_0+h_1}] \\ a_5[\frac{dh_2}{h_2} - \frac{d(h_0+h_1)}{h_0+h_1}] \end{pmatrix}.$$

The fifth column is

$$\begin{pmatrix} 0 \\ -a_1[\frac{dh_1}{h_1} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2}] \\ 0 \\ a_2[\frac{dh_2}{h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2}] \\ (-a_3 - a_5)[\frac{dh_1}{h_1} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2}] - a_4[\frac{dh_2}{h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2}] \\ a_5[\frac{dh_1}{h_1} - \frac{dh_2}{h_2}] \end{pmatrix}.$$

The sixth column is

$$\begin{pmatrix} a_2[\frac{d(h_1+h_2)}{h_1+h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2}] \\ 0 \\ (-a_1 - a_2)[\frac{d(h_1+h_2)}{h_1+h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2}] \\ a_2[\frac{dh_2}{h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2}] \\ a_4[\frac{d(h_1+h_2)}{h_1+h_2} - \frac{dh_2}{h_2}] \\ (-a_3 - a_4)[\frac{d(h_1+h_2)}{h_1+h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2}] - a_5[\frac{dh_2}{h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2}] \end{pmatrix}.$$

Appendix

Let $\{H_i\}_{i \in I}$ be a family of distinct hyperplanes in $\mathbb{P}^n$, $\mathcal{A} = \sum_{i \in I} H_i$ the associated effective divisor and $U = \mathbb{P}^n \setminus \mathcal{A}$ the complementary affine open set. We have the following definition given in [ESV, Definition (**Bad**)].

Definition A.1. (a) Given a linear subspace $L \subset \mathbb{P}^n$, let

$$I_L = \{i \in I \mid L \subset H_i\}.$$

(b) We define the set

$$\mathcal{L}_j(\mathcal{A}) = \{L \subset \mathbb{P}^n \text{ linear} \mid \dim L = j \text{ and} L = \cap_{i \in I_L \setminus \{i_0\}} H_i \text{ for every } i_0 \in I_L\}$$

for $0 \leq j \leq n-2$. Let

$$\mathcal{L}(\mathcal{A}) = \cup_{j=0}^{n-2} \mathcal{L}_j(\mathcal{A}).$$

The loci where $\mathcal{A}$ has non-normal crossings are exactly the linear subspaces contained in $\mathcal{L}(\mathcal{A})$. When there is no possible confusion about the divisors we will write only $\mathcal{L}$. Let X be the variety obtained by considering successive blow ups along the elements of $\mathcal{L}$ in the following way.

Let $\pi^{(r)} = \tau_1 \circ \cdots \circ \tau_r$,

$$X_r \xrightarrow{\tau_r} X_{r-1} \xrightarrow{\tau_{r-1}} \cdots \longrightarrow X_1 \xrightarrow{\tau_1} \mathbb{P}^n, \tag{32}$$

where τ_i is the blow up of X_{i-1} along the proper transform T_{i-1} under $\pi^{(i-1)}$ of the elements of $\mathcal{L}_{i-1}$. As shown in [ESV, Claim], T_{i-1} is the disjoint union of closed nonsingular subschemes. Let $X = X_{n-1}$ and $\pi = \pi^{(n-1)}$. Then X is nonsingular.

Definition A.2. Let $\pi\colon X \longrightarrow \mathbb{P}^n$ be the *standard* resolution of $\mathcal{A}$.

LEMMA A.3. *Let $I' \subset I$ and consider the divisor $H' = \sum_{i \in I'} H_i$. Let $\pi'\colon X' \longrightarrow \mathbb{P}^n$ be the standard resolution of H'. Then there exists a morphism $\gamma\colon X \longrightarrow X'$ such that*

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & \mathbb{P}^n & \end{tikzcd}$$

commutes.

Proof. Let $\mathcal{L}'_j = \mathcal{L}(H')$ be the bad strata of dimension j of H' and let $\mathcal{L}' = \cup_{j=0}^{n-2} \mathcal{L}'_j$. Note that $\mathcal{L}'_j \subset \mathcal{L}_j$. Let $\tau'_j\colon X'_j \longrightarrow X'_{j-1}$ be the j-th blow up of H'. Assume that we have constructed inductively a morphism $\gamma_j\colon X_j \longrightarrow X'_j$. Since X'_{j+1} is obtained by blowing up part of the center of τ_{j+1} there exists $\gamma_{j+1}\colon X_{j+1} \longrightarrow X'_{j+1}$ as well. $\square$

We would like to apply the previous lemma to the special case when $I' = I \setminus \{i_0\}$ with $i_0 \in I$.

We have the diagram

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & \mathbb{P}^n & \end{tikzcd}.$$

In [ESV, Section 2], one finds the following lemma which we prove in [K2] by algebraic methods without referring to A.7.

LEMMA A.4. *Let $H = \sum_{i \in I} H_i$ be a non-trivial configuration of hyperplanes in $\mathbb{P}^n$, $\pi\colon X \longrightarrow \mathbb{P}^n$ a standard resolution and $D = \pi^*(H)$ the reduced pull back divisor of H. Then, for $p > 0$, we have*

$$H^p(X, \Omega_X^q(\log D)) = 0.$$

Let z_i be the projective defining equation for H_i. We fix H_0 with $0 \in I$ as the hyperplane at infinity. Let $x_i = z_i/z_0$ and let $\omega_i = d\log x_i$ be the differential form with a logarithmic pole along H_i with residue 1 and a logarithmic pole along H_0 with residue -1. Let $\omega \in H^0(U, \Omega_U^1)$ be given by

$$\omega = \sum_{i \in I \setminus \{0\}} a_i \omega_i \tag{33}$$

with $a_i \in \mathbb{C}$. The section ω has a logarithmic pole along H_0 with residue $a_0 = -\sum_{i \in I \setminus \{0\}} a_i$. Let $\tilde{\omega} = \pi^*\omega$ where $\pi\colon X \longrightarrow \mathbb{P}^n$ is the *standard* resolution of $\mathcal{A}$ and let $D = \pi^*(\mathcal{A})$. As $H^0(X, \Omega_X^1(\log D))$ injects into $H^0(U, \Omega_U^1)$, we still denote $\pi^*\omega_i$ by ω_i. The form ω defines a connection $d + \omega$ on the rank 1 bundle $\mathcal{O}_X$ which, as $d\omega = 0$, is *integrable*. We have $U = X \setminus D$ and let $j\colon U \longrightarrow X$ the inclusion. Let $\Omega_U^\bullet$ be the de Rham complex with the differential $\nabla = d + \omega$. We have a local constant system V over U given as $V = \ker(\nabla)$.

Let $\Gamma^p = H^0(X, \Omega_X^p(\log(D)))$.

Definition A.5. Let $A^p \subset \Gamma^p$ be given as

$$A^p = \left\{ \bigwedge_{j=1}^{p} \omega_{i_j} \mid i_j \in I \setminus \{0\} \right\} \tag{34}$$

where, as above, ω_i is the pull back of the logarithmic differential form ω_i.

Taking the exterior product by ω from Definition A.5 we obtain the complex

$$0 \longrightarrow A^0 \xrightarrow{\omega} A^1 \xrightarrow{\omega} A^2 \xrightarrow{\omega} \cdots \xrightarrow{\omega} A^n \longrightarrow 0. \tag{35}$$

This complex appeared for the first time in [A].

From [ESV] we use the following theorem; also see [Y].

THEOREM A.6. *Let $\omega \in H^0(U, \Omega^1_U)$ be as in equation* (33), *let $\nabla = d + \omega$ and let $V = \mathrm{Ker}(\nabla)$ be the corresponding local system. Suppose that for every $i \in I$ and every $l \in \mathcal{L}$, the residues a_i and $\sum_{i \in I_l} a_i$ don't lie in $\mathbb{N} \setminus \{0\}$. Then the inclusion*

$$A^\bullet \hookrightarrow H^0(U, \Omega^\bullet_U, \nabla)$$

is a quasiisomorphism.

Brieskorn [B, Lemma 5] proved the following lemma using topology. We give an algebraic proof of this lemma here; we will apply a similar method in Section 3.

LEMMA A.7. *The set A^p generates Γ^p as a $\mathbb{C}$ vector space.*

Proof. The proof will be by induction on $|I|$. For $|I| = 1$ we only have one hyperplane, namely H_0, the one at infinity, so $A^p = \emptyset$. On the other hand, from the exact sequence

$$0 \longrightarrow \Omega^q_{\mathbb{P}^n} \longrightarrow \Omega^q_{\mathbb{P}^n}(\log H_1) \longrightarrow \Omega^{q-1}_{\mathbb{P}^{n-1}} \longrightarrow 0 \tag{36}$$

we have

$$H^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(\log(H_0))) = 0 \tag{37}$$

for $p > 0$.

Let $|I| > 1$. For $I' \subset I$ a proper not empty subset, we can assume that $0 \in I'$ otherwise we can choose another hyperplane as the one at infinity. Let

$$\Gamma'^p = H^0(X, \Omega^p_X(\log \gamma^*(D')))$$

where $\pi'\colon X' \longrightarrow \mathbb{P}^n$ is the standard resolution of $H' = \sum_{i \in I'} H_i$, $D' = \pi'^*(H')$ and $\gamma\colon X \longrightarrow X'$ is the morphism given by Lemma A.3.(a). Let $A'^p = \{\bigwedge^p_{i_j} \omega_{i_j} \mid i_j \in I' \setminus \{0\}\}$. As the induction hypothesis we assume that the claim holds true for any proper subset $I' \subset I$. We fix $i_0 \in I$ with $i_0 \neq 0$ and let $I' = I \setminus \{i_0\}$. We have the exact sequence

$$0 \longrightarrow \Omega^q_X(\log \gamma^*(D')) \longrightarrow \Omega^q_X(\log D) \longrightarrow \Omega^{q-1}_{D_{i_0}}(\log \gamma^*(D') \mid_{D_{i_0}}) \longrightarrow 0. \tag{38}$$

Applying Lemma A.4 to the exact sequence of cohomology obtained from (38) we have the following exact sequence

$$\begin{aligned} 0 \longrightarrow\ & H^0(X, \Omega^p_X(\log \gamma^*(D'))) \longrightarrow H^0(X, \Omega^p_X(\log(D))) \\ & \longrightarrow H^0(D_{i_0}, \Omega^{p-1}_{D_{i_0}}(\log(D')|_{D_{i_0}})) \longrightarrow 0. \end{aligned} \tag{39}$$

The left map in (39) is given by the natural inclusion and the right one is given by

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_q} \mapsto \begin{cases} 0 & \text{if} \quad i_j \neq i_0 \text{ for } 1 \leq j \leq p \\ \omega_{i_1} \wedge \cdots \wedge \widehat{\omega_{i_j}} \wedge \cdots \wedge \omega_{i_q}|_{D_{i_0}} & \text{if} \quad i_j = i_0 \text{ for } 1 \leq j \leq p. \end{cases} \tag{40}$$

By induction on the dimension the restriction of this map to $A'^{n-1} \wedge \omega_{i_0}$ is surjective and one obtains

$$A^p = A'^p + A'^{p-1} \wedge \omega_{i_0}. \tag{41}$$

Induction on $|I|$ proves the claim. $\square$

Remark A.8. Theorem A.6 follows from Lemma A.7 and Lemma A.4. Since the latter is obtained algebraically this proof is different from the one in [ESV], which is based on Brieskorn's Lemma.

Remark A.9. The sum (41) is a direct sum for the case when $l \cap H_{i_0} \neq l$ for every $l \in \mathcal{L}$, i.e., when H_{i_0} does not contain "bad loci".

Proof. The result follows directly from Proposition 2.12, since one can see that there exists no non trivial relation equal to zero involving elements of A'^p and $A'^{p-1} \wedge \omega_{i_0}$. $\square$

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