AN EXTENSION OF A FINE-CYCLIC ADDITIVITY THEOREM FOR A FUNCTIONAL[1]

BY

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1. Introduction

E. J. Mickle and T. Rado in [2], and C. J. Neugebauer in [5], have given cyclic and fine-cyclic additivity theorems for a class of functionals defined for continuous mappings from a Peano space to a metric space, generalizing well-known cyclic and fine-cyclic additivity theorems for Lebesgue area. Using the results of the above papers, we obtain an extension of Neugebauer's result by replacing the requirement that the middle space be of finite degree of multicoherence by the requirement that each proper cyclic element be approximatable by a sequence of irreducible K-chains, as defined in §3, in a manner analogous to the cyclic chain approximation method used in [2].

2. Summary of analytic definitions and known results

The basic definitions of [5] are as follows:

Let P be a Peano space, and let P^* be a metric space. If $X \subset P$, denote by (T, X) a continuous mapping from X into P^*, and let $\mathfrak{T}(X)$ be the collection of all such mappings. Let $\mathfrak{A}$ denote the set of all open sets of P and let $\mathfrak{T}^*$ be the collection $\{\mathfrak{T}(A) : A \in \mathfrak{A}\}$, and let $\mathfrak{T} = \mathfrak{T}(P)$. An *unrestricted factorization* of a mapping $(T, A) \in \mathfrak{T}^*$ consists of a middle space M, and two continuous mappings s, f, where $f : A \to M$ and $s : M \to P^*$, so that $(T, A) = sf$. Two mappings (T_1, A_1), (T_2, A_2) constitute a *partition* of a mapping $(T, P) \in \mathfrak{T}$ provided there are a finite set of points F in P^* and a pair of nonempty closed sets E_1, E_2 of P such that, for $i = 1, 2$,

(1) $P = E_1 \cup E_2, \quad E_i \subset A_i$.

(2) $T_i(x) = T(x)$ for $x \in E_i, \quad T_i$ maps $A_i - E_i$ into F.

(3) $T(E_1 \cap E_2) \subset F$.

Let Φ be a nonnegative functional defined on $\mathfrak{T}^*$ (possibly with infinite values), satisfying

(a) Φ is lower semicontinuous on $\mathfrak{T}$, in the sense that if $\{(T_n, P)\}$ is a sequence of mappings in $\mathfrak{T}$ converging on P uniformly to a mapping (T, P), then $\Phi(T, P) \leqq \liminf \Phi(T_n, P)$.

(b) If (T_1, A_1), (T_2, A_2) form a partition of (T, P), then $\Phi(T, P) = \Phi(T_1, P_1) + \Phi(T_2, P_2)$.

(c) If $(T, A) \in \mathfrak{T}^*$ admits of an unrestricted factorization $(T, A) = sf$,

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where M is a dendrite, then $\Phi(T, A) = 0$. We agree that for the empty set $\emptyset$, $\Phi(T, \emptyset) = 0$.

Let us call a 0-nodal decomposition, of a Peano space M, a decomposition of M into two closed connected sets B_1 and B_2 such that $M = B_1 \cup B_2$, and $B_1 \cap B_2 = Z$, a finite set, with $B_1 \neq Z \neq B_2$, and denote such a decomposition by (B_1, Z, B_2). B_1 and B_2 will be called 0-nodal sets, and we agree that M is a 0-nodal set.

In [3] and [4], a B-set of M is defined to be a nondegenerate continuum of M such that $B = M$ or each component of $M - B$ has finite frontier, a fine-cyclic element is a B-set which cannot be separated by any finite set, and a local A-set B is a nondegenerate closed subset of M such that $B = M$, or there is a connected open subset G of M containing B such that if $\{O\}$ is the collection of components of $G - B$, then, for $O \in \{O\}$, $G \cap \mathfrak{F}(O)$ is a single point, and if O', O'' are two elements of the frontier $\mathfrak{F}\{O\}$ such that

$$G \cap \mathfrak{F}(O') \cap \mathfrak{F}(O'') = \emptyset,$$

then $\bar{O}' \cap \bar{O}'' = \emptyset$. B is then called a (G, A)-set of M.

A retraction t from M onto a local A-set B of M is a continuous mapping such that

(1) there exists a connected open set G containing B such that B is a (G, A)-set of M;

(2) $t \mid G$ is the identity on B and sends every component of $G - B$ into its frontier relative to G;

(3) $t(M - G)$ is a subset of a dendrite $D \subset B$.

Neugebauer gives in [5] the following result basic to this paper.

(2.1) *If sf is an unrestricted factorization of $(T, P) \in \mathfrak{T}$, with middle space M, and (B_1, Z, B_2) is a 0-nodal decomposition of M, then there are retractions t_i from M onto B_i, $i = 1, 2$, such that*

$$\Phi(T, P) = \Phi(st_1 f, P) + \Phi(st_2 f, P).$$

Under the above conditions, and indeed somewhat less stringent conditions, Mickle and Rado in [2] proved the following theorem, extended in [5], and to be extended herein:

(2.2) *For each true cyclic element C of M, there is a unique monotone retraction r_c from M onto C, such that $\Phi(T, P) = \sum \Phi(sr_c f, P)$ where the summation is extended over all proper cyclic elements C of M.*

3. K-sets and irreducible K-chains

We will henceforth assume that M is a Peano space. A D_0-chain is the nonempty intersection of 0-nodal sets. (If a D_0-chain D is the intersection of 0-nodal sets each of whose boundaries consist of not more than n points, D will sometimes be called a D_n-chain.) A D_0-chain D is said to be *prime* if it satisfies the following condition: For each 0-nodal decomposition (B_1, Z, B_2), either $D \subset B_1$ or $D \subset B_2$. If X is not empty, $D_0(X)$ is the intersection of all 0-nodal sets containing X. A D_0-chain D is called *true* if $D = D_0(X)$ where X is connected and nondegenerate, or where X is a single point which is not an essential member of a finite set separating M. The properties of D_0-chains are discussed in [7]; in particular true D_0-chains of a continuum are continua.

A true D_0-chain is called a K-set if it is the intersection of a finite number of 0-nodal sets. It is clear that K is a K-set if and only if $K \cap (\overline{M - K})$ is finite or empty. It is shown in [7] that a true prime D_0-chain which is a K-set is a fine-cyclic element, and conversely, if K is a fine-cyclic element in a Peano space, and also a K-set, it is immediate that it is a true prime D_0-chain, for if no finite set of points separates K, no finite set can separate K in M. Thus whenever M is such that each true prime D_0-chain is a K-set, we may use interchangeably the terms "true prime D_0-chain" and "fine-cyclic element". Neugebauer showed in [3] that if M is cyclic and of finite degree of multicoherence, a fine-cyclic element is a K-set.

The following theorem is proved, in a more general setting, in [7].

(3.1) *If (B_1, Z, B_2) and (B_1', Z', B_2') are 0-nodal decompositions of M such that $B_1' \cap (B_1 - Z) \neq \emptyset \neq B_2 \cap B_2'$, then for each nondegenerate component Q of $B_1 \cap B_1'$, there is a 0-nodal decomposition (B_1'', Z'', B_2'') such that $Z'' \subset Q = B_1''$.*

It is also noted that the finite set of 0-nodal decompositions $\{(B_{1i}, Z_i, B_{2i})\}$ such that $\bigcup Z_i \subset K = \bigcap B_{1i}$ may be chosen so that for each $i \neq j$, $B_{2i} \cap B_{2j} = \emptyset$, so that the intersection of each subcollection of the sets $\{B_{1i}\}$ is a K-set.

(3.2) *If K is a K-set, there is only a finite number of components of $M - K$ with nondegenerate frontier.*

Proof. Otherwise, if U is the union of K with a set of pairwise disjoint neighborhoods of the finite set $\mathfrak{F}(M - K)$, there would be a limit point in $M - U$ of a sequence of points chosen one from each of the components of $M - K$ which meet $M - U$. This contradicts the fact that each component of $M - K$ is open.

Neugebauer in [4] has shown that B-sets which satisfy the property in (3.2) are local A-sets, so that K-sets are local A-sets.

(3.3) *If every true prime D_0-chain of M is a D_n-chain, $n > 0$, then for each $\varepsilon > 0$, there is only a finite number of true prime D_0-chains with diameter greater than ε.*

Proof. If the theorem is false, there is a sequence of distinct true prime D_0-chains $\{Q_k\}$ each of which has diameter not less than ε, and which converges to a limit continuum Q. We may assume that, for each k, $Q \cap Q_k$ is finite.

If $D_0(Q)$ is not prime, there is a 0-nodal decomposition (B_1, Z, B_2) such that Q meets $B_1 - Z$ and $B_2 - Z$, so that for large k, Q_k would also meet $B_1 - Z$ and $B_2 - Z$, contrary to the definition of Q_k, so that $D_0(Q)$ is prime. By assumption, there is an integer k such that $D_0(Q) = D_k(Q)$. Let $\delta = \varepsilon/3k$, and let $\eta > 0$ be such that each pair of points whose distance is less than η lie in a connected set of diameter less than δ. If $V_\eta(X)$ denotes an η-neighborhood of X, let k be an integer such that $Q_k \subset V_\eta(Q)$. There is an n-nodal decomposition (B_1, Z, B_2) such that $Q_k \subset B_1$, $Q \subset B_2$, and

$$Z = \{z_1, z_2, \cdots, z_m\}, \qquad m \leqq n.$$

There is a point $p \in Q_k - \bigcup_{i=1}^m V_\delta(z_i)$. Since

$$V_\eta(p) \cap Q \neq \emptyset \quad \text{but} \quad V_\varepsilon(p) \cap Z = \emptyset,$$

we have a contradiction of the uniform local connectedness of S. This completes the proof.

We remark that each K-set is a D_n-chain, for some integer n, so that (3.3) holds under the assumption that the true prime D_0-chains are K-sets.

An *irreducible chain* in M between points a and b of M is a true D_0-chain which properly contains no true D_0-chain which contains a and b. It is shown in [7] that

(3.4) *If I is an arc from a to b, such that for each true prime D_0-chain Q in $D_0(I)$, $Q \cap I$ is a nondegenerate continuum, and if, when Q_1, Q_2 are two true prime D_0-chains of $D_0(I)$ with nonempty intersection, then $Q_1 \cap Q_2 \cap I \neq \emptyset$; then $D_0(I)$ is irreducible.*

(3.5) *If I is a continuum irreducible from a to b such that for each true prime D_0-chain $Q \subset D_0(I)$, Q is a K-set, and $Q \cap I$ is an arc, then I is an arc.*

For the remainder of the paper we will assume that each true prime D_0-chain is a K-set.

(3.6) *Let I' be an arc from a to b. Then $D_0(I')$ contains an arc I from a to b such that $D_0(I)$ is an irreducible chain.*

Proof. By (3.3), we may enumerate the true prime D_0-chains of $D_0(I')$ in the order of nonincreasing diameter. Let $\{Q'_n\}$ be this sequence. If there are no members of this set, then by [7, (6.3)], $D_0(I') = I'$, and is clearly irreducible. For each n, $Q'_n \cap I'$ is nondegenerate, since otherwise, by (3.1) we can arrive at a 0-nodal decomposition (B_1, Z, B_2) such that $Q'_n \subset B_1$, $I' \subset B_2$. If I' has its natural order from a to b, there is a first point a_1 and a last point b_1 of I' in Q'_1. Since Q'_1 is arcwise connected, there is an arc I'_1 from a_1 to b_1 in Q'_1. Let I_1 be the arc consisting of the part I' from a to a_1, and the part of I' from b_1 to b. If n_2 is the first integer such that $Q'_{n_2} \subset D_0(I_1)$, we may repeat the process, obtaining an arc I_2 such that $I_2 \cap Q'_{n_2}$ is an arc, and $I_2 \subset D_0(I_1)$. Continuing inductively, we obtain a continuum

$I'' = \lim I_n$ such that $I'' \subset D_0(I_n)$ for every n, and such that if $Q' \subset D_0(I'')$, $Q' \cap I''$ is an arc. From (3.5), it follows that I'' is an arc from a to b.

For the second stage of the construction, let $\{Q''_n\}$ be the subsequence of $\{Q'_n\}$ such that $Q''_n \subset D_0(I'')$ for each n. Since $Q''_n \cap I''$ is a continuum, we may say that Q''_n lies to the left of Q''_m if $Q''_n \cap I''$ precedes $Q''_m \cap I''$ in the natural order of I'' from a to b. If there is a least integer n such that $Q_n \cap Q_1 \neq \emptyset$, if Q''_n lies to the left of Q''_1, let us replace the part of I'' from the first point of I'' in Q''_n to the last point of I'' in Q''_1 by an arc in $Q''_n \cup Q''_1$ whose parts in Q''_n and in Q''_1 are arcs, and similarly if Q''_n lies to the right of Q''_1. If the new arc is denoted by I_1, let us repeat the process if there is an integer $m > n$ such that

$$Q''_m \cap Q''_1 \neq \emptyset, \quad \text{but} \quad Q''_m \cap Q''_1 \cap I'_1 = \emptyset,$$

obtaining an arc I_{12} from a to b. On account of the fact that the diameters of Q''_m tend to zero, there can be at most a finite number of integers n such that $Q''_n \cap Q''_1 \neq \emptyset$. To see this we need only observe that if Q''_n meets Q''_1 on the left, the arc of I_{11} from a to Q''_n is at positive distance from Q''_1, and hence only a finite number of the sequence $\{Q''_n\}$ can meet this arc and Q''_1 as well, and similarly, only a finite number of the sequence can meet the part of I_{11} to the right of Q''. Thus after a finite number of steps, we arrive at an arc I''_1 such that if $Q'_n \subset D_0(I''_1)$, $Q''_n \cap I''_1$ is an arc, and such that there are at most two integers n such that $Q''_n \cap Q''_1 \neq \emptyset$, in which case $Q''_n \cap Q'_1 \cap I''_1 \neq \emptyset$. If we apply this process inductively we obtain a continuum I which has the properties:

(1) If $Q \subset D_0(I)$, $Q \cap I$ is an arc.

(2) If $Q_1, Q_2 \subset D_0(I)$, and $Q_1 \cap Q_2 \neq \emptyset$, $Q_1 \cap Q_2 \cap I \neq \emptyset$.

(3) If $Q \subset D_0(I)$, there are at most two true prime D_0-chains Q_1, Q_2 such that $Q_i \cap Q \neq \emptyset$.

From (3.5) it follows that I is an arc. From (3.4), $D_0(I)$ is an irreducible chain. This completes the proof.

It is well known (see [6] or [8]) that a nondegenerate Peano space P has a cyclic chain approximation, in that there is a sequence $\{C_n\}$ of cyclic chains such that $C_{n+1} \cap (\bigcup_{i=1}^{n} C_j)$ is a single point, and $\overline{(\bigcup_{j=1}^{\infty} C_j)} = P$. In [7], it is shown that separable metric continua possess, in a sense, approximations by irreducible chains, but in general even a Peano space may not have a readily usable approximation. Let us call an irreducible chain $D_0(I)$, where I is an arc and $D_0(I)$ is a K-set, an irreducible K-chain, or, simply, a *K-chain*, and restrict ourselves to Peano spaces for which each true prime D_0-chain is a K-set. Such a space M is said to *have a K-chain approximation* if in each K-set in M, and each pair of points in the boundary of K, there is a K-chain joining these points. The term is justified by the next result, where $\delta(X)$ denotes the diameter of x.

(3.7) *If a Peano space M has a K-chain approximation, there is a sequence $\{K_n\}$ of K-chains such that if $H_n = \bigcup_{k=1}^{n} K_k$, then*

(1) *H_n is a K-set, for each n.*
(2) $\overline{\bigcup H_n} = M$.
(3) *For each n, there is a finite set of 0-nodal decompositions*

$$\{(A_{ni}, Z_{ni}, B_{ni}) : i = 1, 2, \cdots, k_n\}$$

such that

$$\bigcup_{i=1}^{k_n} Z_{ni} \subset H_n = \bigcap_{i=1}^{k_n} B_{ni},$$

and such that if $d_n = \max_{i=1,2,\cdots,k_n} \delta(A_{ni})$, *then* $\lim d_n = 0$.

Proof. Let $\{p_n\}$ be a sequence of non-0-endpoints, dense in M, where a 0-endpoint is a point p such that $D_0(p) = p$, but there is no 0-nodal decomposition (A, Z, B) such that $p \in Z$. We may assume that there is at least one 0-nodal decomposition of M. It is clear that the sequence $\{p_n\}$ exists, for if $\{p'_n\}$ is any dense denumerable sequence, for each $p'_n \in \{p'_n\}$ which is a 0-endpoint, there is a sequence of non-0-endpoints which converges to p'_n, so that each 0-endpoint may be replaced by non-0-endpoints. By (3.6), there is a K-chain K_1 joining p_1 and p_2, and if n_1 is the least index such that $p_{n_1} \notin K_1$, there is a K-chain K_2 joining a point of K_1 to p_{n_1}, with $K_1 \cap K_2$ a finite nonempty set, since there is a 0-nodal decomposition (A, Z, B) such that $p_{n_1} \in A - Z$, and $Z \subset K_1 \subset B$. Let us add K-chains $K_2, K_3, \cdots, K_k$, so that $H_j = K_1 \cup \cdots \cup K_j$, $2 \leqq j \leqq k$ is a K-set, and so that $Z \subset H_k$.

By induction, the sequences $\{K_n\}$ and $\{H_n\}$ are defined, so that H_n satisfies (1), for each positive integer n. It is clear from (3.6) that (2) is also satisfied; each non-0-endpoint is an element of H_n, for some positive integer n.

Suppose p is a 0-endpoint not in $\bigcup_{n=1}^{\infty} H_n$, let V be any open set containing p. By definition of 0-endpoint and (3.1), there is an 0-nodal decomposition (A, Z, B) such that the compact set $M - V$ is contained in $B - Z$, so that $A \subset V$, with $p \in A - Z$. Further, there are a 0-nodal decomposition (A', Z', B') such that $p \in A' - Z'$ and $A' \subset A - Z$ and an integer n such that $Z' \subset H_n$. If $\{(A_{ni}, Z_{ni}, B_{ni})\}$ are 0-nodal decompositions such that

$$\bigcup_i Z_{ni} \subset H_n = \bigcap_i B_{ni},$$

then if A_{ni} is the nodal set containing p, $A_{ni} \subset A \subset V$. This completes the proof.

4. Finitely generalized retractions

In the attack on the main problem, we need a trivial modification of (2.1). Let us denote by "finitely generalized dendrite" a continuum H of finite degree of multicoherence which is the union of a finite number of dendrites, and by "finitely generalized retraction" from M onto B a retraction t, in which condition (3) for retractions (see §2) is replaced by

(3′) $t(M - G)$ is a subset of a finitely generalized dendrite $H \subset B$.

(4.1) *If sf is an unrestricted factorization of* $(T, P) \in \mathfrak{T}$ *with middle space* M, *if* (B_1, Z, B_2) *is a 0-nodal decomposition of* M, *and if* D_i, $i = 1, 2$, *are* *finitely generalized dendrites with $Z_i \subset D_i \subset B_i$, then there are finitely generalized retractions t_i from M onto B_i, such that $t_i(M - B_i) \subset D_i$, and such that*

$$\Phi(T, P) = \Phi(st_1 f, P) + \Phi(st_2 f, P).$$

Proof. We appeal to the development in [5]. If $i = 1$ or 2, there is a connected open set G_i containing B_i, such that B_i is a (G_i, A)-set of M, and there is, in the same manner as in [5], a finitely generalized retraction t_i such that $t_i(G_i - B_i) \subset Z$, $t_i(M - B_i) \subset D_i$. If we let D_i' be a dendrite such that $Z \subset D_i' \subset D_i$, and $A_i = f^{-1}(G_i)$, which we may assume nonempty, it follows at once that $(st_1 f, A_1)$ and $(st_2 f, A_2)$ constitute a partition of (T, P), so that, if Φ satisfies (c) of §2, then

$$\Phi(T, P) = \Phi(st_1 f, A_1) + \Phi(st_2 f, A_2).$$

If we consider the maps $t' = t_1 t_2$ on B_1, t_1 on B_2, then it is easily seen that $(st_1 f, A_1)$ and $(st'f, A_2)$ form a partition of $(st_1 f, P)$, and also that $(st'f, A_1)$ and $(st'f, A_2)$ form a partition of $(st'f, P)$. It follows from [5, §4, (ii)], since $D_1 \cup D_2$ is of finite degree of multicoherence, that $\Phi(st'f, P) = 0$. Thus if Φ satisfies the conditions listed in §2, $\Phi(st'f, A_2) = \Phi(st'f, A_1) = 0$, and since $\Phi(st_1 f, P) = \Phi(st_1 f, A_1) + \Phi(st'f, A_2)$, we have

$$\Phi(st_1 f, P) = \Phi(st_1 f, A_2).$$

Similarly

$$\Phi(st_2 f, P) = \Phi(st_2 f, A_2).$$

Accordingly,

$$\Phi(T, P) = \Phi(st_1 f, P) + \Phi(st_2 f, P).$$

For the remainder of the paper, we will designate "finitely generalized retractions from M onto B" more simply as "retractions from M onto B".

If K is a K-set, there is, by the remark following (3.1), a finite number k of 0-nodal decompositions $\{(B_{1i}, Z_i, B_{2i})\}$ such that if $i \neq j$, $B_{1i} \cap B_{1j} = \emptyset$; for $1 \leqq j \leqq k$, $\bigcap_{i=1}^{j} B_{1i}$ is a K-set, and $\bigcup_{i=1}^{k} Z_i \subset K = \bigcap_{i=1}^{k} B_{2i}$. It is a simple exercise in induction, using (4.1) to conclude that

(4.2) *There are retractions $t_1, t_2, \cdots, t_k$ from M onto $B_{11}, B_{12}, \cdots, B_{1k}$ respectively, and t_{k+1} from M onto K such that, if D_j is any finitely generalized dendrite with $Z_j \subset D_j \subset B_{ij}$, and D_{k+1} is any finitely generalized dendrite with $\bigcup_{j=1}^{k} Z_j \subset D_{k+1} \subset K$, then, for $1 \leqq j \leqq k$,*

(1) $t_j(M - B_{1j}) \subset D_j, \quad t_{k+1}(M - K) \subset D_{k+1}$;

(2) $\Phi(T, P) = \sum_{j=1}^{k+1} \Phi(st_j f, P)$.

(4.3) *If M has an approximation by a sequence $\{K_n\}$ of K-chains, as in (3.7), then there is a sequence $\{r_n\}$ of retractions such that r_n is from M onto H_n and $\{r_n\}$ converges uniformly to the identity.*

Proof. Let n be a positive integer. By (3) of (3.7) there is a finite set $\{(A_{ni}, Z_{ni}, B_{ni})\}$ of 0-nodal decompositions such that

$$\bigcup_i Z_{ni} \subset H_n = \bigcap_i B_{ni}.$$

Moreover, if $m > n$, and $H_m \cap (A_{ni} - Z_{ni}) \neq \emptyset$, then, by assumption, if $A_{mj} \subset A_{ni}$ properly, there is a dendrite F_{ni}, contained in $\overline{A_{ni} - A_{mj}}$, which can be extended to a finitely generalized dendrite F'_{ni} containing also all points of each Z_{mj} for which $A_{mj} \subset A_{ni}$ properly. By (4.2), there is a retraction r_m from M onto H_n, with $r_m(M - H_m)$ a subset of $\bigcup_{i=1}^{k'(n)} F_{ni}$, which is clearly a generalized dendrite, contained in a finite sum of ordinary dendrites. Since max $\delta(A_{ni})$ tends to zero as n tends to infinity, the proof is complete.

(4.4) *If M has an approximation by a sequence $\{K_n\}$ of K-chains, as in* (3.7), *and r_n is the retraction from M onto H_n, then if Φ satisfies the conditions of* §2, $\Phi(T, P) = \lim_{n\to\infty} \Phi(sr_n f, P)$.

Proof. By (4.2), if $n > m$, $\Phi(T, P) \geqq \Phi(sr_n f, P) \geqq \Phi(sr_m f, P)$, so that $\Phi(T, P) \geqq \limsup \Phi(sr_n f, P)$. By (4.3), $sr_n f$ converges uniformly to $T = sf$, so that $\Phi(T, P) \leqq \liminf \Phi(sr_n f, P)$. Accordingly $\Phi(T, P) = \lim \Phi(sr_n f, P)$. (If $\{K_n\}$ is finite, (4.2) alone will suffice.)

We observe that $sr_n f$ has middle space contained in H_n. Again by (4.2), if t_j is the composition of r_n followed by a retraction t'_{nj} from H_n onto K_j, $1 \leqq j \leqq n$, $\Phi(sr_n f, P) = \sum_{j=1}^{n} \Phi(st_j f, P)$, so that we have proved

(4.5) *If M has an approximation by a sequence $\{K_j\}$ of K-chains, then there are retractions t_j from M onto K_j such that* $\Phi(T, P) = \sum_{j=1}^{\infty} \Phi(st_j f, P)$.

(4.6) *If M is an irreducible chain $D_0(I)$, where I is an arc, then if $\{Q_j\}$ is the sequence of true prime D_0-chains of M, there are retractions $\bar{t}_j$ from M onto Q_j such that if Φ satisfies the conditions of* §2, $\Phi(T, P) = \sum_j \Phi(s\bar{t}_j f, P)$.

Proof. Let I be an arc such that $D_0(I)$ is an irreducible chain in M from a to b. It is clear that $D_0(I)$ can be written as the sum of a countable set of closed nondegenerate arcs $\{I_n\}$, each of which is contained in a unique fine-cyclic element $P \subset D_0(I)$, a countable set of open arcs $\{I'_n\}$, each of which meets no $P \subset D_0(I)$ (and is maximal with respect to that property), and a set of points $\{C_\alpha\}$ which are not in any of the sets of $\{I_n\}$, $\{I'_n\}$. It follows at once that each such point C_α is the limit of a null sequence of arcs of the null sequence $\{I_n\}$, and that each point of $I - (\bigcup I_n \cup \{a, b\})$ is a cutpoint of $D_0(I)$, by an argument similar to that in [7, (6.4)]. If $T \in \mathfrak{T}$ is a mapping from P onto P^*, such that the middle space M is contained in $D_0(I)$, the theorem of Mickle and Rado, (2.2), allows us to consider a single cyclic element C of $D_0(I)$, which can contain as endpoints at most two of the cut-points of $D_0(I)$, so that each point $I \cap C$ other than these two is contained in at least one, and at most two, fine-cyclic elements of C. Further, if P_1 and P_2 are fine-cyclic elements of C such that $P_1 \cap P_2 \neq \emptyset$, there is a 0-nodal decomposition of C into two irreducible chains C_1 and C_2 (which indeed can be extended to a 0-nodal decomposition of M), so that (4.1) permits restriction of attention to a mapping $T_1 \in \mathfrak{T}$ which has middle space C_1, possessing at most one cutpoint of $D_0(I)$.

If the number of fine-cyclic elements is finite, then by (4.2), T_1 is fine-cyclically additive on C_1. If not, from the above, there is a null sequence $\{P'_n\}$ of fine-cyclic elements of C_1, such that $P'_n \cap P'_m \neq \emptyset$ if and only if $|n - m| = 1$, and such that $\lim P'_n = c \in C_1$. By [4, §8], there is a retraction r_n from C_1 onto $Q'_n = \bigcup_{j=1}^{n} P'_j$ such that $r(C_1 - Q'_n)$ is a subset of a dendrite in P'_n. Since $\lim P'_n = c$, it is clear that the sequence $\{r_n\}$ converges uniformly to the identity mapping. Suppose $T_1 = s_1 f_1$, and $f_1(P) \subset C_1$. Then the mapping $s_1 r_n f_1$ has middle space contained in Q'_n, so that, if t'_n is the retraction from C_1 onto P'_n, we have by (4.2)

$$\Phi(s_1 r_n f, P) = \textstyle\sum_{j=1}^{n} \Phi(s_1 t'_j r_n f, P) = \sum_{j=1}^{n} \Phi(s_1 t'_j r_j f_1, P)$$

since $t'_j r_n$ is a retraction from C_1 onto P'_n such that $t'_j r_n(C_1 - P'_n)$ is a subset of a dendrite of P'_n, so that on Q'_j, $t'_j r_j = t'_j r_n$, for $j \leqq n$. Thus

$$\Phi(T_1, P) \leqq \liminf \Phi(s_1 r_n f, P) = \textstyle\sum_{j=1}^{\infty} \Phi(s_1 t'_j r_j f_1, P),$$

so that, since by (4.2), $\Phi(T_1, P) \geqq \Phi(s_1 r_n f_1, P)$, we have

$$\Phi(T_1, P) = \textstyle\sum_{j=1}^{\infty} \Phi(s_1 t'_j r_j f_1, P).$$

Finally,

(4.7) *If each true cyclic element of M has an approximation by K-chains, then if $\{Q_i\}$ is the (possibly finite or empty) sequence of fine cyclic elements, then for each integer j, there is a retraction $\bar{r}_j$ from M onto Q_j, such that, if Φ satisfies the conditions of* §2, *we have*

$$\Phi(T, P) = \textstyle\sum_{j=1}^{\infty} \Phi(s\bar{r}_j f, P).$$

Proof. By (2.2), if $\{C_n\}$ is the sequence of true cyclic elements of M, and r_n is the monotone retraction from M onto C_n, $\Phi(T, P) = \sum_n \Phi(sr_n f, P)$. By assumption, if $\{K_{nm}\}$ is a set of K-chains approximating C_n, t'_{nm} a retraction from C_n onto K_{nm} as in (3.7), and $t_{nm} = t'_{nm} r_n$, then, by (4.5), $\Phi(sr_n f, P) = \sum_m \Phi(st_{nm} f, P)$. It now follows from (4.6) that there is, for each fine-cyclic element Q_{nmi} of K_{nm}, a retraction t_{nmi} from M onto Q_{nmi} such that $\Phi(st_{nm} f, P) = \sum_i \Phi(st_{nmi} f, P)$. If we reindex the fine-cyclic elements of M so that we denote them as $\{Q_j\}$, and let $\bar{r}_j = t_{nmi}$, we have the result stated.

It is easy to construct examples showing the increase in generality of this result.

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