IMMERSIONS OF COMPACT METRIC SPACES INTO EUCLIDEAN SPACES[1]

BY

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Introduction

By an application of the Smith classes to the tubular neighborhood of the diagonal of the topological square X^2 of a finitely triangulable space X, W.-T. Wu [8] introduced his immersion classes $\Psi^n(X)$ for every $n = 1, 2, \cdots$ and proved that a necessary condition for X to be immersible into the n-dimensional Euclidean space R^n is $\Psi^n(X) = 0$. By means of this condition, he proved that the n-dimensional skeleton of the unit $(m + 2)$-simplex cannot be immersed in R^m if $n \leqq m \leqq 2n - 1$. His method is purely combinatorial, and hence it cannot be extended to general spaces.

In a recent paper on isotopy invariants [1], the author defined the enveloping space $E_m(X)$ of any given topological space X for each integer $m > 1$. If X is finitely triangulable, then $E_m(X)$ has the same homotopy type as the boundary of a tubular neighborhood of the diagonal in the topological power X^m.

The objective of the present paper is to apply the Smith theory to $E_m(X)$. This leads to the immersion classes $\Psi^n_m(X)$ defined for every topological space X. If X is a metric space, we consider a subspace $E_m(X, \delta)$ of $E_m(X)$ for every real number $\delta > 0$ and prove that the inclusion $E_m(E, \delta) \subset E_m(X)$ is a homotopy equivalence. This enables us to localize the situation and to establish the main theorem that a necessary condition for a compact metric space X to be immersible into R^n is $\Psi^n_2(X) = 0$.

CHAPTER I. GEOMETRICAL CONSTRUCTIONS

1. Residual and enveloping spaces

Let X be an arbitrary topological space, and $m > 1$ a given integer. Consider the m^{th} *(topological) power*

$$W = X^m$$

of the space X; in other words, W denotes the topological product $X \times \cdots \times X$ of m copies of the space X. There is a natural imbedding

$$d : X \to W$$

defined by $d(x) = (x, \cdots, x) \in W$ for every x in X. This imbedding d is called the *diagonal imbedding* of X into its m^{th} power X^m. By means of d, the space X can be identified with a subspace $d(X)$ of X^m, namely, the

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diagonal of the m^{th} power X^m. Thus, we obtain a pair (W, X) of a space $W = X^m$ and a subspace X of W.

The m^{th} *residual space* of the space X is defined to be the subspace

$$R_m(X) = W\backslash X = X^m\backslash X$$

of the m^{th} topological power X^m, where $W\backslash X$ denotes the set-theoretic difference.

Next, let us consider the space $P(W)$ of all paths $\sigma : I \to W$ in the topological power $W = X^m$ of the space X with the usual compact-open topology.

The m^{th} *enveloping space* of the space X is defined to be the subspace

$$E_m(X) = E(W, X) = E(X^m, X)$$

of the space $P(X)$ which consists of all paths $\sigma : I \to W$ such that $\sigma(t) \in X$ if and only if $t = 0$. In other words, a path $\sigma \in P(W)$ is in $E_m(X)$ if and only if it issues from X and never comes back to X again.

As shown in [1], the isotopy types of the spaces $R_m(X)$ and $E_m(X)$ are isotopy invariants of the space X. Hence, every isotopy invariant of $R_m(X)$ or $E_m(X)$ is an isotopy invariant of X. In particular, every homotopy invariant of $R_m(X)$ or $E_m(X)$ is an isotopy invariant of X, [2].

2. Operations of the cyclic group

Let $m > 1$ be a given integer, and let G denote the cyclic group of order m with ξ as a generator. Then G acts on the topological power $W = X^m$ as a group of left operators defined by

$$\xi(x_1, x_2, \cdots, x_{m-1}, x_m) = (x_2 . x_3, \cdots, x_m, x_1)$$

for every point $x = (x_1, \cdots, x_m)$ of X^m. Then the diagonal X of X^m is precisely the set of all fixed points of ξ and hence of all elements in G. Furthermore, ξ maps the subspace $R_m(X)$ of X^m homeomorphically onto itself. It follows that G acts on the m^{th} residual space $R_m(X)$ of X without fixed point provided that m is a prime.

Next, let $\sigma \in E_m(X)$ be arbitrarily given. Since ξ leaves X pointwise fixed and sends $R_m(X)$ into itself, it follows that the composed map

$$\xi \circ \sigma : I \to X^m$$

is in $E_m(X)$. Therefore, the cyclic group G acts on the m^{th} enveloping space $E_m(X)$ of X by means of the operation defined by

$$\xi(\sigma) = \xi \circ \sigma$$

for every $\sigma \in E_m(X)$. Since $\sigma(1) \in R_m(X)$ and ξ has no fixed point in $R_m(X)$, it follows that

$$\xi(\sigma) \neq \sigma \qquad (\sigma \in E_m(X)).$$

Hence G acts on $E_m(X)$ without fixed point provided that m is a prime.

The orbit spaces

$$R_m^*(X) = R_m(X)/G \quad \text{and} \quad E_m^*(X) = E_m(X)/G$$

will be called the m^{th} *cyclic residual space* and the m^{th} *cyclic enveloping space* of X respectively. If $i : X \to Y$ is an imbedding, then i induces imbeddings

$$R_m^*(i) : R_m^*(X) \to R_m^*(Y), \qquad E_m^*(i) : E_m^*(X) \to E_m^*(Y)$$

in the obvious way. Furthermore, if the imbedding i is an isotopy equivalence [2, p. 168], it follows as in [1, p. 343] that the imbeddings $R_m^*(i)$ and $E_m^*(i)$ are isotopy equivalences. Hence the isotopy type of the spaces $R_m^*(X)$ and $E_m^*(X)$ are isotopy invariants of the space X.

3. The natural projections

Consider the m^{th} enveloping space $E_m(X)$ and the m^{th} residual space $R_m(X)$ of a given space X as defined in §1. Let σ be an arbitrary point in $E_m(X)$; then σ is a path $\sigma : I \to X^m$ such that $\sigma(t) \in X$ if and only if $t = 0$. In particular, $\sigma(1)$ is a point of $R_m(X)$. Hence, the assignment $\sigma \to \sigma(1)$ defines a function

$$\pi : E_m(X) \to R_m(X)$$

which will be called the *natural projection* from $E_m(X)$ to $R_m(X)$. Since $E_m(X)$ is a subspace of the space $P(X^m)$ of paths in X^m with the compact-open topology and $R_m(X)$ is a subspace of X^m, it is obvious that π is continuous.

Next, considering the homeomorphisms ξ on both $E_m(X)$ and $R_m(X)$ as defined in §2, we obtain the following diagram:

$$\begin{tikzcd} E_m(X) \arrow[r, "\pi"] \arrow[d, "\xi"'] & R_m(X) \arrow[d, "\xi"] \\ E_m(X) \arrow[r, "\pi"] & R_m(X). \end{tikzcd}$$

From the definitions of the mappings ξ and π, one can easily see that the preceding rectangle is commutative, i.e.,

$$\pi \circ \xi = \xi \circ \pi$$

holds. It follows that π induces a mapping

$$\pi^* : E_m^*(X) \to R_m^*(X)$$

in the orbit spaces. This continuous map will be called the *natural projection* from $E_m^*(X)$ to $R_m^*(X)$.

4. The subspace $E_m(X, \delta)$

Throughout the present section, let X be an arbitrarily given metric space with a distance function

$$d : X \times X \to R.$$

This distance function d in X induces a distance function

$$d : X^m \times X^m \to R$$

in the topological power X^m defined by

$$d(u, v) = \operatorname{Max} \{d(u_i, v_i) \mid i = 1, \cdots, m\}$$

for arbitrary points $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_m)$ in X^m.

For any positive real number δ, let $E_m(X, \delta)$ denote the subspace of the m^{th} enveloping space $E_m(X)$ which consists of all paths $\sigma \in E_m(X)$ satisfying the condition

$$d[\sigma(0), \sigma(t)] < \delta$$

for every $t \in I$. Obviously, $E_m(X, \delta)$ is invariant under the operators G defined on $E_m(X)$ in §2. In fact, ξ sends $E_m(X, \delta)$ onto itself. Therefore, we have the orbit space

$$E^*_m(X, \delta) = E_m(X, \delta)/G.$$

THEOREM 4.1. *There exists a homotopy*

$$h_t : E_m(X) \to E_m(X) \qquad (t \in I)$$

satisfying the following conditions:

(4.1A) *h_0 is the identity map on $E_m(X)$.*

(4.1B) *h_1 sends $E_m(X)$ into $E_m(X, \delta)$.*

(4.1C) *For every $t \in I$, h_t sends the subspace $E_m(X, \delta)$ into itself.*

(4.1D) *For every $t \in I$, $h_t \circ \xi = \xi \circ h_t$.*

Proof. Define a real-valued function κ on the topological product $E_m(X) \times I$ by taking

$$\kappa(\sigma, t) = \delta^{-1} \sup_{s \leqq t} d[\sigma(0), \sigma(s)]$$

for every $\sigma \in E_m(X)$ and every $t \in I$. Continuity of κ is obvious. Furthermore, for any given σ in $E_m(X)$, the function $\kappa_\sigma : I \to I$ defined by

$$\kappa_\sigma(t) = \min [\kappa(\sigma, t), 1] \qquad (t \in I)$$

is continuous and nondecreasing. By the definition of $E_m(X)$, we have $\kappa_\sigma(t) = 0$ if and only if $t = 0$. Hence we may define a continuous real function

$$\mu : E_m(X) \to I$$

by taking $\mu(\sigma)$ to be the unique solution of the equation

$$\kappa_\sigma(t) = 1 - t$$

in the variable $t \in I$. It is easily verified that $0 < \mu(\sigma) < 1$ for each σ in $E_m(X)$.

By means of the continuous real function μ, we may define a homotopy

$$h_t : E_m(X) \to E_m(X) \qquad (t \in I)$$

as follows. For each path $\sigma \in E_m(X)$ and each $t \in I$, $h_t(\sigma) \in E_m(X)$ is defined to be the path in X^m given by

$$[h_t(\sigma)](s) = \sigma[s - st + st\mu(\sigma)]$$

for every $s \in I$. Intuitively speaking, $h_t(\sigma)$ is obtained from σ by omitting the part of σ outside of the point $\sigma[1 - t + t\mu(\sigma)]$.

It follows immediately from the definition of the homotopy h_t that h_0 is the identity map on $E_m(X)$. Hence (4.1A) holds.

By the construction of μ, one can easily verify that

$$d\{[h_1(\sigma)](0), [h_1(\sigma)](t)\} = d\{\sigma(0), \sigma[t\mu(\sigma)]\} = \delta[1 - \mu(\sigma)] < \delta$$

for each $\sigma \in E_m(X)$ and each $t \in I$. Hence $h_1(\sigma)$ is in $E_m(X, \delta)$. This proves (4.1B).

By the definition of h_t, it is clear that h_t sends $E_m(X, \delta)$ into itself. Hence (4.1C) is satisfied.

Since h_t is defined essentially coordinatewise, we obviously have (4.1D). This completes the proof of (4.1).

Because of (4.1D), the homotopy h_t induces a homotopy

$$h_t^* : E_m^*(X) \to E_m^*(X) \qquad (t \in I)$$

of the m^{th} cyclic enveloping space

$$E_m^*(X) = E_m(X)/G.$$

The conditions (4.1A–C) imply that h_0^* is the identity map on $E_m^*(X)$, h_1^* sends $E_m^*(X)$ into the subspace

$$E_m^*(X, \delta) = E_m(X, \delta)/G,$$

and h_t^* sends $E_m^*(X, \delta)$ into itself. Hence, we have the following corollary.

COROLLARY 4.2. *The inclusion map*

$$i^* : E_m^*(X, \delta) \subset E_m^*(X)$$

is a homotopy equivalence.

CHAPTER II. APPLICATION TO IMMERSIONS

5. Immersion classes

By an *imbedding* of a space X into a space Y, we mean a continuous map

$$f : X \to Y$$

which carries X homeomorphically onto a subspace $f(X)$ of Y.

By an *immersion* of a space X into a space Y, we mean a continuous map

$$g : X \to Y$$

such that, for each point $x \in X$, there exists a neighborhood U of x in X such that the restriction $g \mid U$ is an imbedding of U into Y.

Let X be an arbitrarily given space, and $m > 1$ any prime number.

Consider the m^{th} residual space $R_m(X)$ and the m^{th} enveloping space $E_m(X)$ of the given space X together with the periodic homeomorphisms ξ on $R_m(X)$ and $E_m(X)$ induced by the operation

$$(x_1, x_2, \cdots, x_{m-1}, x_m) \rightarrow (x_2, x_3, \cdots, x_m, x_1).$$

The Smith invariants defined in [4] and [8] can be obviously generalized to singular homology. Since the homeomorphisms ξ are free of fixed points, the Smith characteristic classes of the pairs $(R_m(X), \xi)$ and $(E_m(X), \xi)$ are well-defined. Let us denote for each integer $n = 1, 2, 3, \cdots$

$$\Phi_m^n(X) = \chi^n[R_m(X), \xi], \qquad \Psi_m^n(X) = \chi^n[E_m(X), \xi].$$

The classes $\Phi_m^n(X)$ are exactly the *imbedding classes* of X studied by W.-T. Wu in [8]. The classes $\Psi_m^n(X)$ will be called the *immersion classes* of the given space X. In case X is a finite simplicial complex, one can prove that these immersion classes are essentially those introduced by Wu [8] by means of the tubular neighborhood of X in X^m. The method of proof is similar to that used by the author in [1] and hence is left to the interested reader.

For the important special case $m = 2$, we will use the simpler notation:

$$\Phi^n(X) = \Phi_2^n(X), \qquad \Psi^n(X) = \Psi_2^n(X).$$

The class $\Phi^n(X)$ will be called the *n-dimensional imbedding class* of X, and $\Psi^n(X)$ will be called the *n-dimensional immersion class* of X.

The natural projection

$$\pi^* : E_m^*(X) \rightarrow R_m^*(X)$$

induces a homomorphism

$$\pi^{**} : H^n(R_m^*(X); G) \rightarrow H^n(E_m^*(X); G)$$

for each integer n and every abelian coefficient group G. Since π^* is induced by the natural projection

$$\pi : E_m(X) \rightarrow R_m(X)$$

which commutes with ξ, we have the following proposition.

PROPOSITION 5.1. *For every $n = 1, 2, \cdots$, we have*

$$\pi^{**}[\Phi_m^n(X)] = \Psi_m^n(X).$$

6. Homomorphisms induced by imbeddings

Let us consider an arbitrarily given imbedding $i : X \rightarrow Y$ of a space X into any space Y. According to §2, this imbedding i induces imbeddings

$$R_m^*(i) : R_m^*(X) \rightarrow R_m^*(Y),$$

$$R_m^*(i) : E_m^*(X) \rightarrow E_m^*(Y).$$

As continuous maps, these imbeddings induce homomorphisms

$$R_m^{**}(i) : H^n(R_m^*(Y); G) \to H^n(R_m^*(X); G),$$
$$E_m^{**}(i) : H^n(E_m^*(Y); G) \to H^n(E_m^*(X); G),$$

for each dimension n and every abelian coefficient group G. In case i is an isotopy equivalence, then $R_m^*(i)$ and $E_m^*(i)$ are obviously isotopy equivalences, and hence we get the following proposition.

PROPOSITION 6.1. *If $i : X \to Y$ is an isotopy equivalence, then the induced homomorphisms $R_m^{**}(i)$ and $E_m^{**}(i)$ are isomorphisms.*

The following proposition is obvious.

PROPOSITION 6.2. *For an arbitrary imbedding $i : X \to Y$ we always have*

$$R_m^{**}(i)[\Phi_m^n(Y)] = \Phi_m^n(X), \qquad E_m^{**}(i)[\Psi_m^n(Y)] = \Psi_m^n(X).$$

It follows from (6.1) and (6.2) that the immersion classes $\Psi_m^n(X)$, as well as the imbedding classes $\Phi_m^n(X)$, of any given topological space X are isotopy invariants [2].

7. Homomorphisms induced by immersions

In the present section, we are concerned with an arbitrarily given immersion $j : X \to Y$ of a compact metric space X into any topological space Y.

For each point x of X, choose an open neighborhood U_x of x in X such that $j \mid U_x$ is an imbedding. Since X is compact, the open cover

$$\mathfrak{C} = \{U_x \mid x \in X\}$$

has a finite subcover $\mathfrak{F}$; in other words, there exists a finite number of points $x_1, \cdots, x_q$ in X such that the subfamily

$$\mathfrak{F} = \{U_{x_1}, \cdots, U_{x_q}\}$$

of $\mathfrak{C}$ covers the space X. Let $\varepsilon > 0$ denote a Lebesgue number of $\mathfrak{F}$, that is to say, ε is a positive real number such that every subset of X with diameter not greater than ε is contained in at least one member of $\mathfrak{F}$.

Let $\delta = \frac{1}{2}\varepsilon$, and consider the subspace $E_m(X, \delta)$ of the m^{th} enveloping space $E_m(X)$ of the metric space X as defined in §4.

Let $\sigma \in E_m(X, \delta)$ be arbitrarily given. Since $\sigma : I \to X^m$ is a path in the m^{th} topological power X^m of X, we may compose σ with the m^{th} topological power

$$j^m : X^m \to Y^m$$

of the given immersion $j : X \to Y$ and obtain a path

$$j^m \circ \sigma : I \to Y^m.$$

By the choice of the real number $\delta > 0$, one can easily see that $j^m \circ \sigma$ is in the m^{th} enveloping space $E_m(Y)$. In fact, j^m defines an imbedding

$$E_m(j) : E_m(X, \delta) \to E_m(Y).$$

Since $E_m(j)$ commutes with the periodic homeomorphisms ξ, it induces an imbedding

$$E_m^*(j) : E_m^*(X, \delta) \to E_m^*(Y).$$

For each dimension n and every abelian coefficient group G, $E_m^*(j)$ induces a homomorphism

$$E_m^{**}(j, \delta) : H^n(E_m^*(Y); G) \to H^n(E_m^*(X, \delta); G).$$

The following lemma is obvious.

LEMMA 7.1. *For each $n = 1, 2, \cdots$, we have*

$$E_m^{**}(j, \delta)[\Psi_m^n(Y)] = \chi^n[E_m(X, \delta), \xi].$$

Next, consider the inclusion map

$$i^* : E_m^*(X, \delta) \subset E_m^*(X)$$

induced by the inclusion map

$$i : E_m(X, \delta) \subset E_m(X)$$

which commutes with the periodic homeomorphisms ξ on $E_m(X, \delta)$ and $E_m(X)$.

For each dimension n and every abelian coefficient group G, i^* induces a homomorphism

$$i^{**} : H^n[E_m^*(X); G] \to H^n[E_m^*(X, \delta); G].$$

The following lemma is obvious.

LEMMA 7.2. *For each $n = 1, 2, \cdots$, we have*

$$i^{**}[\Psi_m^n(X)] = \chi^n[E_m(X, \delta), \xi].$$

Since i^* is a homotopy equivalence by (4.2), we have the following lemma.

LEMMA 7.3. *The homomorphism i^{**} is an isomorphism.*

By means of the inverse of i^{**}, we may define a homomorphism

$$E_m^{**}(j) = (i^{**})^{-1} \circ E_m^{**}(j, \delta) : H^n[E_m^*(Y); G] \to H^n[E_m^*(X); G]$$

for each dimension n and every abelian coefficient group G. One can easily verify that $E_m^{**}(j)$ is independent of the choice of the positive real number δ used in the construction.

Combining (7.1) and (7.2), we get the following theorem.

THEOREM 7.4. *For each $n = 1, 2, \cdots$, we have*

$$E_m^{**}(j)[\Psi_m^n(Y)] = \Psi_m^n(X).$$

8. Main theorem

THEOREM 8.1. *If a compact metric space X can be immersed in the n-dimensional Euclidean space R^n, then* $\Psi^n(X) = 0$.

Because of (5.1) and (7.4), this theorem is a consequence of the following lemma which was known to W.-T. Wu [8].

LEMMA 8.2. $\Phi^n(R^n) = 0$.

Proof. Consider the unit $(n-1)$-sphere S^{n-1} in R^n, and define a continuous map

$$f : R_2(R^n) \to S^{n-1}$$

as follows. Let (x, y) be an arbitrary point of the second residual space $R_2(R^n)$. Then x and y are two distinct points in R^n and hence determine a directed line $\overrightarrow{xy}$. From the origin O of R^n, draw a half line along the direction $\overrightarrow{xy}$. This half line meets S^{n-1} at a unique point $f(x, y)$. The assignment $(x, y) \to f(x, y)$ defines a continuous map f from $R_2(R^n)$ into S^{n-1}.

Now consider the homeomorphisms

$$\xi : R_2(R^n) \to R_2(R^n), \qquad \xi : S^{n-1} \to S^{n-1}$$

defined by $\xi(x, y) = (y, x)$ for every point $(x, y) \in R_2(R^n)$ and $\xi(z) = -z$ for every $z \in S^{n-1}$. Then we have

$$R_2^*(R^n) = R_2(R^n)/\xi, \qquad P^{n-1} = S^{n-1}/\xi,$$

where P^{n-1} denotes the $(n-1)$-dimensional real projective space.

Since $f \circ \xi = \xi \circ f$, f induces a continuous map

$$f^* : R_2^*(R^n) \to P^{n-1}.$$

For each dimension q and every abelian coefficient group G, f^* induces a homomorphism

$$f^{**} : H^q(P^{n-1}; G) \to H^q[R_2^*(R^n); G].$$

As in [4] and [8], we obtain

$$\Phi^q(R^n) = f^{**}[\chi^q(S^{n-1}, \xi)].$$

For the special case $q = n$, we have

$$\chi^n(S^{n-1}, \xi) = 0$$

since P^{n-1} is of dimension $n - 1$. This implies

$$\Phi^n(R^n) = 0$$

and completes the proof of (8.2).

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