LIKELIHOOD RATIOS FOR STOCHASTIC PROCESSES RELATED BY GROUPS OF TRANSFORMATIONS

BY

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1. Introduction

If $x(t)$ and $y(t)$ are stochastic processes with the same parameter set, they induce measures m_x and m_y on a suitably chosen space of sample functions. It is an important problem of statistics to find conditions guaranteeing the existence of the Radon-Nikodym derivative (or likelihood ratio) dm_x/dm_y and to find formulas for computing it. These derivatives are also helpful in describing one process in terms of the other, in particular, in carrying almost everywhere properties from one process to another which is less well known.

This problem has been studied most in the case where $x(t)$ and $y(t)$ are closely related to a Brownian-motion process (see, for example, [1], [2], [7], [10], and [11]). Prokhorov [9] and Skorokhod [12] have investigated the case where $x(t)$ and $y(t)$ are solutions of a diffusion equation (again, of course, closely related to Brownian motion), and Skorokhod [13] has also investigated the case where $x(t)$ and $y(t)$ are processes with independent increments. The most important case in engineering applications is that for which the processes are Gaussian. This has been attacked by, among others, Grenander [6], Slepian [14], Feldman [5], and Woodward [15].

In most of the above work the special nature of the processes involved is relied on, in particular, the independence or near independence of many of the random variables arising in the computations. In this paper we shall develop a technique relying less on such computations and more on assumed geometrical relationships between the processes. This technique has already been applied in [8] to the mean value problem, $y(t) = x(t) + f(t)$ for a fixed $f(t)$ when $x(t)$ is the solution of a diffusion equation.

Throughout Sections 2 and 3 we shall make the following assumptions. We assume given a set X, a σ-algebra S of subsets of X, a probability measure P on (X, S), an algebra F of bounded, real-valued S-measurable functions containing the constant functions, and a one-parameter group T_α of automorphisms of F. F and T_α are to satisfy

(1) T_α preserves bounds and $T_\alpha f(x)$ has a continuous derivative which is bounded uniformly in α and x for every f in F and x in X.

(2) If f_n is a uniformly bounded sequence from F with $\lim f_n(x) = 0$ for all x, then $\lim T_\alpha f_n(x) = 0$ for all x.

(3) There exists a function ϕ in some $L_p(P)$, $1 \leqq p < \infty$, satisfying, for

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every f in F

$$\int \phi f \, dP = \frac{\partial}{\partial \alpha} \int (T_\alpha f) \, dP \bigg|_{\alpha=0} .$$

Examples of such situations are given in Section 4 of this paper.

We shall write Df for $\frac{\partial}{\partial \alpha} T_\alpha f \Big|_{\alpha=0}$. By the Stone-Weierstrass theorem, for every f and g in F the functions max (f, g) and min (f, g) are in $\bar{F}$, the uniform closure of F. $\bar{F}$ contains f^p for every positive f in $\bar{F}$, and T_α can be extended to $\bar{F}$. The functionals $l_\alpha : l_\alpha(f) = \int T_\alpha f \, dP$ defined on $\bar{F}$ can be extended to Daniell integrals $l_\alpha(f) = \int f \, dP_\alpha$ where P_α are probability measures on subfields S_α of S. Both $\bar{F}$ and F are dense subsets of $L_p(P_\alpha)$ for every α. We shall assume in what follows that ϕ is S_0-measurable (replacing it by its conditional expectation on S_0 with respect to P_0 if necessary).

It is easily verified that if the P_α are absolutely continuous with respect to P_0, the transformations $V(\alpha)$ defined on F by

$$V(\alpha)f = \left[\frac{dP_\alpha}{dP_0}\right]^{1/p} T_{-\alpha} f$$

can be extended to a group of isometries of $L_p(P_0)$ into itself, and that, at least formally, the generator of $V(\alpha)$ contains the operator A defined on F by $Af = (1/p)\phi f - Df$. In Section 2 we shall construct approximations to the semigroups $V(\alpha)$, $\alpha \geqq 0$, and $V(-\alpha)$, $\alpha \geqq 0$, and in Section 3 we shall find conditions under which these semigroups are isometries. Section 4 is devoted to applications of these results.

2. The semigroups $V_+(\alpha)$ and $V_-(\alpha)$

For any f in F and $\alpha \geqq 0$ we define a transformation of $\bar{F}$ into bounded S_0-measurable functions by

$$V_f(\alpha)g = \left(\exp \int_0^\alpha T_{-\beta} f \, d\beta\right) T_{-\alpha} g.$$

LEMMA 2.1. *$V_f(\alpha)$ takes $\bar{F}$ into $\bar{F}$, and $V_f(\alpha) V_f(\beta) = V_f(\alpha + \beta)$. If g is in F, $V_f(\alpha)g$ is in the domain of $\bar{D}$, the closure of D, and*

$$\bar{D} V_f(\alpha)g = f V_f(\alpha)g - \frac{\partial}{\partial \alpha} V_f(\alpha)g.$$

We have, for g in F,

$$\frac{\partial}{\partial \alpha} \int V_f(\alpha) g \, dP_0 = \int (f - \phi) V_f(\alpha) g \, dP_0 .$$

Proof. Since the derivative of $T_\alpha f(x)$ is bounded uniformly in α and x, $\sigma_n = (\alpha/n) \sum_{k=0}^n T_{-k\alpha/n} f$ converges uniformly to $\int_0^\alpha T_{-\beta} f \, d\beta$, and hence

$$V_f(\alpha)g = \lim_n \sum_{k=0}^n \frac{1}{k!} \left(\int_0^\alpha T_{-\beta} f \, d\beta\right)^k T_{-\alpha} g$$

is in $\bar{F}$. It is easily verified that

$$T_\gamma \left(\exp \int_0^\alpha T_{-\beta} f \, d\beta\right) = \exp \int_0^\alpha T_{-\beta+\gamma} f \, d\beta$$

and hence that $V_f(\alpha) V_f(\beta) g = V_f(\alpha + \beta) g$. It follows from the continuity and boundedness of $DT_{-\alpha} f$ that $D\sigma_n^k$ converges boundedly to

$$k \left(\int_0^\alpha T_{-\beta} f \, d\beta\right)^{k-1} \int_0^\alpha DT_{-\beta} f \, d\beta = (f - T_{-\alpha} f) k \left(\int_0^\alpha T_{-\beta} f \, d\beta\right)^{k-1}.$$

Thus, for g in F and

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \left(\int_0^\alpha T_{-\beta} f \, d\beta\right)^k T_{-\alpha} g,$$

Ds_n converges boundedly to

$$(f - T_{-\alpha} f) V_f(\alpha) g + (V_f(\alpha) 1) DT_{-\alpha} g = f V_f(\alpha) g - \frac{\partial}{\partial \alpha} V_f(\alpha) g,$$

which proves the second assertion. Finally,

$$\int \phi V_f(\alpha) g \, dP_0 = \lim \int \phi s_n \, dP_0 = \lim \int Ds_n \, dP_0$$

$$= \int f V_f(\alpha) g \, dP_0 - \int \left(\frac{\partial}{\partial \alpha} V_f(\alpha) g\right) dP_0$$

which completes the proof of Lemma 2.1.

LEMMA 2.2. *For any sequence f_n from F converging to*

$$(1/p)\phi_N = (1/p) \min(\phi, N)$$

and bounded above, and any $\alpha \geqq 0$, the operators $V_{f_n}(\alpha)$ converge to an operator $V_N(\alpha)$ on $\bar{F}$. Each $V_N(\alpha)$ has a unique extension to $L_p(P_0)$ satisfying

(1) $\| V_N(\alpha) \| \leqq 1$.

(2) *$V_N(\alpha)$, $\alpha \geqq 0$ is a strongly continuous semigroup with $V_N(0) = I$.*

(3) *$V_N(\alpha) f$ is nonnegative if f is.*

(4) *$V_N(\alpha)(fg) = (V_N(\alpha) f) T_{-\alpha} g$ for every f in $L_p(P_0)$ and g in $\bar{F}$.*

(5) *The generator A_N of $V_N(\alpha)$ is the closure of the operator*

$$f \to (1/p)\phi_N f - Df$$

defined on F.

Proof. For any f and g in F, with f bounded above by M, we have

$$\frac{\partial}{\partial \alpha} \int (V_f(\alpha) 1 - V_g(\alpha) 1)^2 \, dP$$

$$= \frac{\partial}{\partial \alpha} \int (V_{2f}(\alpha) 1 - 2V_{(f+g)}(\alpha) 1 + V_{2g}(\alpha) 1) \, dP$$

$$= \int [(2f - \phi)V_{2f}(\alpha)1 - 2(f + g - \phi)V_{f+g}(\alpha)1 + (2g - \phi)V_{2g}(\alpha)1]\, dP$$

$$= \int [(2f - \phi_N) + (\phi_N - \phi)](V_f(\alpha)1 - V_g(\alpha)1)^2\, dP$$

$$+ \int (f - g)(2V_{f+g}(\alpha)1 - 2V_{2g}(\alpha)1)\, dP$$

$$\leqq \{\| 2f - \phi_N \| + \| f - g \|\} 4e^{2\alpha M},$$

so that

$$\int (V_f(\alpha)1 - V_g(\alpha)1)^2\, dP \leqq \{\| 2f - \phi_N \| + \| f - g \|\} 4\alpha e^{2\alpha M}.$$

Hence assuming that the f_n's are bounded above by M, and using $| x - y |^p \leqq | x^p - y^p |$ which holds for positive x and y and $p \geqq 1$ gives, if $\sup | g(x) | \leqq 1$,

$$\| V_{f_n}(\alpha)g - V_{f_m}(\alpha)g \|^p \leqq \| V_{f_n}(\alpha)1 - V_{f_m}(\alpha)1 \|^p$$

$$\leqq \int | V_{pf_n}(\alpha)1 - V_{pf_m}(\alpha)1 |\, dP$$

$$\leqq 2e^{\alpha M} \int | V_{pf_n/2}(\alpha)1 - V_{pf_m/2}(\alpha)1 |\, dP$$

$$\leqq 2e^{\alpha M} \left(\int | V_{pf_n/2}(\alpha)1 - V_{pf_m/2}(\alpha)1 |^2 \right)^{1/2}$$

$$\leqq 4\sqrt{\alpha} e^{2\alpha M} (\| pf_n - \phi_N \| + (p/2) \| f_n - f_m \|)^{1/2}.$$

This proves that $V_{f_n}(\alpha)g$ converges uniformly for α in a bounded interval and fixed g in $\bar{F}$ to an element $V_N(\alpha)g$ in $L_p(P_0)$.

For any positive g in F,

$$\frac{\partial}{\partial \alpha} \int V_{pf_n}(\alpha)g\, dP = \int (pf_n - \phi)V_{pf_n}(\alpha)g\, dP \leqq 2e^{\alpha M} \| pf_n - \phi_N \|,$$

so that

$$\int (V_N(\alpha)1)^p T_{-\alpha}\, g\, dP \leqq \int g\, dP.$$

This extends easily to g in $\bar{F}$; in particular it is true for g^p if g is positive and in $\bar{F}$ so $\| V_N(\alpha)g \| \leqq \| g \|$. Hence $V_N(\alpha)$ can be extended to an operator on $L_p(P_0)$ satisfying (1). Properties (3) and (4) are proved by simple continuity arguments. For f in F,

$$V_N(\alpha)\, V_N(\beta)f = \lim_n V_N(\alpha)\, V_{f_n}(\beta)f = \lim_n (V_N(\alpha)1)\, T_{-\alpha}\, V_{f_n}(\beta)f$$

$$= \lim_n (V_{f_n}(\alpha)1)\, T_{-\alpha}\, V_{f_n}(\beta)f = \lim_n V_{f_n}(\alpha)\, V_{f_n}(\beta)f$$

$$= V_N(\alpha + \beta)f,$$

and because of (1), this implies that $V_N(\alpha)$ is a semigroup. Again, for f in F, because $V_{f_n}(\alpha)f$ converges to $V_N(\alpha)f$ uniformly in α,

$$\| V_N(\alpha)f - V_N(\beta)f \|$$
$$\leqq \| V_N(\alpha)f - V_{f_n}(\alpha)f \| + \| V_N(\beta)f - V_{f_n}(\beta)f \| + \| V_{f_n}(\alpha)f - V_{f_n}(\beta)f \|$$

can be made arbitrarily small, proving that $V_N(\alpha)$ is strongly continuous and completing the proof of (4). In proving (5), it will be sufficient to show (see [4, Corollary 16, p. 627]) that

$$(\lambda - A_N) \int_0^\infty e^{-\alpha\lambda} V_N(\alpha)f\, d\alpha = f,$$

and because of (1), we need only show this for f in F. From Lemma 2.1,

$$A_N V_{f_n}(\alpha)f = \left(\frac{1}{p}\phi_N - f_n\right) V_{f_n}(\alpha)f + \frac{\partial}{\partial\alpha}(V_{f_n}(\alpha)f),$$

so

$$(\lambda - A_N) V_{f_n}(\alpha)f = \left(f_n - \frac{1}{p}\phi_N\right) e^{-\alpha\lambda} V_{f_n}(\alpha)f - \frac{\partial}{\partial\alpha}(e^{-\alpha\lambda} V_{f_n}(\alpha)f),$$

and using Riemann approximating sums gives

$$(\lambda - A_N) \int_0^b e^{-\alpha\lambda} V_{f_n}(\alpha)g\, d\alpha$$
$$= \left(f_n - \frac{1}{p}\phi_N\right) \int_0^b e^{-\alpha\lambda} V_{f_n}(\alpha)f\, d\alpha + f - e^{-b\lambda} V_{f_n}(b)f.$$

The proof is completed by letting f_n converge to $(1/p)\phi_N$ and be bounded from above, and then letting b go to ∞.

THEOREM 2.1. *$V_N(\alpha)$ converges strongly to a strongly continuous semigroup $V_+(\alpha)$ satisfying*
(1) $\| V_+(\alpha) \| \leqq 1$.
(2) $V_+(\alpha)(fg) = (V_+(\alpha)f)T_{-\alpha}\, g$ *for g in $\bar{F}$.*
(3) *$V_+(\alpha)$ preserves positivity.*
(4) *The generator of $V_+(\alpha)$ contains the operator A defined on F by*

$$Af = (1/p)\phi f - Df.$$

Proof. For positive f_n in $L_p(P_0)$, $V_N(\alpha)f$ is a nondecreasing, nonnegative sequence with $\| V_N(\alpha)f \| \leqq \| f \|$ and hence, converges for such f and trivially then for all f in $L_p(P_0)$. Properties (1), (2), and (3) are immediate. For f in F,

$$\| V_N(\alpha)f - V_N(\beta)f \| \leqq \int_\alpha^\beta \| V_N(\gamma)A_N f \|\, d\gamma \leqq | \beta - \alpha |\, \| A_N f \|$$
$$\leqq | \beta - \alpha |\, (\sup | f(x) |\, \| \phi \| + \| Df \|)$$

so that

$$\| V_+(\alpha)f - V_+(\beta)f \| \leqq \| V_+(\alpha)f - V_N(\alpha)f \| \\ + \| V_+(\beta)f - V_N(\beta)f \| + \| V_N(\alpha)f - V_N(\beta)f \|$$

can be made arbitrarily small by choosing $| \beta - \alpha |$ small enough and then N large enough. This proves the strong continuity of $V_+(\alpha)$. The semigroup property of $V_+(\alpha)$ now follows straightforwardly from the fact that the $V_N(\alpha)$ are semigroups with $\| V_N(\alpha) \| \leqq 1$. For any f in F, since A_N is the generator of $V_N(\alpha)$,

$$V_+(\alpha)f = \lim_N V_N(\alpha)f = f + \lim_N \int_0^\alpha V_N(\beta)A_N f\, d\beta = f + \int_0^\alpha V(\beta)Af\, d\beta,$$

and thus

$$\lim \frac{V_+(\varepsilon)f - f}{\varepsilon} = \lim \frac{1}{\varepsilon} \int_0^\varepsilon V(\gamma)Af\, d\gamma = Af.$$

This establishes (5) and completes the proof of Theorem 2.1.

Theorem 2.1 also holds, of course, with T_α, D, and ϕ replaced by $T'_\alpha = T_{-\alpha}$, $D' = -D$, and $\phi' = -\phi$, giving a strongly continuous semigroup $V_-(\alpha)$ satisfying

(1)′ $\| V_-(\alpha) \| \leqq 1$,
(2)′ $V_-(\alpha)(fg) = (V_-(\alpha)f)T_\alpha g$ for g in $\bar{F}$,
(3)′ $V_-(\alpha)$ preserves positivity, and
(4)′ $-A$ is contained in the generator of $V_-(\alpha)$.

THEOREM 2.2. *$V_-(\alpha)V_+(\alpha)f(x) = e_\alpha(x)f(x)$, where $e_\alpha = V_-(\alpha)V_+(\alpha)1$ is an $L_p(P_0)$ continuous family of functions with $e_0 = 1$. The e_α are non-increasing in α, $0 \leqq e_\alpha \leqq 1$, and*

$$e_\alpha = 1 - \lim_{n\to\infty} \frac{1}{p} \int_0^\alpha V_-(\gamma)[(\phi - \phi_n)V_n(\gamma)1]\, d\gamma.$$

If $e_\alpha = 1$ for some $\alpha > 0$, then $V_-(\beta)V_+(\beta) = V_+(\beta)V_-(\beta) = I$ for all β.

Proof. If f is in $\bar{F}$, then

$$V_-(\alpha)V_+(\alpha)f = V_-(\alpha)[(V_+(\alpha)1)T_{-\alpha}f] = (V_-(\alpha)V_+(\alpha)1)f$$

by properties (3) and (3)′ above, and this equation extends immediately to all f in $L_p(P_0)$. The $L_p(P_0)$ continuity of e_α follows from the strong continuity of the semigroups. It is also apparent from this equation for e_α that $e_0 = 1$ and $e_\alpha \geqq 0$. For any f in F, $V_f(\alpha)1$ can be approximated boundedly by elements s_n from F as in Lemma 2.1 with

$$\lim_n As_n = (1/p)\phi V_f(\alpha)1 - \bar{D}V_f(\alpha)1,$$

so that

$$\frac{\partial}{\partial\alpha} V_-(\alpha)V_f(\alpha)1 = V_-(\alpha)\left[\left(f - \frac{1}{p}\phi\right)V_f(\alpha)1\right],$$

and hence

$$V_-(\alpha)V_f(\alpha)1 = 1 - \frac{1}{p}\int_0^\alpha V_-(\beta)[(\phi - pf)V_f(\beta)1]\, d\beta.$$

The formula of the theorem is obtained by letting f be bounded from above and converge to $(1/p)\phi_n$ and then letting n go to ∞. It is clear from this formula that $e_\alpha \leqq 1$ and e_α is nonincreasing. Suppose finally that $e_\alpha = 1$ for some $\alpha > 0$, so that $V_-(\beta)V_+(\beta) = I$ for $\beta \leqq \alpha$. If G is the generator of $V_+(\beta)$ and f is in the domain of G, then $\| (V_-(\varepsilon)f - f)/\varepsilon + Gf \| \leqq \| V_-(\varepsilon)((f - V_+(\varepsilon)f)/\varepsilon + Gf) \| + \| V_-(\varepsilon)Gf - Gf \|$ which goes to 0. Thus the generator of $V_-(\beta)$ contains $-G$ and therefore equals $-G$ (again by [4, Corollary 16, p. 627]). For any f in the domain of G,

$$\frac{\partial}{\partial\beta} V_-(\beta)V_+(\beta)f = V_-(\beta)[-G + G]V_+(\beta)f = 0,$$

and this completes the proof.

THEOREM 2.3. *If $e_\alpha = 1$ for some $\alpha > 0$, then*

(1) *For any α and β, $S_\alpha = S_\beta$ and P_α and P_β are mutually absolutely continuous.*

(2) *T_α has an extension to $L_p(P_0)$ which is linear, preserves bounds, and satisfies $T_\alpha(fg) = (T_\alpha f)(T_\alpha g)$ whenever f, g, and fg are in $L_p(P_0)$.*

(3) *$V_+(\alpha)f = (dP_\alpha/dP)^{1/p}T_{-\alpha}f$ and $V_-(\alpha)f = (dP_{-\alpha}/dP)^{1/p}T_\alpha f$ for all f in $L_p(P_0)$, and all $\alpha \geqq 0$.*

(4) *There is a measurable version of $T_\alpha\phi$ which satisfies*

$$\log\frac{dP_\alpha}{dP} = \int_0^\alpha T_{-\beta}\,\phi\, d\beta.$$

Proof. From Theorem 2.2, $V(\alpha) : V(\alpha) = V_+(\alpha)$ if $\alpha \geqq 0$ and $V(\alpha) = V_-(-\alpha)$ if $\alpha \leqq 0$, is a group of isometries. For any positive f in $\bar{F}$

$$\int f\, dP_\alpha = \int [T_\alpha(f^{1/p})]^p\, dP_0 = \int [V(\alpha)T_\alpha(f^{1/p})]^p\, dP_0 = \int (V(\alpha)1)^p f dP_0,$$

which shows that P_α is absolutely continuous with respect to P_0, $S_0 \subset S_\alpha$, and that $(V(\alpha)1)^p = dP_\alpha/dP_0$. Now suppose that f_n is a decreasing sequence of nonnegative functions from $\bar{F}$ which converges to 0 almost everywhere with respect to P_α. Then $T_\alpha f_n$ decreases to 0 almost everywhere with respect to P_0, and

$$\int f_n\, dP_0 = \int (f_n^{1/p})^p\, dP_0 = \int (V(-\alpha)(f_n^{1/p}))^p\, dP_0 = \int (V(-\alpha)1)^p T_\alpha f_n\, dP_0$$

converges to 0, completing the proof of (1). According to (1), we have $0 < V(-\alpha)1 < \infty$ almost everywhere P_0, so we can define

$$\bar{T}_\alpha f = V(-\alpha)f/V(-\alpha)1$$

for all f in $L_p(P_0)$. $\bar{T}_\alpha$ is clearly a linear positivity-preserving extension of T_α, and, since $\bar{T}_\alpha 1 = 1$, it also preserves bounds. If g is in $\bar{F}$, then $\bar{T}_\alpha(fg) = (V(-\alpha)f)T_\alpha g/V(-\alpha)1 = (\bar{T}_\alpha f)(\bar{T}_\alpha g)$, and letting g converge boundedly to an arbitrary bounded S_0-measurable function completes the proof of (2). We shall write T_α for $\bar{T}_\alpha$ from now on. (3) is clear from the definition of T_α. If f_n is a sequence from F converging to ϕ_N, with $f_{n+1} \leqq f_n + 1/n$ and $\sum \| f_{n+1} - f_n \| < \infty$, then $V(\beta)f_n$ converges almost everywhere to $V(\beta)\phi_N$ so that $T_{-\beta} f_n$ converges almost everywhere to $T_{-\beta}\phi_N$. Thus $T_{-\beta}\phi_N$ is $d\beta \times dP_0$-measurable, and for almost all x,

$$\int_0^\alpha T_{-\beta}\,\phi_N(x)\,d\beta = \lim \int_0^\alpha T_{-\beta} f_n(x)\,d\beta = p \log (V_N(\alpha)1)(x).$$

The proof follows, on letting N go to ∞, from the monotonicity of ϕ_N and $V_N(\alpha)1$.

From the above theorem

$$T_{-\alpha}\,\phi = \frac{\partial}{\partial\alpha} \log \frac{dP_\alpha}{dP_0},$$

so by the Cramer-Rao inequality [6, pp. 247–248], if ϕ is in $L_2(P_0)$ and $\sup_{a \leqq \alpha \leqq b} | T_{-\alpha}\,\phi\, dP_\alpha/dP_0 |$ is in $L_1(P_0)$, then for any estimate α^* of α with bias $b(\alpha) = \int \alpha^*(x)\,dP_\alpha - \alpha$ and any α in the interval $[a, b]$, we have

$$\int (\alpha^* - \alpha)^2\,dP_\alpha \geqq \left(1 + \frac{db}{d\alpha}\right)^2 \Big/ \int \phi^2\,dP_0 .$$

Before leaving this section we note that the constructions involved in the proof of Theorem 2.1 only made use of the T_α for $\alpha \leqq 0$, so that this theorem is applicable to the case where T_α is only a semigroup. This is stated formally in the next theorem.

THEOREM 2.4. *If* (1) *through* (3) *of Section* 1 *are satisfied except that* T_α *is defined only for* $\alpha \geqq 0$, *then there exists a strongly continuous semigroup* $V(\alpha)$ *satisfying*

(1) $\| V(\alpha) \| \leqq 1$.

(2) $V(\alpha)(fg) = (V(\alpha)f)T_\alpha g$ *for all* g *in* $\bar{F}$.

(3) $V(\alpha)$ *preserves positivity.*

(4) *The generator of* $V(\alpha)$ *contains the operator* A *defined on* F *by*

$$Af = -(1/p)\phi f + Df .$$

If $V(\alpha)$ *is an isometry, then* P_0 *is absolutely continuous with respect to* P_α.

Proof. All but the last statement follow from Theorem 2.1 with T_α, D, and ϕ replaced by $T_{-\alpha}$, $-D$, and $-\phi$ respectively. If $V(\alpha)$ is an isometry and f is a positive function in $\bar{F}$, then

$$\int f\,dP_0 = \int [V(\alpha)(f^{1/p})]^p\,dP_0 = \int (V(\alpha)1)^p T_\alpha f\,dP_0 .$$

Hence, if $T_\alpha f_n$ decreases to 0 almost everywhere, $\int f_n \, dP_0$ also goes to 0, which proves the last statement.

3. Conditions guaranteeing that $V_-(\alpha)V_+(\alpha) = V_+(\alpha)V_-(\alpha) = I$

In this section we shall derive various sets of conditions which are sufficient to insure that $V_-(\alpha)$ and $V_+(\alpha)$ are the two halves of a group of isometries. Relatively simple examples, one of which is given below, show that this is not always the case. When $p = 2$, the operator iA is symmetric, and, of course, if its defect indices are 0, the semigroups are the two halves of a group of unitary operators. In the examples given here, and in all other cases known to the author, the defect indices of iA are equal; but, as will be seen below, it is possible that none of the skew-adjoint extensions of A generates the desired group of unitaries, and in fact no such group need exist.

The following class of examples will illustrate the range of possibilities under the assumptions of Section 1. We take X to be the unit circle, S the Borel sets, P of the form $m(x)\,dx$, F the continuously differentiable functions, and T_α to be rotation, i.e.,

$$\begin{aligned} T_\alpha f(x) &= f(x - \alpha) && \text{if} \quad x - \alpha \geqq \pi, \\ &= f(2\pi + x - \alpha) && \text{if} \quad x - \alpha < \pi. \end{aligned}$$

If $m(x)$ is assumed to be continuously differentiable, then $Df = -f'$ and $\phi = m'/m$ satisfy (1) through (3) of Section 1 provided

$$\int_{-\pi}^{\pi} \left| \frac{m'}{m} \right|^p m \, dx < \infty.$$

In the simplest case, $m(x) = 1/2\pi$, $\phi = 0$, the closure of iA is self-adjoint if $p = 2$, and $V_+(\alpha) = T_\alpha$.

Next we take $m(x) = c \exp(-1/(\pi^2 - x^2))$. The map $f \to \sqrt{m}\, f$ carries $L_2(m(x)\,dx)$ isometrically onto $L_2(dx)$ and takes A into $-d/dt$. However, it carries F into (essentially) the set of continuously differentiable functions vanishing at π, so the defect indices of iA in this case are $(1, 1)$. Since A is not maximal, it is properly contained in the generators of $V_+(\alpha)$ and $V_-(\alpha)$. It is easily shown by calculation that

$$V_+(\alpha)f(x) = \left(\frac{m(x - \alpha)}{m(x)}\right)^{1/2} f(x - \alpha) \qquad \text{if} \quad x - \alpha \geqq -\pi$$

and is 0 otherwise;

$$V_-(\alpha)f(x) = \left(\frac{m(x + \alpha)}{m(x)}\right)^{1/2} f(x + \alpha) \qquad \text{if} \quad x + \alpha \leqq \pi$$

and is 0 otherwise; and $e_\alpha(x) = 1$ if $-\pi + \alpha \leqq x \leqq \pi - \alpha$ and is 0 otherwise. $\phi(x)$ is in $L_p(m(x)\,dx)$ for every p, $1 \leqq p < \infty$, and it is clear that e_α is the same no matter which p is chosen. It will be shown in the discussion of the next case that this result is also independent of the form of $m(x)$ beyond the fact that it has exactly one 0. Before going to that case, we note that an iA with defect indices (n, n) can be constructed in the same way by choosing an $m(x)$ with exactly n zeros.

In both of the above cases the P_α were mutually absolutely continuous; but for an m which is positive on $-\pi + a < x < \pi - a$ and vanishes elsewhere, this is not so. The map $f \to \sqrt{m}\, f$ now carries F into the set of continuously differentiable functions vanishing outside the interval from $-\pi + a$ to $\pi - a$, and iA again has defect indices $(1, 1)$. $V_+(\alpha)$ has the same form as above except that T_α is replaced by T'_α, rotation through the circle short-circuited by identifying $-\pi + a$ and $\pi - a$. T'_α is also the group generated by the unique positivity preserving skew-adjoint extension of A. If $m'(x) = m\left(\dfrac{\pi - a}{\pi} x\right)$ gives rise to a group of isometries, then so must $m(x)$, but this is impossible by Theorem 2.3. This justifies the statement made above about the nondependence of the second case on the form of $m(x)$.

The next theorem shows, as might be expected, that there is little to be gained by considering cases other than $p = 1$.

THEOREM 3.1. *If $V_+(\alpha)$ and $V_-(\alpha)$ are the two halves of a group of isometries for some $p > 1$, then the same is true for every q, $p \geqq q \geqq 1$. If they are a group of isometries for $p = 1$ and ϕ is in $L_p(P_0)$, then they are a group of isometries for every q, $1 \leqq q \leqq p$.*

Proof. We shall write ${}_pV_+(\alpha)$ and ${}_pV_-(\alpha)$ for the semigroups constructed in $L_p(P_0)$. It is clear from the construction of ${}_qV_+(\alpha)$ that ${}_qV_+(\alpha)1 = ({}_pV_+(\alpha)1)^{p/q}$, so if ${}_pV_+(\alpha)1 = (dP_\alpha/dP_0)^{1/p}$, ${}_qV_+(\alpha)$ is an isometry. Similarly, ${}_qV_-(\alpha)$ is an isometry so $\| e_\alpha \| = \| {}_qV_-(\alpha)\, {}_qV_+(\alpha)1 \| = 1$ and $e_\alpha = 1$. Conversely, if ${}_1V_+(\alpha)$ and ${}_1V_-(\alpha)$ are the two halves of a group of isometries in $L_1(P_0)$, then

$$ {}_pV_+(\alpha)f = (dP_\alpha/dP_0)^{1/p}T_{-\alpha} f \quad \text{and} \quad {}_pV_-(\alpha)f = (dP_{-\alpha}/dP_0)^{1/p}T_\alpha f $$

are isometries, and hence, ${}_pV_-(\alpha) = ({}_pV_+(\alpha))^{-1}$.

THEOREM 3.2. *If $\bar{A}$, the closure of A, is the generator of $V_+(\alpha)$, or if $-\bar{A}$ is the generator of $V_-(\alpha)$,* then $V_-(\alpha) = V_+(\alpha)^{-1}$. *In particular, if $(\lambda - \bar{A})F$ is dense in $L_p(P_0)$ for some $\lambda \neq 0$, then $\bar{A}$ generates $V_+(\alpha)$, and $V_-(\alpha) = V_+(\alpha)^{-1}$. Conversely, if $V_-(\alpha) = V_+(\alpha)^{-1}$, then the generator of $V(\alpha)$ $[V(\alpha) = V_+(\alpha)$ if $\alpha \geqq 0$, $V(\alpha) = V_-(\alpha)$ if $\alpha \leqq 0]$ is the operator A_0 with domain $\bigcup_{-\infty<\alpha<\infty} V(\alpha)F$ defined by $A_0\, V(\alpha)f = V(\alpha)Af$.*

Proof. If $\bar{A}$ generates $V_+(\alpha)$, then $V_+(\alpha)f$ is in the domain of $\bar{A}$, so

$$ \frac{\partial}{\partial \alpha} V_-(\alpha)V_+(\alpha)f = V_-(\alpha)[-\bar{A} + \bar{A}]V_+(\alpha)f = 0. $$

If $(\lambda - A)F$ is dense and G is the generator of $V_+(\alpha)$, then for g in the dense set $(\lambda - G)F \supset (\lambda - A)F$, $(\lambda - G)^{-1}g$ is in the domain of $\lambda - A$, and

$$(\lambda - A)\int_0^\infty e^{-\alpha\lambda}\, V_+(\alpha)\, g\, d\alpha$$

$$= (\lambda - G)\,[(\lambda - G)^{-1}(\lambda - A)]\int_0^\infty e^{-\alpha\lambda}\, V_+(\alpha)\, g\, d\alpha$$

$$= (\lambda - G)\int_0^\infty e^{-\alpha\lambda}\, V_+(\alpha)\, g\, d\alpha = g,$$

which proves the second assertion. Suppose now that $V_-(\alpha) = V_+(\alpha)^{-1}$. We first show that A_0 is well defined. If $V(\alpha)f = V(\beta)g$, then $A_0\, V(\alpha)f = V(\alpha)Af = V(\alpha)AV(\beta - \alpha)g = V(\alpha)V(\beta - \alpha)Ag = A_0\, V(\beta)g$. For f in F, $(\lambda - A_0)\sum_{k=0}^{n^2}(1/n)e^{-\lambda k/n}V_+(k/n)f = \sum_{k=0}^{n^2}(1/n)e^{-\lambda k/n}V_+(k/n)(\lambda - A)f$ so $\int_0^\infty e^{-\alpha\lambda}V_+(\alpha)f\,d\alpha$ is in the domain of $\bar{A}_0$, and

$$(\lambda - \bar{A}_0)\int_0^\infty e^{-\alpha\lambda}\, V_+(\alpha)\, f\, d\alpha = \int_0^\infty e^{-\alpha\lambda}\, V_+(\alpha)\,(\lambda - A)\, f\, d\alpha = f,$$

which proves that $\bar{A}_0$ is the generator of $V_+(\alpha)$, and hence of $V(\alpha)$.

The above theorem can be improved if $p = 1$.

THEOREM 3.3. *In case $p = 1$, the following conditions are equivalent:*
(1) *$(\lambda - A)F$ is dense for some $\lambda \neq 0$.*
(2) *$\bar{A}$ is the generator of $V_+(\alpha)$ or $-\bar{A}$ is the generator of $V_-(\alpha)$.*
(3) *$V_-(\alpha) = (V_+(\alpha))^{-1}$.*

Proof. From the previous theorem, (1) and (2) are equivalent, and they imply (3). (3) implies by Theorem 2.3 that

$$\lim\left\|\int_0^K V_-(\alpha)[(\phi - \phi_n)V_n(\alpha)1]\,d\alpha\right\| = \lim\int_0^K \|(\phi - \phi_n)V_n(\alpha)1\|\,d\alpha = 0.$$

Hence if f is in F,

$$\lim_n\,(\lambda - A)\int_0^K e^{-\alpha\lambda}\, V_n(\alpha)\, f\, d\alpha$$

$$= \lim_n\left[(\phi_n - \phi)\int_0^K e^{-\alpha\lambda}\, V_n(\alpha)f\,d\alpha + f - e^{-K\lambda}V_n(K)f\right] = f - e^{-K\lambda}V(K)f,$$

so $\int_0^K e^{-\alpha\lambda}V(\alpha)f\,d\alpha$ is in the domain of $\bar{A}$, and

$$\lim_{K\to\infty}\,(\lambda - \bar{A})\int_0^K e^{-\alpha\lambda}\, V(\alpha)\, f\, d\alpha = (\lambda - \bar{A})\int_0^\infty e^{-\alpha\lambda}\, V(\alpha)\, f\, d\alpha = f.$$

THEOREM 3.4. *If, for some $\varepsilon > 0$ either*

$$\liminf\int_{[x\mid\phi(x)>n]}\phi^p\,dP \quad \textit{or} \quad \liminf\int_{[x\mid\phi(x)<-n]}(-\phi)^p\,dP$$

is $O(e^{-\varepsilon n})$, then $V_-(\alpha) = (V_+(\alpha))^{-1}$.

Proof. We prove the theorem under the first hypothesis. From Theorem 2.3,

$$\| 1 - e_\varepsilon \| \leqq \liminf \int_0^\varepsilon \| (\phi - \phi_n) V_n(\gamma) 1 \| \, d\gamma$$

$$\leqq \liminf \| \phi - \phi_n \| \int_0^\varepsilon e^{n\gamma} \, d\gamma = 0.$$

THEOREM 3.5. *If there are a sequence* (f_n) *from* F *converging to* ϕ *in* $L_p(P_0)$ *and an* $a > 0$ *such that*

$$\liminf_{N\to\infty} \liminf_{n\to\infty} \int_0^a \int_{[T_{-\alpha} f_n > N]} T_{-\alpha} f_n \, dP \, d\alpha = 0,$$

then $V_-(\alpha) = (V_+(\alpha))^{-1}$.

Proof.

$$e_a - 1 = \lim_N \int_0^a V_-(\alpha)[(\phi - \phi_N) V_N(\alpha) 1] \, d\alpha$$

$$= \lim_N \lim_n \int_0^a V_-(\alpha)[f_n - (f_n)_N) V_N(\alpha) 1] \, d\alpha$$

$$= \lim_N \lim_n \int_0^a (V_-(\alpha) V_N(\alpha) 1)[T_{-\alpha} f_n - (T_{-\alpha} f_n)_N] \, d\alpha$$

$$\leqq \lim_N \lim_n \int_0^a (T_{-\alpha} f_n - (T_{-\alpha} f_n)_N) \, d\alpha,$$

where we have written $(f_n)_N$ and $(T_{-\alpha} f_n)_N$ for $\min (f_n , N)$ and $(T_{-\alpha} f_n , N)$ respectively. Hence by Fatou's lemma

$$\int (e_a - 1) \, dP \leqq \liminf_N \liminf_n \int_{[T_{-\alpha} f_n > N]} \int_0^a T_{-\alpha} f_n \, d\alpha \, dP,$$

from which the theorem follows.

4. Applications

In this section we discuss applications of the theory developed in Sections 2 and 3.

A. *Translation of a random analytic function*

P is to be the probability measure associated with the stochastic process $x(t)$, $-\infty < t < \infty$, given by $x(t) = \sum_{n=0}^{\infty} \zeta_n a_n t^n/n!$ where ζ_n are positive real numbers satisfying $\sum_{n=0}^{\infty} (\zeta_{n+1}/\zeta_n)^2 < \infty$, and the a_n are independent, identically distributed, random variables with density $g(a) \, da$ satisfying $\int_{-\infty}^{\infty} a^2 g(a) \, da < \infty$. Note that the random variables $y_n = \zeta_n a_n t^n/n!$ are independent and $\sum \int y_n^2 \, dP \log^2 n < \infty$, so [3, Theorem 4.2, p. 157] the series for $x(t)$ converges with probability 1, and applying this argument to a sequence $t_n \to \infty$, that $x(t)$ has an infinite radius of convergence with probability 1. We further assume that g is continuously differentiable and, setting $\xi = g'/g$, that $\int_{-\infty}^{\infty} \xi^2(a)g(a)\, da < \infty$.

F is the set of polynomials in functions of the form

$$h(x) = \int \exp\left(i \sum_{j=0}^{n} \lambda_j\, x(t_j)\right) H(d\lambda_1, \cdots, d\lambda_n)$$

with

$$\int \left(1 + \sum_j \lambda_j^2\right) \mid H(d\lambda_1, \cdots, d\lambda_n) \mid < \infty.$$

Conditions (1) and (2) of Section 1 are easily verified for T_α given by

$$T_\alpha\, h(x) = \int \exp\left(i \sum_{j=0}^{n} \lambda_j\, x(t_j + \alpha)\right) H(d\lambda_1, \cdots, d\lambda_n)$$

and

$$T_\alpha\, Q(h_1, \cdots, h_k) = Q(T_\alpha\, h_1, \cdots, T_\alpha\, h_k).$$

The random variables $b_n = \xi(a_n) a_{n+1}$ form an orthogonal sequence in $L_2(P)$, so that

$$\phi(x) = -\sum_{n=0}^{\infty} (\zeta_{n+1}/\zeta_n)\, \xi(a_n)\, a_{n+1}$$

is in $L_2(P)$ because of the assumption made on the ζ_n's. The following lemma shows that condition (3) is satisfied for this ϕ.

LEMMA 4.1. *For f in F, $\int \phi f\, dP = \int Df\, dP$.*

Proof. Since $\sum_{n=1}^{\infty} (\zeta_n/(n-1)!) a_n\, t_j^{n-1}$ converges in $L_2(P)$ to $x'(t_j)$,

$$\begin{aligned}\int D \exp\left(i \sum_j \lambda_j\, x(t_j)\right) dP &= i \int \left(\sum_j \lambda_j\, x'(t_j)\right) \exp i \left(\sum_j \lambda_j\, x(t_j)\right) dP \\ &= i \sum_{n=0}^{\infty} \frac{\zeta_{n+1}}{\zeta_n}\, w_n \int a_{n+1} \exp\left(i \sum_j \lambda_j\, x(t_j)\right) dP\end{aligned}$$

where $w_n = (\zeta_n/n!) \sum_j \lambda_j\, t_j^n$. Using the independence of the a_n's,

$$\begin{aligned}&\int D \exp\left(i \sum_j \lambda_j\; x(t_j)\right) dP \\ &\quad = i\left[\sum_{n=0}^{\infty} \frac{\zeta_{n+1}}{\zeta_n}\, w_n \frac{\int a_{n+1} \exp\,(iw_{n+1}\, a_{n+1})\, dP}{\int \exp\,(iw_{n+1}\, a_{n+1})\, dP}\right] \int \exp\left(i \sum_j \lambda_j\; x(t_j)\right) dP.\end{aligned}$$

Again, since the defining sequence for ϕ is L_2-convergent,

$$\begin{aligned}\int \phi(x) \exp\left(i \sum_j \lambda_j\, x(t_j)\right) dP = &- \left[\sum_{n=0}^{\infty} \frac{\zeta_{n+1}}{\zeta_n} \frac{\int \xi(a_n) \exp\,(iw_n\, a_n)\, dP}{\int \exp\,(iw_n\, a_n)\, dP}\right. \\ &\left.\cdot \frac{\int a_{n+1} \exp\,(iw_{n+1}\, a_{n+1}) dP}{\int \exp\,(iw_{n+1}\, a_{n+1})\, dP}\right] \int \exp\left(i \sum_j \lambda_j\, x(t_j)\right) dP,\end{aligned}$$

and integrating

$$\int \xi(a_n) \exp(iw_n a_n)\, dP = \int_{-\infty}^{\infty} g'(a) \exp(iw_n a)\, da$$

by parts completes the proof of the lemma for $f(x) = \exp(i\sum_j \lambda_j x(t_j))$. The extension to general f is straightforward.

Before going on to a discussion of the Gaussian case, we note that the above analysis can be carried through with only minor changes without the assumption that the a_n's are identically distributed. In the Gaussian case $g(a) = (1/\sqrt{2}\,\pi\sigma) \exp(-a^2/2\sigma)$, $\xi(a) = -a/\sigma$, so

$$\phi(x) = (1/\sigma) \textstyle\sum_{n=0}^{\infty} (\zeta_{n+1}/\zeta_n) a_n\, a_{n+1}\,.$$

We wish to find a sequence f_k satisfying the conditions of Theorem 3.5 for this case. In the next two lemmas and the theorem that concludes this part of Section 4, we shall use the following notation: $x^{(k)}$ is the k^{th} derivative of the function x; for any function G, G_n is the function whose value is n, $G(x)$, or $-n$ if $G(x)$ is greater than n, between $-n$ and n, or less than $-n$, respectively; and $\Delta(\delta)$ is the operator $\Delta(\delta)f(a) = \delta^{-1}(f(a + \delta) - f(a))$.

LEMMA 4.2. *If the a_n's are Gaussian,*

$$\int_0^a \int \left| \sum_{k=0}^{\infty} \frac{1}{\zeta_k^2} x^{(k)}(-\alpha) x^{(k+1)}(-\alpha) \right| dP\, d\alpha < \infty.$$

Proof. Leaving out the first term, which is obviously integrable, and applying Schwartz's inequality to the rest, we obtain

$$\begin{aligned}\int_0^a \int & \left| \sum_{k=1}^{\infty} \frac{1}{\zeta_k^2} x^{(k)}(-\alpha) x^{(k+1)}(-\alpha) \right| dP\, d\alpha \\ & \leqq \int_0^a \int \left(\sum_{k=1}^{\infty} \frac{1}{\zeta_{k-1}^2} (x^{(k)}(-\alpha)^2) \right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{\zeta_{k-1}^2}{\zeta_k^4} (x^{(k+1)}(-\alpha))^2 \right)^{1/2} dP\, d\alpha \\ & \leqq C \int_0^a \int \left(\sum_{k=1}^{\infty} \frac{1}{\zeta_{k-1}^2} (x^{(k)}(-\alpha))^2 \right) dP\, d\alpha,\end{aligned}$$

where we have used the fact that ζ_{k-1}/ζ_k is bounded. Using the power series expansions for $x^{(k)}$ and evaluating, shows that this last integral is dominated by

$$a\sigma \sum_{k=1}^{\infty} \frac{1}{\zeta_{k-1}^2} \sum_{n=0}^{\infty} \zeta_{n+k}^2 \frac{a^{2n}}{(n!)^2} \leqq a\sigma \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \left(\frac{\zeta_{n+k}}{\zeta_{k-1}} \right)^2 \right) \frac{a^{2n}}{(n!)^2}\,.$$

Since ξ_{n+k}/ζ_k is always less than 1 for k beyond some k_0, the k summations are bounded, and the proof is complete.

LEMMA 4.3. *There is a sequence f_k from F satisfying the requirements of Theorem* 3.5.

Proof. We choose $n(k)$, $m(k)$, and $\Delta_k = \Delta(\delta(k))$ to satisfy

$$\int \left| \sum_{q=m(k)+1}^{\infty} \frac{\zeta_{q+1}}{\zeta_q} a_q\, a_{q+1} \right| dP < \frac{1}{k}, \tag{1}$$

$$(2) \qquad \int \left| \sum_{q=0}^{m(k)} \frac{\zeta_{q+1}}{\zeta_q} a_q a_{q+1} \right|_{n(k)} dP < \frac{1}{k},$$

$$(3) \qquad \int_0^a \int \left| \left(\sum_{q=0}^{m(k)} \frac{1}{\zeta_q^2} (\Delta_k^q x(-\alpha)) (\Delta_k^{q+1} x(-\alpha)) \right)_{n(k)} - \left(\sum_{q=0}^{m(k)} \frac{1}{\zeta_q^2} x^{(q)}(-\alpha) x^{(q+1)}(-\alpha) \right)_{n(k)} \right| dP \, d\alpha < \frac{1}{k},$$

$$(4) \quad \int \left| \left(\sum_{q=0}^{m(k)} \frac{1}{\zeta_q^2} (\Delta_k^q x(0))(\Delta_k^{q+1} x(0)) \right)_{(nk)} - \left(\sum_{q=0}^{m(k)} \frac{\zeta_{q+1}}{\zeta_q} a_q a_{q+1} \right)_{n(k)} \right| dP < \frac{1}{k},$$

and f_k to be within $1/k$ of

$$\left(\sum_{q=0}^{m(k)} \frac{1}{\zeta_q^2} (\Delta_k^q x(0))(\Delta_k^{q+1} x(0)) \right)_{n(k)}.$$

f_k converges to ϕ by (1), (2), and (4). Finally, writing $\int^N$ for the integral taken over the set where the integrand is greater than N,

$$\begin{aligned} &\liminf_{N\to\infty} \liminf_{k\to\infty} \int_0^a \int^N T_{-\alpha} f_k \, dP \, d\alpha \\ &\qquad \leqq \liminf_{N\to\infty} \liminf_{k\to\infty} \int_0^a \int^N \sum_{q=0}^{m(k)} \frac{1}{\zeta_q^2} (\Delta_k^q x(-\alpha)(\Delta_k^{q+1} x(-\alpha) \, dP \\ &\qquad \leqq \liminf_{N\to\infty} \liminf_{k\to\infty} \int_0^a \int^N \sum_{q=0}^{m(k)} \frac{1}{\zeta_q^2} x^{(q)}(-\alpha) x^{(q+1)}(-\alpha) \, dP \, d\alpha, \end{aligned}$$

and this equals 0 by Lemma 4.2.

THEOREM 4.1. *If the a_n are Gaussian,*

$$\log \frac{dP_\alpha}{dP_0}(x) = \frac{1}{2\sigma} \sum_{n=0}^{\infty} \frac{1}{\zeta_n^2} [(x^{(n)}(0))^2 - (x^{(n)}(-\alpha))^2].$$

Proof. By Theorem 2.3, T_α can now be extended to $L_2(P_0)$. We have $T_{-\alpha} a_n = (1/\zeta_n) x^{(n)}(-\alpha)$, so

$$T_{-\alpha} \phi(x) = \frac{1}{\sigma} \sum_{n=0}^{\infty} \frac{1}{\zeta_n^2} x^{(n)}(-\alpha) x^{(n+1)}(-\alpha)$$

giving

$$\begin{aligned} \log \frac{dP_\alpha}{dP_0}(x) &= \frac{1}{\sigma} \sum_{n=0}^{\infty} \frac{1}{\zeta_n^2} \int_0^\alpha x^{(n)}(-\beta) x^{(n+1)}(-\beta) \, d\beta \\ &= \frac{1}{2\sigma} \sum_{n=0}^{\infty} \frac{1}{\zeta_n^2} [(x^{(n)}(0))^2 - (x^{(n)}(-\alpha))^2]. \end{aligned}$$

B. *Approximating ϕ by a martingale*

It sometimes happens that there are subfields S_n of S invariant under the T_α on which the conditional expectations of ϕ can be calculated. Suppose, for example, that X is a real Hilbert space, and that for some sequence (x_i) from

X all the functions $l_i : l_i(x) = (x, x_i)$ are in $L_2(P)$. F is the set of all functions of the form $f(x) = \hat{f}(l_1(x), \cdots, l_n(x))$ for a bounded $\hat{f}$ with continuous bounded first derivatives. T_α is defined on F by

$$T_\alpha f(x) = \hat{f}(\lambda_1^\alpha l_1(x), \cdots, \lambda_n^\alpha l_n(x))$$

for some bounded sequence λ_i of positive numbers. In particular, if the x_i are orthonormal, then $T_\alpha f(x) = f(\tau^\alpha x)$ where τ is the transformation with eigenvalues (λ_i) and eigenvectors (x_i). Let the joint distribution of $l_1, \cdots, l_n$ be given by a density function $p_n(a_1, \cdots, a_n)$. We assume

(a1) The functions q_{nj} :

$$q_{nj}(a_1, \cdots, a_n) = \frac{\partial}{\partial a_j} (a_j p_n(a_1, \cdots, a_n)) \Big/ p_n(a_1, \cdots, a_n)$$

exist and are in $L_2(p_n(a_1, \cdots, a_n)\, da_1 \cdots da_n)$, and, for every

$a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n,$

$$\lim_{a_j \to \pm\infty} a_j p_n(a_1, \cdots, a_n) = 0.$$

$\phi_n(x) = -\sum_{j=1}^n \log \lambda_j q_{nj}(l_1(x), \cdots, l_n(x))$ is in $L_2(P)$, and $\int \phi_n f\, dP = \int Df\, dP$ for every f in F which depends only on $l_1, \cdots, l_n$. Since this inner product relation defines ϕ_n uniquely, the conditional expectation of ϕ_{n+1} on the field generated by $l_1, \cdots, l_n$ equals ϕ_n. This implies that $\int \phi_n^2\, dP$ is nondecreasing and we assume

(a2) $$\lim \int \phi_n^2\, dP < \infty.$$

The sequence (ϕ_n) is a martingale, and [3, Theorem 4.1, p. 319] there is a function ϕ in $L_2(P)$ which is the limit almost everywhere of the ϕ_n and satisfies $\int \phi f\, dP = \int Df\, dP$ for all f in F. Hence the conditions of Section 1 are satisfied if (a1) and (a2) hold.

The following is an example of this type. (X, S, P) is the measure space associated with a stochastic process $[x(t); t \in I]$. Let (t_i) be a sequence of points such that $x(t_i)$ is dense in $L_2(P)$, and suppose that the joint distribution of $x(t_1), \cdots, x(t_n)$ is given by a density $p_n(a_1, \cdots, a_n)$. F is the set of functions of the form $f(x) = \hat{f}(x(t_1), \cdots, x(t_n))$ for bounded $\hat{f}$ with continuous bounded derivatives, and for some fixed function $m(t)$, we define T_α by $T_\alpha f(x) = \hat{f}(x(t_1) + \alpha m(t_1), \cdots, x(t_n) + \alpha m(t_n))$. Hence, if we define τ_α by $(\tau_\alpha x)(t) = x(t) + \alpha m(t)$, then $T_\alpha f(x) = f(\tau_\alpha x)$. We assume

(b1) The functions q_{nj} :

$$q_{nj}(a_1, \cdots, a_n) = \frac{\partial}{\partial a_j} (p_n(a_1, \cdots, a_n)) \Big/ p_n(a_1, \cdots, a_n)$$

exist and are in $L_2(p_n(a_1, \cdots, a_n)\, da_1, \cdots, da_n)$.

As before, the function $\phi_n(x) = -\sum_{j=1}^{n} m(t_j) q_{nj}(x(t_1), \cdots, x(t_n))$ satisfies $\int \phi_n f\, dP = \int Df\, dP$ for every f in F, measurable on the field generated by $x(t_1), \cdots, x(t_n)$; and the conditional expectation of ϕ_{n+1} on this field is ϕ_n. We also assume

$$\text{(b2)} \qquad \lim \int \phi_n^2\, dP < \infty.$$

As before, (b1) and (b2) imply the conditions of Section 1 with $\phi = \lim \phi_n$.

In cases of this type the likelihood ratios are usually also known on the subfields. They, of course, also form a martingale, and it may well be easier to work with them directly than to attempt to calculate ϕ.

We have chosen to work in L_2, but the entire theory would clearly work just as well in L_p for $p > 1$. In L_1 however, some additional condition would be required to insure $\int (\lim \phi_n) f\, dP = \lim \int \phi_n f\, dP$.

C. *The addition of a Gaussian indeterminacy in* α

We suppose given (X, S, P), F, and T_α satisfying (1) and (2) of Section 1. For $\sigma > 0$, let $K_\sigma(\alpha) = (2\pi\sigma)^{-1/2} \exp(-\alpha^2/2\sigma)$, and let P^σ be the measure gotten by completing the functionals

$$(*) \qquad \int f\, dP^\sigma = \int_{-\infty}^{\infty} K_\sigma(\alpha) \left(\int T_\alpha f\, dP \right) d\alpha$$

defined for f in $\bar{F}$. We shall show that replacing P by P^σ always leads to the situation described in Theorem 2.3.

LEMMA 4.4. *There is a* ϕ^σ *in* $L_1(P^\sigma)$ *satisfying*

$$\int \phi^\sigma f\, dP^\sigma = \int Df\, dP^\sigma \qquad \textit{for all } f \textit{ in } F.$$

Proof. For f in F,

$$\int Df\, dP^\sigma = \lim_{\varepsilon \to 0} \int \frac{T_\varepsilon f - f}{\varepsilon}\, dP^\sigma$$

$$= \int_{-\infty}^{\infty} K_\sigma(\alpha) \left(\frac{\partial}{\partial \alpha} \int T_\alpha f\, dP \right) d\alpha = \frac{1}{\sigma} \int_{-\infty}^{\infty} \alpha K_\sigma(\alpha) \left(\int T_\alpha f\, dP \right) d\alpha,$$

so for any $B > 0$ and f in F,

$$\left| \int Df\, dP^\sigma \right| \leqq \frac{B}{\sigma} \int_{-B}^{B} K_\sigma(\alpha) \left| \int T_\alpha f\, dP \right| d\alpha + 2K_\sigma(B)$$

$$\leqq \frac{B}{\sigma} \int |f|\, dP^\sigma + 2K_\sigma(B).$$

The linear functional $l(f) = \int Df\, dP^\sigma$ can now be extended to $\bar{F}$, and the inequality still holds. For positive f in $\bar{F}$, define

$$l^+(f) = \sup\nolimits_{0 \leqq g \leqq f} l(g) \quad \text{and} \quad l^-(f) = \sup\nolimits_{0 \leqq g \leqq f} -l(g).$$

It is clear that $l^+(f_1 + f_2) \geqq l^+(f_1) + l^+(f_2)$, and since for any g satisfying $0 \leqq g \leqq f_1 + f_2$, the functions $g_i = gf_i/(f_1 + f_2)$ are in $\bar{F}$ and satisfy $0 \leqq g_i \leqq f_i$, the opposite inequality also obtains. Now if f_n is any decreasing sequence of nonnegative functions from $\bar{F}$ and $\lim \int f_n \, dP^\sigma = 0$, then

$$\lim l^+(f_n) \leqq \lim \left(\frac{B_n}{\sigma} \int f_n \, dP^\sigma + 2K_\sigma(B_n)\right) = 0$$

if we choose $B_n = \left(\int f_n \, dP^\sigma\right)^{-1/2}$. This plus $l^+(1) < \infty$ proves that there is a function ϕ_+^σ in $L_1(P^\sigma)$ satisfying $\int \phi_+^\sigma f \, dP^\sigma = l^+(f)$. Similarly we can show the existence of a ϕ_-^σ in $L_1(P^\sigma)$ satisfying $\int \phi_-^\sigma f \, dP^\sigma = l^-(f)$. Whenever $0 \leqq g \leqq f$, we also have $0 \leqq f - g \leqq f$ so $l^+(f) \geqq l(f) - l(g)$, and hence $l^+(f) - l^-(f) \geqq l(f)$. The opposite inequality can be proved in the same way showing that $\int (\phi_+^\sigma - \phi_-^\sigma) f \, dP^\sigma = \int Df \, dP^\sigma$.

LEMMA 4.5. $$\int (\phi - \phi_n) \, dP^\sigma = O(e^{-n}).$$

Proof. For any f in $\bar{F}$ of absolute bound 1 and any $B > 0$

$$\left| \int \phi_\sigma f \, dP \right| \leqq \frac{B}{\sigma} \int |f| \, dP^\sigma + 2K_\sigma(B),$$

and by a continuity argument, this holds for all S_0-measurable f of absolute bound 1. Putting f equal to the characteristic function of the set A_n where $\phi_\sigma(x) > n$ and setting $B = \sigma(n - 1)$ gives

$$P^\sigma(A_n) \leqq 2(2\pi\sigma)^{-1/2} \exp\left(-\sigma(n - 1)^2/2\right),$$

and hence

$$\int (\phi - \phi_n) \, dP^\sigma \leqq C \sum_{k=n}^{\infty} (k + 1) \exp\left(\frac{-\sigma(k - 1)^2}{2}\right)$$

which is obviously $O(e^{-n})$.

THEOREM 4.2. *If (X, S, P), F, and T satisfy conditions* (1) *and* (2) *of Section* 1 *and P^σ is defined for any $\sigma > 0$ by* (*), *then the conclusions of Theorem* 2.3 *hold for P_α^σ, $-\infty < \alpha < \infty$. All the measures P_α^σ, $0 < \sigma < \infty$; $-\infty < \alpha < \infty$ are mutually absolutely continuous. If in addition there is a ϕ in $L_1(P)$ satisfying $\int \phi f \, dP = \int Df \, dP$ for all f in F, then each P_α is absolutely continuous with respect to each P_β^σ, and we have*

$$\int \left| \frac{dP_\alpha}{dP_\alpha^\sigma} - 1 \right| dP_\alpha^\sigma \leqq \left(\frac{2\sigma}{\pi}\right)^{1/2} \| \phi \|.$$

Proof. The conclusions of Theorem 2.3 hold by virtue of the preceding two lemmas. If (f_n) is a decreasing sequence of nonnegative functions from $\bar{F}$, then $\lim \int f_n \, dP_\alpha^\sigma = 0$ if and only if $\lim \int T_\alpha f_n \, dP = 0$ for almost all α which proves the mutual absolute continuity of the P_α^σ. If a ϕ exists satisfying $\int \phi f \, dP = \int Df \, dP$, then $\int T_\alpha f \, dP$ is differentiable everywhere, and its derivative is bounded by $\| \phi \| \sup_x | f(x) |$, so in order for $\lim \int f_n \, dP_\alpha^\sigma$

to be 0, $\int T_\alpha f_n \, dP$ would have to go to 0 everywhere which shows the absolute continuity of the P_β with respect to the P_α^σ. In this case, for f in F of absolute bound 1

$$\left| \int T_\alpha f \, dP^\sigma - \int T_\alpha f \, dP \right| = \left| \int_{-\infty}^{\infty} K_\sigma(\gamma) \int_0^\gamma \left(\int \phi T_{\beta+\alpha} f \, dP \right) d\beta \, d\gamma \right|$$
$$\leqq \| \phi \| \int_{-\infty}^{\infty} | \gamma | K_\sigma(\gamma) \, d\gamma = \left(\frac{2\sigma}{\pi} \right)^{1/2} \| \phi \|.$$

Hence, for any measurable f of absolute bound 1

$$\left| \int f \left(1 - \frac{dP_\alpha}{dP_\alpha^\sigma} \right) dP_\alpha^\sigma \right| \leqq \left(\frac{2\sigma}{\pi} \right)^{1/2} \| \phi \|,$$

which implies the inequality of the theorem.

Bibliography

1. R. H. Cameron and R. E. Fagan, *Nonlinear transformations of Volterra type in Wiener space,* Trans. Amer. Math. Soc., vol. 75 (1953), pp. 552–575.
2. R. H. Cameron and W. T. Martin, *Transformation of Wiener integrals under a general class of linear transformations,* Trans. Amer. Math. Soc., vol. 58 (1945), pp. 184–219.
3. J. L. Doob, *Stochastic processes,* New York, Wiley, 1953.
4. N. Dunford and J. T. Schwartz, *Linear operators,* New York, Interscience Publishers, 1958.
5. J. Feldman, *Equivalence and perpendicularity of Gaussian processes,* Pacific J. Math., vol. 8 (1958), pp. 699–708.
6. U. Grenander, *Stochastic processes and statistical inference,* Ark. Mat., vol. 1 (1950), pp. 195–277.
7. L. Gross, *Integration and nonlinear transformations in Hilbert space,* Trans. Amer. Math. Soc., vol. 94 (1960), pp. 404–440.
8. T. S. Pitcher, *Likelihood ratios for diffusion processes with shifted mean values,* Trans. Amer. Math. Soc., vol. 101 (1961), pp. 168–176.
9. Yu. V. Prokhorov, *Convergence of random processes and limit theorems in probability theory,* Theory of Probability and its Applications, vol. I (1956), pp. 157–214.
10. T. I. Seidman, *Linear transformations of a functional integral. I,* Comm. Pure Appl. Math., vol. 12 (1959), pp. 611–621.
11. I. E. Segal, *Distributions in Hilbert space and canonical systems of operators,* Trans. Amer. Math. Soc., vol. 88 (1958), pp. 12–41.
12. A. V. Skorokhod, *On the differentiability of measures which correspond to stochastic processes, I: Processes with independent increments,* Theory of Probability and its Applications, vol. II (1957), pp. 407–432.
13. ———, *On the differentiability of measures which correspond to Markov processes,* Teor. Veroyatnost. i Primenen., vol. 5 (1960), pp. 45–53 (in Russian with English summary).
14. D. Slepian, *Some comments on the detection of Gaussian signals in Gaussian noise,* IRE Transactions on Information Theory, vol. IT-4 (1958), pp. 65–68.
15. D. A. Woodward, *A general class of linear transformations of Wiener integrals,* Trans. Amer. Math. Soc., vol. 100 (1961), pp. 459–480.

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