GENERALIZATIONS OF A THEOREM OF N. BLACKBURN ON p-GROUPS

BY
CHARLES R. HOBBY

Let G be a finite p-group, and let $G = G_1 \supset G_2 \supset \cdots$ be the descending central series of G. N. Blackburn [1] has shown that if G_2 can be generated by two elements, then G_2 is a metacyclic group of class at most two. We generalize this theorem in two ways. We first show (Theorem 1) that Blackburn's theorem holds if "G_2" is replaced by $\Psi(G)$ where $\Psi(G)$ is any one of a large class of characteristic subgroups of G. Secondly, we show (Theorem 2) that if G_n can be generated by n elements then the Frattini subgroup of G_n coincides with the subgroup generated by the p^{th} powers of elements of G_n. If p is odd, this result for $n = 2$ is equivalent to Blackburn's theorem. An application of Theorem 2 to the problem of bounding the length of the derived series of a p-group is given in Remark 2.

All groups considered are finite p-groups. We use the following notation: $P(G)$ is the subgroup generated by the p^{th} powers of elements of G; $\Phi(G)$ is the Frattini subgroup of G; $G^{(k)}$ is the k^{th} derived group of G; $G = G_1 \supseteq G_2 \supseteq \cdots$ is the descending central series of G; $1 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots$ is the ascending central series of G; $|G|$ is the order of G; $(h, k) = h^{-1}k^{-1}hk$; (H, K) is the subgroup generated by the set of all (h, k) for $h \in H$ and $k \in K$. The group G is said to be metacyclic if G contains a cyclic normal subgroup N such that G/N is cyclic.

We denote by Ψ a rule which assigns a unique subgroup $\Psi(G)$ to every p-group G. We consider only those rules for which

(1) $\Psi(G)$ is a characteristic subgroup of G,

(2) $\Psi(G) \subseteq \Phi(G)$, and

(3) $\Psi(G/N) = \Psi(G)/N$ whenever $N \subseteq \Psi(G)$ and N is normal in G.

For example, one could let $\Psi(G) = G_n$ for any $n \geqq 2$.

We shall need two lemmas.

LEMMA 1. *If $|N| \leqq p^n$ and N is normal in G, then $N \subseteq Z_n$.*

Proof. This result is well known for $n = 1$, and the general case follows by an easy induction.

The next lemma follows from the fact that the automorphism group of a cyclic group is abelian.

LEMMA 2. *Every cyclic normal subgroup of G is centralized by $G^{(1)}$.*

THEOREM 1. *If $\Psi(G)$ can be generated by two elements, then $\Psi(G)$ is meta-*

Received May 11, 1960.

cyclic and $(G^{(1)}, \Psi(G)^{(1)}) = \langle 1 \rangle$. *In particular,* $\Psi(G)$ *has class at most* 2 *if* $\Psi(G) \subseteq G^{(1)}$.

Remark 1. If $\Psi(G)$ is not contained in $G^{(1)}$, then the class of $\Psi(G)$ is not bounded. For example, if $p \neq 2$ let G be the group generated by x, y with defining relations

$$x^{p^{2k}} = y^{p^{2k+1}} = 1, \qquad (y, x) = y^p.$$

If $p = 2$, let $G = \langle x, y \rangle$ with defining relations

$$x^{2^{3k-1}} = y^{2^{3k+1}} = 1, \qquad (y, x) = y^2.$$

If we let $\Psi(G)$ be $\Phi(G)$, then $\Psi(G)$ is generated by x^p and y^p, yet $\Psi(G)$ has class k.

Proof of Theorem 1.[1] We need only show that $\Psi(G)$ is metacyclic, for then $\Psi(G)^{(1)}$ is a cyclic normal subgroup of G, and the theorem follows from Lemma 2.

Let $\Psi = \Psi(G)$. Then Ψ is metacyclic if and only if $\Psi/\Phi(\Psi_2)\Psi_3$ is metacyclic [2, Theorem 2.3]. Thus we may assume $\Phi(\Psi_2)\Psi_3 = \langle 1 \rangle$, and hence $\Psi^{(1)}$ has order p and is contained in the center of G. If Ψ is not metacyclic, and if $p \neq 2$, then [2, Theorem 2.6] we have $P(\Psi) \not\supseteq \Psi^{(1)}$. Therefore, if $\bar{G} = G/P(\Psi)$, $\Psi(\bar{G})$ is a non-abelian group of order p^3. This is impossible [3, Remark 1], so Ψ is metacyclic if $p \neq 2$.

We suppose henceforth that $p = 2$, Ψ is not metacyclic, and G is a group of minimal order for which the theorem is false. Let $\Psi = \langle a_1, b_1 \rangle$ and $c = (a_1, b_1)$. Then $\Psi^{(1)} = \langle c \rangle$ is a group of order 2 in the center of G. It is easy to see that the group H generated by the fourth powers of elements of Ψ is normal in G and has trivial intersection with $\Psi^{(1)}$. It follows from the minimality of G that $H = \langle 1 \rangle$. Thus $a_1^4 = b_1^4 = 1$. Also,

$$\Phi(\Psi) = \langle a_1^2 \rangle \times \langle b_1^2 \rangle \times \langle c \rangle = Z_1(\Psi).$$

Let $\bar{a}$ be a nontrivial element of the center of $G/Z_1(\Psi)$ which is contained in $\Psi(G/Z_1(\Psi)) = \Psi(G)/Z_1(\Psi)$, and let a be a coset representative of $\bar{a}$. Then $(g, a) \in Z_1(\Psi)$ for every $g \in G$. Also, since $Z_1(\Psi)$ has exponent 2, $(g, a^2) = 1$. Thus $a^2 \in Z_1(G)$, and it follows from the minimality of G that $a^2 = 1$.

It is clear that we can pick $b \in \Psi(G)$ such that $\Psi(G) = \langle a, b \rangle$, where $(a, b) = c$ and $a^2 = b^4 = 1$. Then $Z_1(\Psi) = \langle b^2 \rangle \times \langle c \rangle$, and $(G, \langle a \rangle) \subseteq Z_1(\Psi)$. It follows that if $g \in G$ then

$$a^g = ab^{2s}c^t, \qquad \text{and} \qquad b^g = a^u b^{1+2v} c^w$$

for appropriate integers s, t, u, v, w. A computation shows that

$$\text{(I)} \qquad a^{g^2} = ac^{us}, \qquad \text{and} \qquad b^{g^2} \equiv b^{1+2us} \mod \langle c \rangle.$$

[1] I am indebted to the referee for simplifying this proof.

In a 2-group $P(G) = \Phi(G)$, so there exist elements $x_1, \cdots, x_n \in G$ such that $b = x_1^2 x_2^2 \cdots x_n^2$. By (I),

$$a^{x_i^2} = ac^{\alpha_i}, \qquad b^{x_i^2} \equiv b^{1+2\alpha_i} \mod \langle c \rangle$$

for some α_i. Hence

$$a^b = ac^{\alpha_1+\cdots+\alpha_n}, \qquad b^b \equiv b^{1+2(\alpha_1+\cdots+\alpha_n)} \mod \langle c \rangle.$$

It follows that $a^b = a$, a contradiction.

THEOREM 2. *If G_n can be generated by n elements, then $P(G_n) = \Phi(G_n)$.*

Proof. It suffices to show that $P(G_n) \supseteq G_n^{(1)}$; hence we assume $P(G_n) = \langle 1 \rangle$ and show that $G_n^{(1)} = \langle 1 \rangle$. Also, we need only consider the case where $G_n^{(1)}$ has order p. But then G_n is normal in G and has order at most p^{n+1}; hence there exists a subgroup N of G_n such that $G_n : N = p$ and N is normal in G, where $|N| \leqq p^n$. It follows from Lemma 1 that $N \subseteq Z_n$. Thus N is contained in the center of G_n, and we see that G_n is abelian. This completes the proof.

Remark 2. It is known [2, Theorem 2.6] that, for odd primes p, a p-group H with two generators is metacyclic if and only if $P(H) = \Phi(H)$. It follows from Theorem 2 for $n = 2$ and p odd that G_2 is metacyclic, and consequently (Lemma 2) is of class 2. Thus Theorem 2 is a natural generalization of Blackburn's result.

If p is odd, it follows from $P(G_n) = \Phi(G_n)$ that $G_n^{(k)} = \langle 1 \rangle$ if G_n has exponent p^k [4, Theorem 3]. This result, together with Theorem 2, gives a bound on the derived length of G in terms of n and the exponent of G_n whenever G_n can be generated by n elements.

REFERENCES

1. N. BLACKBURN, *On prime-power groups in which the derived group has two generators*, Proc. Cambridge Philos. Soc., vol. 53 (1957), pp. 19–27.
2. ———, *On prime-power groups with two generators*, Proc. Cambridge Philos. Soc., vol. 54 (1958), pp. 327–337.
3. C. HOBBY, *The Frattini subgroup of a p-group*, Pacific J. Math., vol. 10 (1960), pp. 209–212.
4. ———, *A characteristic subgroup of a p-group*, Pacific J. Math., vol. 10 (1960), pp. 853–858.

CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA