SOME INEQUALITIES FOR POLYNOMIALS AND RELATED ENTIRE FUNCTIONS

BY

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1. Inequalities for polynomials

Throughout this section let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n. The following results are immediate.

THEOREM A.

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leqq n^2 \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \tag{1}$$

THEOREM B. *For* $R > 1$

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq R^{2n} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \tag{2}$$

If $p(z)$ has no zeros in $|z| < 1$, Theorem A can be sharpened.

THEOREM C. *If* $p(z)$ *has no zeros in* $|z| < 1$, *then*

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leqq \frac{n^2}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \tag{3}$$

Theorem C was proved by N. G. de Bruijn [4].
We prove a corresponding modification of Theorem B.

THEOREM 1. *If* $p(z)$ *has no zeros in* $|z| < 1$, *then*

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \frac{R^{2n}+1}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta \tag{4}$$

for $R > 1$.

Proof of Theorem 1. If $q(z) = z^n \overline{p(1/\bar{z})}$, then $|q(z)| = |p(z)|$ for $|z| = 1$. Since $p(z) \neq 0$ for $|z| < 1$, it follows that $|q(z)| \leqq |p(z)|$ for $|z| < 1$. On replacing z by $1/z$ we deduce that for $|z| > 1$,

$$|p(z)| \leqq |q(z)|.$$

Now $q(z) = \sum_{\nu=0}^{n} \bar{a}_{n-\nu} z^\nu$; hence

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \frac{1}{2} \int_0^{2\pi} |q(Re^{i\theta})|^2 \, d\theta + \frac{1}{2} \int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta$$
$$= \pi \sum_{\nu=0}^{n} (R^{2\nu} + R^{2n-2\nu}) |a_\nu|^2.$$

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The greatest of the quantities $R^{2\nu} + R^{2n-2\nu}$, $\nu = 1, \cdots, n$, is $R^{2n} + 1$. Therefore

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \frac{R^{2n}+1}{2} 2\pi \sum_{\nu=0}^{n} |a_\nu|^2 = \frac{R^{2n}+1}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta,$$

which was the assertion.

In (4) equality holds for $p(z) = \alpha + \beta z^n$ where $|\alpha| = |\beta|$. We also prove

THEOREM 2. *If the geometric mean of the moduli of the zeros of $p(z)$ is at least equal to 1, then*

$$(5) \qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \left\{\frac{(\sqrt{2})^{2n}+1}{2}\right\}^{(\log R)/\log\sqrt{2}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta$$

for $R < \sqrt{2}$, and

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \frac{R^{2n}+1}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta$$

for $R \geqq \sqrt{2}$. More precisely,

$$(6) \qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \left(\frac{k^{2n}+1}{2}\right)^{(\log R)/\log k} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta$$

for $R < k$, and

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \frac{R^{2n}+1}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta$$

or $R \geqq k$ where $k > 1$ is a root of the equation

$$2k^{2n-2} = 1 + k^{2n}.$$

Proof of Theorem 2. We observe that the hypothesis implies $|a_0| \geqq |a_n|$, so that

$$(7) \qquad |a_n|^2 R^{2n} + |a_0|^2 \leqq \tfrac{1}{2}(R^{2n} + 1)(|a_n|^2 + |a_0|^2)$$

for $R > 1$. Besides, in general

$$(8) \qquad |a_{n-\nu}|^2 R^{2n-2} + |a_\nu|^2 R^{2\nu} \leqq \tfrac{1}{2}(R^{2n} + 1)(|a_{n-\nu}|^2 + |a_\nu|^2)$$

if both

$$|a_{n-\nu}|^2 R^{2n-2\nu} \leqq \tfrac{1}{2}(R^{2n} + 1)|a_{n-\nu}|^2, \qquad |a_\nu|^2 R^{2\nu} \leqq \tfrac{1}{2}(R^{2n} + 1)|a_\nu|^2$$

hold. For $\nu = 1, \cdots, n - 1$, therefore, (8) holds if $R \geqq \sqrt{2}$. Consequently for $R \geqq \sqrt{2}$

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta$$

$$\begin{aligned} &= 2\pi(|a_n|^2 R^{2n} + \cdots + |a_{n-\nu}|^2 R^{2n-2\nu} + \cdots + |a_\nu|^2 R^2 + \cdots + |a_0|^2) \\ &\leqq \tfrac{1}{2}(R^{2n} + 1)\, 2\pi(|a_n|^2 + \cdots + |a_0|^2) \\ &= \tfrac{1}{2}(R^{2n} + 1) \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \end{aligned}$$

Now $\log \int_0^{2\pi} |p(re^{i\theta})|^2 \, d\theta$ is a convex function of $\log r$, and therefore for $1 < R < \sqrt{2}$

$$\begin{aligned} &\left(\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta\right)^{\log\sqrt{2}} \\ &\leqq \left(\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right)^{\log(\sqrt{2}/R)} \left(\int_0^{2\pi} p(\sqrt{2}e^{i\theta})|^2 \, d\theta\right)^{\log R} \\ &\leqq \left(\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right)^{\log(\sqrt{2}/R)} \left\{\frac{(\sqrt{2})^{2n} + 1}{2} \int_0^{2\pi} p(e^{i\theta})|^2 \, d\theta\right\}^{\log R} \\ &= \left\{\frac{(\sqrt{2})^{2n} + 1}{2}\right\}^{\log R} \left(\int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta\right)^{\log\sqrt{2}}, \end{aligned}$$

or

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \left\{\frac{(\sqrt{2})^{2n} + 1}{2}\right\}^{(\log R)/\log\sqrt{2}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta.$$

2. Inequalities for polynomials (continued)

The following result is immediate.

THEOREM D. *If $p(z)$ is a polynomial of degree n and $\rho < 1$, then*

$$\int_0^{2\pi} |p(\rho e^{i\theta})|^2 \, d\theta \geqq \rho^{2n} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \tag{9}$$

Equality in (9) holds only for $p(z) = cz^n$.

The conclusion of Theorem D can also be written as

$$\begin{aligned} &\frac{1}{(1-\rho)^2}\left\{\int_0^{2\pi} |p(\rho e^{i\theta})|^2 \, d\theta - \frac{1+\rho^{2n}}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta\right\} \\ &\qquad\qquad \geqq -\frac{1}{2}\frac{1-\rho^{2n}}{(1-\rho)^2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \end{aligned} \tag{10}$$

In case $p(z)$ has all its zeros in $|z| < 1$, we may expect in analogy with Theorem 1 that for every $\rho < 1$

$$\int_0^{2\pi} |p(\rho e^{i\theta})|^2 \, d\theta \geqq \frac{1+\rho^{2n}}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \tag{11}$$

But if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ has all its zeros on the unit circle, then

$$|a_\nu| = |a_{n-\nu}|, \qquad \nu = 0, 1, \cdots, n.$$

Consequently for $\rho < 1$

$$\begin{aligned}\int_0^{2\pi} |p(\rho e^{i\theta})|^2 d\theta &= 2\pi(|a_0|^2 + |a_1|^2\rho^2 + \cdots + |a_{n-1}|^2 \rho^{2n-2} + |a_n|^2 \rho^{2n}) \\ &= 2\pi\{(|a_0|^2 + |a_n|^2 \rho^{2n}) + (|a_1|^2 \rho^2 + |a_{n-1}|^2 \rho^{2n-2}) + \cdots\} \\ &= 2\pi\left\{(|a_0|^2 + |a_n|^2) \frac{1+\rho^{2n}}{2} + (|a_1|^2 + |a_{n-1}|^2) \frac{\rho^2 + \rho^{2n-2}}{2} + \cdots\right\} \\ &\leqq 2\pi\left\{(|a_0|^2 + |a_n|^2) \frac{1+\rho^{2n}}{2} + (|a_1|^2 + a_{n-1}|^2) \frac{1+\rho^{2n}}{2} + \cdots\right\} \\ &= \frac{1+\rho^{2n}}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta.\end{aligned}$$

In fact, strict inequality holds unless $p(z) = \alpha + \beta z^n$ where $|\alpha| = |\beta|$. Thus (11) is not necessarily true. We can however prove

THEOREM 3. *If $p(z)$ is a polynomial of degree n not having zeros in $|z| < 1$ then*

$$\liminf_{\rho\to 1-} \frac{1}{(1-\rho)^2}\left\{\int_0^{2\pi} |p(\rho e^{i\theta})|^2 d\theta - \frac{1+\rho^{2n}}{2}\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right\} > -\infty.$$

Since $(1 - \rho^{2n})/(1 - \rho)^2 \to \infty$ as $\rho \to 1-$, this result is an improvement on (10).

Proof of Theorem 3. Since $|p(z)| \geqq |q(z)| = |z^n\overline{p(1/\bar{z})}|$ for $|z| = \rho < 1$ we have

$$\begin{aligned}\int_0^{2\pi} |p(\rho e^{i\theta})|^2 d\theta &\geqq \frac{1}{2}\int_0^{2\pi} |q(\rho e^{i\theta})|^2 d\theta + \frac{1}{2}\int_0^{2\pi} |p(\rho e^{i\theta})|^2 d\theta \\ &= \pi\{(|a_0|^2 + |a_n|^2)(1+\rho^{2n}) + (|a_1|^2 + |a_{n-1}|^2)(\rho^2 + \rho^{2n-2}) + \cdots\} \\ &= \pi[\{(|a_0|^2 + |a_n|^2)(1+\rho^{2n}) + (|a_1|^2 + |a_{n-1}|^2)(1+\rho^{2n}) + \cdots\} \\ &\qquad - \{(|a_1|^2 + |a_{n-1}|^2)(1-\rho^2)(1-\rho^{2n-2}) + \cdots \\ &\qquad + (|a_m|^2 + |a_{n-m}|^2)(1-\rho^{2m})(1-\rho^{2n-2m}) + \cdots\}].\end{aligned}$$

But $(1 - \rho^{2m})(1 - \rho^{2n-2m})/(1 - \rho)^2 \to 2m(2n - 2m)$ as $\rho \to 1-$, and therefore the theorem follows.

We can also prove

THEOREM 4. *If the geometric mean of the moduli of the zeros of* $p(z)$ *is at least equal to* 1, *then*

$$\liminf_{\rho\to(1/k)-} \frac{1}{1/k-\rho}\left\{\int_0^{2\pi} |p(\rho e^{i\theta})|^2\,d\theta - \frac{1+\rho^{2n}}{2}\int_0^{2\pi} |p(e^{i\theta})|^2\,d\theta\right\} > -\infty,$$

where k (>1) *is a root of the equation*

$$2k^{2n-2} = 1 + k^{2n}.$$

3. Inequalities for periodic entire functions of exponential type

Throughout this section let $f(z)$ be an entire function of exponential type τ, periodic with period 2π. Since $f(z)$ has the form [1, p. 109]

$$f(z) = \sum_{k=-n}^{n} a_k e^{ikz}, \qquad n \leqq \tau,$$

the following two theorems are immediate.

THEOREM A′.

$$\int_{-\pi}^{\pi} |f'(x)|^2\,dx \leqq \tau^2 \int_{-\pi}^{\pi} |f(x)|^2\,dx.$$

THEOREM B′. *For all* y

$$\int_{-\pi}^{\pi} |f(x+iy)|^2\,dx \leqq e^{2\tau|y|} \int_{-\pi}^{\pi} |f(x)|^2\,dx.$$

If $f(z)$ is $O(e^{\varepsilon|z|})$ on the positive imaginary axis for some ε less than 1, then $f(z)$ has the form

$$f(z) = \sum_{k=0}^{n} a_k e^{ikz}, \qquad n \leqq \tau.$$

Hence Theorems C and 1 may be restated as follows. (We use $h_f(\theta)$ to denote the indicator of $f(z)$.)

THEOREM C′. *If* $f(z) \neq 0$ *for* $\operatorname{Im} z > 0$, *and if* $h_f(\pi/2) = 0$, *then*

$$\int_{-\pi}^{\pi} |f'(x)|^2\,dx \leqq \frac{\tau^2}{2} \int_{-\pi}^{\pi} |f(x)|^2\,dx.$$

THEOREM 1′. *If* $f(z) \neq 0$ *for* $\operatorname{Im} z > 0$, *if* $h_f(\pi/2) = 0$, *and if* $y < 0$, *then*

$$\int_{-\pi}^{\pi} |f(x+iy)|^2\,dx \leqq \frac{1}{2}\,(e^{2\tau|y|}+1) \int_{-\pi}^{\pi} |f(x)|^2\,dx.$$

A result in a different direction is the following.

THEOREM 5. *If* $f(z)$ *is real on the real axis, then for any real* y

$$\int_{-\pi}^{\pi} |f(x+iy)|^2\,dx \leqq \cosh 2\tau y \int_{-\pi}^{\pi} |f(x)|^2\,dx.$$

Proof of Theorem 5. Clearly $f(z)$ is of the form

$$f(z) = \sum_{k=-n}^{n} a_k e^{ikz}, \qquad n \leqq \tau,$$

where $a_{-k} = \bar{a}_k$ for $k = 1, 2, \cdots, n$. Consequently

$$\int_{-\pi}^{\pi} | f(x+iy) |^2 \, dx = 2\pi \textstyle\sum_{k=-n}^{n} | a_k |^2 e^{-2ky}$$

$$= 2\pi | a_0 |^2 + 2\pi \textstyle\sum_{k=1}^{n} | a_k |^2 (e^{-2ky} + e^{2ky})$$

$$\leqq 2\pi | a_0 |^2 + (2 \cosh 2yn) \, 2\pi \textstyle\sum_{k=1}^{n} | a_k |^2$$

$$\leqq (\cosh 2yn) \, 2\pi \textstyle\sum_{k=-n}^{n} | a_k |^2$$

$$\leqq \cosh 2yn \int_{-\pi}^{\pi} | f(x) |^2 \, dx.$$

4. Inequalities for entire functions of exponential type belonging to L^2 on the real axis

Throughout this section suppose $f(z)$ is an entire function of exponential type τ belonging to L^2 on the real axis. In this section we give theorems for such a function analogous to the theorems of the preceding section. The following three theorems are analogues of Theorems A′, B′, and 5, respectively.

THEOREM A″.

$$\int_{-\infty}^{\infty} | f'(x) |^2 \, dx \leqq \tau^2 \int_{-\infty}^{\infty} | f(x) |^2 \, dx.$$

THEOREM B″. *For all* y

$$\int_{-\infty}^{\infty} | f(x+iy) |^2 \, dx \leqq e^{2\tau|y|} \int_{-\infty}^{\infty} | f(x) |^2 \, dx.$$

THEOREM 5′. *If* $f(z)$ *is real on the real axis, then for all real* y

$$\int_{-\infty}^{\infty} | f(x+iy) |^2 \, dx \leqq \cosh 2\tau y \int_{-\infty}^{\infty} | f(x) |^2 \, dx.$$

Theorem A″ is due to Boas [1, p. 211], Theorem B″ to Plancherel and Pólya [6], and Theorem 5′ to Boas [2, p. 32].

We shall prove the following analogues of Theorems C′ and 1′.

THEOREM 6. *If* $f(z) \neq 0$ *for* Im $z > 0$, *and if* $h_f(\pi/2) = 0$, *then*

$$\int_{-\infty}^{\infty} | f'(x) |^2 \, dx \leqq \frac{\tau^2}{2} \int_{-\infty}^{\infty} | f(x) |^2 \, dx.$$

THEOREM 7. *If* $f(z) \neq 0$ *for* Im $z > 0$, *if* $h_f(\pi/2) = 0$, *and if* $y < 0$, *then*

$$\int_{-\infty}^{\infty} | f(x+iy) |^2 \, dx \leqq \frac{1}{2} (e^{2\tau|y|} + 1) \int_{-\infty}^{\infty} | f(x) |^2 \, dx.$$

Entire functions $f(z)$ of exponential type, not vanishing for Im $z > 0$, and satisfying $h_f(\pi/2) = 0$ were first studied by Boas [3]. Theorems 6 and 7 compare respectively with Theorems 2 and 1 of his paper.

Proof of Theorem 6. To prove Theorem 6 consider $\omega(z) = e^{i\tau z}\overline{f(\bar{z})}$, an entire function of exponential type $\geqq \tau$. Since $f(z)$ has no zeros for Im $z > 0$, $h_f(\pi/2) = 0$, and $h_f(-\pi/2) = \tau$, the function $\omega(z)$ has no zeros for Im $z < 0$, $h_\omega(-\pi/2) = \tau$, and $h_\omega(\pi/2) = 0$. Thus $\omega(z)$ belongs to the class P discussed in [1, p. 129]. Since $|f(x)| = |\omega(x)|$ for $-\infty < x < \infty$, it follows by a theorem of Levin [1, p. 226] that

$$|f'(x) \leqq |\omega'(x)| \tag{12}$$

for $-\infty < x < \infty$.

Since $f(z)$ belongs to L^2 on the real axis, we have by the Paley-Wiener Theorem [5, pp. 499–501]

$$f(z) = \int_0^\tau e^{izt}\varphi(t)\,dt, \qquad \varphi \in L^2.$$

Now

$$\omega(x+iy) = e^{i(x+iy)\tau}\int_0^\tau e^{-i(x+iy)t}\overline{\varphi(t)}\,dt;$$

hence by (12)

$$\begin{aligned}\int_{-\infty}^{\infty} |f'(x)|^2\,dx &\leqq \frac{1}{2}\int_{-\infty}^{\infty} |\omega'(x)|^2\,dx + \frac{1}{2}\int_{-\infty}^{\infty}|f'(x)|^2\,dx \\ &= \pi\int_0^\tau (\tau - t)^2|\varphi(t)|^2\,dt + \pi\int_0^\tau t^2|\varphi(t)|^2\,dt \\ &\leqq \tau^2\pi\int_0^\tau |\varphi(t)|^2\,dt = \frac{\tau^2}{2}\int_{-\infty}^{\infty}|f(x)|^2\,dx.\end{aligned}$$

Proof of Theorem 7. To prove Theorem 7, consider the same function $\omega(z)$. The function $g(z) = f(z)e^{-i\tau z/2}$ has no zeros for $y > 0$, and $h_g(-\pi/2) = h_g(\pi/2) = \tau/2$. By another theorem of Levin [1, p. 129] we have $|g(z)| \leqq |g(\bar{z})|$ for $y < 0$. Thus for $y < 0$,

$$\begin{aligned}|f(z)| &\leqq |e^{i\tau z/2}|\,|f(\bar{z})e^{-i\tau\bar{z}/2}| \\ &= |e^{i\tau z/2}|\,|\overline{f(\bar{z})}e^{i\tau z/2}| \\ &= |\overline{f(\bar{z})}e^{i\tau z}| = |\omega(z)|.\end{aligned}$$

It follows that for $y < 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x+iy)|^2\,dx &\leqq \frac{1}{2}\int_{-\infty}^{\infty}|\omega(x+iy)|^2\,dx + \frac{1}{2}\int_{-\infty}^{\infty}|f(x+iy)|^2\,dx \\ &= \pi\int_0^\tau e^{-2y(\tau-t)}|\varphi(t)|^2\,dt + \pi\int_0^\tau e^{-2yt}|\varphi(t)|^2\,dt \\ &\leqq (e^{2\tau|y|}+1)\pi\int_0^\tau |\varphi(t)|^2\,dt \\ &= \frac{1}{2}(e^{2\tau|y|}+1)\int_{-\infty}^{\infty}|f(x)|^2\,dx.\end{aligned}$$

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