THE STRUCTURE OF CONTOURS OF A FRÉCHET SURFACE

BY

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1. Introduction

The concept of a contour for a Fréchet surface was introduced by Cesari [1] who made use of contours to establish the Cesari-Cavalieri inequality, to develop some of the most fundamental properties of Fréchet surfaces, and to investigate variational problems for surface integrals. In a recent paper [3], Cesari and the author introduced several methods for smoothing contours by deleting certain inessential portions from them and proved the equivalence of several such methods. The method of contours is of value chiefly because it provides a means for constructing on the surface a conveniently disposed family of continuous curves. The counterimages of these curves, which are called contours, lie in the two-dimensional set over which the surface is defined, and it is the principal purpose of this paper to show that a representative mapping defining the surface can be found for which almost all of the contours have a simple structure. To this end we rely heavily on the methods and results of the previous paper [3] on smoothing methods for contours, and in Section 2 a brief exposition of smoothing methods will be given. In Section 3 we establish certain properties of smoothed contours and show that, in computing the length of the image of a smoothed contour, either an outer or inner border may be used. The principal result is established in Section 4 in which it is shown that for a nondegenerate surface of the type of the disk, a representation can be found for which almost all contours are arcs, points, or simple closed curves. This constitutes a considerable improvement over a previous result of the author [4] in which he showed that a countable dense set of contours had this property.

2. Notations and definitions

Let Q be a bounded, closed, simply connected, planar Jordan region, and let $T:Q \rightarrow E_N$ be a continuous mapping from Q into euclidean N-space. Then T defines a Fréchet surface S. We assume that S has finite Lebesgue area, and we denote by $[S]$ the set of points in E_N occupied by the surface. It may also be assumed that Q is the unit square in the (u, v) coordinate plane, $Q = \{p = (u, v) \mid 0 \leqq u, v \leqq 1\}$.

Let f be a real-valued continuous function defined on $[S]$ with upper and lower bounds t_1, t_2 respectively. For $t_1 \leqq t \leqq t_2$ we define $D^{-}(t)$, $D^{+}(t)$,

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$C(t)$, respectively, as the set of points $p \in Q$ for which

$$f(T(p)) < t, \qquad f(T(p)) > t, \qquad f(T(p)) = t.$$

The set $C(t)$ is the *contour* associated with f, T, t, and the set of boundary points $[D^-(t)]^* - D^-(t)$, $[D^+(t)]^* - D^+(t)$ are called respectively the lower and upper borders of $D^-(t)$, $D^+(t)$ in Q. Let $\alpha(t)$ be a component of $D^-(t)$, and let γ be a component of $[\alpha(t)]^* - \alpha(t)$. As in [1] the set $A = A(\alpha, \gamma)$ is the set of points of Q which lie in α plus those which are separated from γ by other components of $[\alpha(t)]^* - \alpha(t)$ plus those components of $[\alpha(t)]^* - \alpha(t)$ which separate points of Q from γ. $A(\alpha, \gamma)$ is an open simply connected set in Q with γ as a connected portion of its boundary. Evidently γ is a portion of the lower border of $D^-(t)$, and in [3] several methods of replacing γ by a smoother set were defined and their equivalence proved. Since these smoothing methods will be extensively used in the following sections, we shall give a brief exposition of them here.

The definition of the smoothing methods depends strongly upon the theory of ends and prime ends for open simply connected domains as developed by Cesari in [1]. Consider the set γ and the corresponding open set $A(\alpha, \gamma)$. Any point p of γ accessible from $A(\alpha, \gamma)$ by an arc defines one or more ends. An arc b lying in A with end point $p \in \gamma$ defines an end, and two different arcs b_1, b_2 define the same end if for every neighborhood N of p either (i)

$$(b_1 - (p)) \cap (b_2 - (p)) \neq 0,$$

or (ii) there exists in N an arc c joining $b_1 - (p)$ to $b_2 - (p)$ such that $b_1 \cup b_2 \cup c$ bound a simply connected Jordan region in $N \cap A$. If η_1, η_2, η_3, η_4 are ends, then η_1, η_3 separate η_2, η_4 if there exists a cross cut made up of defining arcs for η_1, η_3 which separates defining arcs for η_2, η_4 in A. In terms of this separation an order can be defined on γ which orders the ends in a cyclic or linear order, and by a method of completion as described in [1], prime ends can be defined in such a way that to every point of γ, accessible or inaccessible from $A(\alpha, \gamma)$, there is associated at least one prime end. The boundary point on γ associated with the end η will be denoted by w_η, and the points of γ associated with a prime end ω will be denoted by E_ω.

The smoothing method which will chiefly be used in the following sections is described as follows. Let Γ be the set of all maximal continua of constancy for T in Q, and let $\sigma' \subset \Gamma$ be the set of all such continua which intersect γ. Consider the set $\{\omega\}_{A,\gamma}$ of all prime ends of A corresponding to points on γ. This set can be linearly or cyclically ordered as defined in [1]. Let $\omega_1 < \omega_2$ be two prime ends in $\{\omega\}_{A,\gamma}$. Let ω', ω'' be given $\omega_1 \leqq \omega' < \omega'' \leqq \omega_2$. Assume that $E_{\omega'} \cap E_{\omega''} \neq 0$. If there exists an end ω''' with $\omega' < \omega''' < \omega''$, let $\sigma'(\omega', \omega'')$ be the subset of σ' obtained by deleting from σ' all elements which intersect any E_ω for $\omega' < \omega < \omega''$. Let $\sigma_0(\omega_1, \omega_2)$ be the intersection of all sets of the form $\sigma'(\omega', \omega'')$ for all ω', ω'' with $\omega_1 \leqq \omega' < \omega'' \leqq \omega_2$. This set will be the *smoothed contour* between ω_1 and ω_2 in σ'. It was shown in [3] that in the hyperspace topology of Γ, $\sigma_0(\omega_1, \omega_2)$ is an arc. We can order the elements of $\sigma_0(\omega_1, \omega_2)$ by the order of the prime ends ending on the elements of $\sigma_0(\omega_1, \omega_2)$, and when the order on a smoothed contour is mentioned, we make no distinction between the ordering of the elements of $\sigma_0(\omega_1, \omega_2)$ and the ordering of the prime ends ending on them since for a smoothed contour no ambiguity will arise.

Another method of smoothing contour which was defined [3] and shown to be equivalent to the above method for appropriately chosen elements $\omega_1, \omega_2 \in \{\omega\}_{\alpha,\gamma}$ is defined as follows. Again, let σ' be the set of all elements $g \in \Gamma$ for which $g \cap \gamma \neq 0$.

Let $K = \bigcup_{g \in \sigma'} g \subset Q$. Let D_i, $i = 1, 2, 3, \cdots$, be the complementary domains of K in Q which do not intersect $A(\alpha, \gamma)$. Let σ_i be the family of all $g \in \sigma'$ which intersect both $A(\alpha, \gamma)$ and D_i, and let $\sigma_0 = \bigcup_i \sigma_i$ in Γ. Let $K' = \bigcup_{g \in \sigma_0} g \subset Q$, and let $D_0 \supset A(\alpha, \gamma)$ be the component of $Q - K'$ which contains $A(\alpha, \gamma)$. Let γ_i', $i = 0, 1, 2, \cdots$, be respectively the boundaries of D_i and Q_0, and for $i = 1, 2, 3, \cdots$ let γ_i'' be the subset of γ_0' given by

$$\bigcup_{g \in \sigma_i} (g \cap \bar{D}_0).$$

The set σ_0 is the smoothed contour corresponding to γ, and the γ_i', γ_i'' are said to lie respectively on the outside and the inside of the smoothed contour σ_0. In case ω_1, ω_2 correspond to elements of the same σ_i, then the portion of σ_i between ω_1 and ω_2 is the same smoothed contour $\sigma_0(\omega_1, \omega_2)$ as described above. The ordering on σ_0 can be constructed by ordering the σ_i in a consistent manner, either from the outside or inside of the contour.

3. Relations between the sides of a smoothed contour

Let us assume that the contour γ has been smoothed according to the second method described in the last section and that the sets σ_i, γ_i', γ_i'', etc. have been constructed as above. In this section we describe a method of defining lengths of the images of the various portions of the contour and show that the lengths coincide whether computed from inside or outside the contour. The lengths $l(\gamma_0)$, $l(\gamma_i')$, $l(\gamma_i'')$ are defined as follows. In each σ_i let

$$g_1(i) < g_2(i) < \cdots < g_k(i)$$

be an ordered set of elements of σ_i. Form the sum

$$\sum_{j=1}^{k_0-1} | T(g_j(i)) - T(g_{j+1}(i)) |,$$

and define $l(\gamma_i'')$ as the supremum of all such sums for all possible choices of the $\{g_j(i)\}$.

THEOREM 1. *For each* $i = 1, 2, \cdots$, $l(\gamma_i') = l(\gamma_i'')$, *and* $l(\gamma_i')$ *is the same as the Cesari generalized length of* γ_i'.

Proof. The theorem follows from the definitions of the ordering of the σ_i given in [1]. The ordering is the same considered relative to D_i or to D_0

since each $g \in \sigma_i$ intersects both D_i and D_0. Thus all sums in the definition of $l(\gamma_i')$ are defined in the same way in forming the sums for $l(\gamma_i'')$. Hence $l(\gamma_i') = l(\gamma_i'')$. To prove the second part of the theorem, consider an ordered set of ends $\{\eta_l(i)\}$, $\eta_1(i) < \eta_2(i) < \cdots < \eta_{l_i}(i)$. If $w_l(i)$ is the end point of $\eta_l(i)$ on γ_i', then $w_l(i) \in g_j(i)$ for some j. If $w_{l_1}(i)$, $w_{l_2}(i)$ lie in the same set $g_j(i)$, then

$$T(w_{l_1}(i)) = T(w_{l_2}(i)) = T(g_j(i)).$$

Thus in the sum $\sum_{l=1}^{l_i-1} | T(w_l(i)) - T(w_{l+1}(i)) |$, if $w_l(i)$, $w_{l+1}(i) \in g_j(i)$ for some j, $T(w_l(i)) - T(w_{l+1}(i)) = 0$ and may be omitted from the sum defining the lengths. However, by [3], γ_i' is a boundary corresponding to a smoothed contour, and if $w_l(i)$, $w_{l+1}(i) \in g_j(i)$, then all points corresponding to ends η between $\eta_l(i)$ and $\eta_{l+1}(i)$ also lie in $g_j(i)$. Hence

$$\sum_{l=1}^{l_i-1} | T(w_l(i)) - T(w_{l+1}(i)) | = \sum_{j=1}^{k_i-1} | T(g_j(i)) - T(g_{j+1}(i)) |$$

where the sum is taken over all $g_j(i)$ which contain points $w_l(i)$. Thus the length defined in terms of ends is not greater than that defined as above. The opposite inequality is trivial, and hence the lengths are the same.

THEOREM 2. *If $l(\gamma) < \infty$ and if σ_i, σ_j are subsets of Γ corresponding to D_i, D_j as above, then σ_i, σ_j can have at most two elements $g \in \Gamma$ in common.*

Proof. Suppose that $\sigma_i \cap \sigma_j$ consists of at least three elements, g_1, g_2, g_3. Then $g_k \cap \bar{D}_i \neq 0$, $g_k \cap \bar{D}_j \neq 0$, $k = 1, 2, 3$. Assume that $g_1 < g_2 < g_3$ in the ordering on σ_i. By definition of the ordering on γ_i with respect to D_i, this implies that there exist two arcs b_1^i, b_2^i (possibly indefinite, i.e., the homeomorphic image of a half-open interval; see [2])which intersect at a single point in D_i, terminate respectively on g_1, g_3, and lie entirely in D_i with the exception of points on g_1, g_3, and such that if b_3^i is an arc (possibly indefinite) ending on g_2, then a subarc of $b_2^i - g_2$ lies in the open set bounded by $b_1^i \cup b_2^i \cup \beta_i$ where $\beta_i \subset \gamma_i$ is the subset of γ_i consisting of points of γ_i on elements $g \in \sigma_i$ between g_1 and g_3 inclusive. Similarly for D_j there exist two such arcs, b_1^j, b_2^j contained in D_j except for end points on g_1, g_2. Then $b_1^j \cup b_2^j \cup g_1 \cup g_3 \cup b_1^i \cup b_2^i$ bound an open set G which contains g_2. However $G \subset D_i \cup D_j \cup K'$. Since g_2 contains boundary points of D_0, $G \cap D_0 \neq 0$. However this is impossible since $(D_i \cup D_j \cup K') \cap D_0 = 0$.

THEOREM 3. *Let γ be a contour for which $l(\gamma) < \infty$, and let γ_i', γ_i'', $i = 1, 2, \cdots$, be as defined above. Then $\sum_{i=1}^{\infty} l(\gamma_i') = \sum_{i=1}^{\infty} l(\gamma_i'') = l(\gamma_0') \leqq l(\gamma)$.*

Proof. Let σ_i, σ_j be two sets as defined above. By the previous theorem, $\sigma_i \cap \sigma_j$ can consist of at most two elements, g_1, $g_2 \in \Gamma$. Assume first that $\sigma_i \cap \sigma_j$ consists of exactly two elements g_1, $g_2 \in \Gamma$, $g_1 < g_2$. By the previous theorem, the family $G_1 = \{g \mid g \in \sigma_i \cup \sigma_j, g_1 \leqq g \leqq g_2\}$ must be a subfamily of one of the sets σ_i, σ_j, and the family G_2 complementary to G_1 in $\sigma_i \cup \sigma_j$ must be in the other. Hence $\sigma_i = G_1$, $\sigma_j = G_2$. Thus the length

$$l(\gamma_i' \cup \gamma_j') = l(\gamma_i') + l(\gamma_j')$$

by definition since it is the length of the image of at most three arcs in Γ with only end points in common.

Let $\sigma_i \cap \sigma_j = g_1 \in \Gamma$. If σ_i and σ_j are arcs in Γ with common end point g_1, then again, the above argument shows that $l(\gamma_i' \cup \gamma_j') = l(\gamma_i') \cup l(\gamma_j')$. If this is not the case, then one of the sets must consist of g_1 only since if $g_1, g_2 \in \sigma_j$, then either the arc consisting of elements g for which $g_1 \leqq g \leqq g_2$ or the set $\{g \mid g \leqq g_1, g \geqq g_2, g \in \sigma_i \cup \sigma_j\}$ must lie entirely in σ_j. If σ_i consists of more than one element, then in the first case, σ_i can contain no element $g > g_2$, and in the second case, no element g with $g_1 < g < g_2$, since otherwise $\sigma_i \cap \sigma_j$ would contain more than one element. Hence if $g_1, g_2 \in \sigma_j$, $\sigma_i = g_1$, and $l(\gamma_i' \cup \gamma_j') = l(\gamma_j') = l(\gamma_i') + l(\gamma_j')$.

If $\sigma_i \cap \sigma_j = 0$, the equality above is obvious.

Since there can be at most a countable family of the open sets D_i in Q, there can be at most a countable family of the sets σ_i. Hence by enumerating the σ_i and performing successively the above steps, it can be seen that $l(\bigcup_{i=1}^{\infty} \gamma_i') = l(\gamma_0') = \sum_{i=1}^{\infty} l(\gamma_i')$. The inequality at the conclusion of the theorem is obvious.

4. The structure of contours for nondegenerate surfaces

A continuous mapping $T: Q \to E_N$ defines a nondegenerate surface if no maximal continuum of constancy for T in Q separates Q or the plane. In this case it is known that there exists a mapping T', Fréchet equivalent to T which is light (i.e. all continua of constancy for T' are single points). Assume that T is light, and let $C(t)$ be a contour defined by T, f, t in Q whose image is of finite length in the sense of Cesari [1]. This implies that all sets of the form E_ω are continua of constancy and hence points in this case. Let α be a component of $D^-(t)$, and let γ be a component of $\alpha^* - \alpha$. Since all sets $E_\omega \subset \gamma$ are single points, in this case, all such sets are accessible from α, and each prime end is an end. We shall show that in this case, almost all contour components have an exceptionally simple form.

THEOREM 4. *Let $T: Q \to E_N$ be a light mapping defining a Fréchet surface S, and let $f: [S] \to$ Reals be a continuous function. Let $\{\gamma\}_f$ be the set of all components of contours corresponding to f in Q and whose images on S are of finite length. Then there exists a countable set $\{\gamma_1, \gamma_2, \cdots\}$ such that if $\gamma \in \{\gamma\}_f$, $\gamma \neq \gamma_i$, $i = 1, 2, 3, \cdots$, then γ is a point, a simple arc, or a simple closed curve.*

Proof. Let $C(t) \subset Q$ be a contour with image of finite length on S. Let $D^-(t)$ be the corresponding open set as defined in Section 2, and let α be a component of $D^-(t)$. Let γ be a component of $\alpha^* - \alpha$. Assume that on γ there exist a point p and two distinct ends, $\eta_1 \neq \eta_2$ from $A(\alpha, \gamma)$ to γ which have common end point p. Let b_1, b_2 be defining arcs for η_1, η_2 respectively which have common end point p and such that

$$b_1 - (p) \in A(\alpha, \gamma), \qquad b_2 - (p) \in A(\alpha, \gamma),$$

and $(b_1 - (p)) \cap (b_2 - (p)) = p' \in A(\alpha, \gamma)$, p' a point. Thus $b_1 \cup b_2$ is a simple closed curve β in Q which intersects γ only at p and lies otherwise in $A(\alpha, \gamma)$. Also since $\eta_1 \neq \eta_2$ there exist ends between η_1 and η_2, and by definition of the ordering of the ends, there exist points of γ lying interior to the closed curve β. Assume $\eta_1 < \eta_2$. To prove the theorem several cases must be considered.

Case I. There exist ends η_1', η_2' with $\eta_1' < \eta_1 < \eta_2 < \eta_2'$ ending on γ with $w_{\eta_1'} \neq w_{\eta_2'}$, $w_{\eta_1'} \neq p$, $w_{\eta_2'} \neq p$ such that defining arcs exist for η_1', η_2' which together form a cross cut c for $A(\alpha, \gamma)$ and such that (i) $\beta - (p)$ lies in the component of $A(\alpha, \gamma) - c$ which contains as boundary points the point $w_\eta \in \gamma$ where $\eta_1 < \eta < \eta_2$; (ii) there exist no end η_1'' between η_1 and η_1' for which $w_{\eta_1''} = w_{\eta_2'}$ and no end η_2'' between η_2 and η_2' with $w_{\eta_2''} = w_{\eta_1'}$.

This case is illustrated in Figure 1. Let η_0 be an end with $\eta_1 < \eta_0 < \eta_2$. Since $\eta_1 < \eta_2$, such ends exist. By using the first smoothing method described in Section 2, let the set γ be smoothed between η_1 and η_0. By [3] the smoothed contour between η_1 and η_0 is an arc in the hyperspace topology in Γ. However, since all elements of Γ are single points in Q, the hyperspace topology of Γ coincides with the ordinary topology in Q. Hence the smoothed portion of γ is an arc in Q. Let also γ be smoothed between η_1' and η_1 and between η_2' and η_2. This yields three arcs in Q with one end point in common but with the other end points all distinct. Such a configuration contains a triod, i.e., three arcs which are distinct except for one point which is a common end point of all three. By a theorem of R. L. Moore [5], there can exist at most countably many distinct triods in the plane. Thus there can exist at most countably many contours having components γ which satisfy Case I.

Case I′. This case is the same as Case I, except that here

$$\eta_1 < \eta_1' < \eta_2' < \eta_2 .$$

The same arguments hold in this case when the obvious modifications in order relations in the proof are made. (See Figure 2.)

Case II. η_1 is the first end on γ, and there exist an end $\eta_2' > \eta_2$ with $w_{\eta_1'} \neq p$ and a cross cut c from Q^* to $w_{\eta_2'}$ which contains a defining arc for η_2' and such that $\beta - (p)$ lies in the component of $A(\alpha, \gamma) - c$ which includes among its boundary points the points $w_\eta \in \gamma$ for which $\eta_1 < \eta < \eta_2$.

Let η_0 be an end with $\eta_1 < \eta_0 < \eta_2$. Let the portion of γ between η_0 and η_2 and the portion between η_2 and η_2' be smoothed. This yields, as in Case I, two arcs τ_1, τ_2, τ_1 with initial point w_{η_0}, τ_2 with initial point $w_{\eta_1'} \neq w_{\eta_0}$, and each with terminal point p. In case $\tau_1 \cap \tau_2$ contains a point other than p, then $\tau_1 \cup \tau_2$ contains a triod. In case $\tau_1 \cap \tau_2 = p$, the arcs intersect only on Q^*. A configuration consisting of two arcs with this property we shall call a

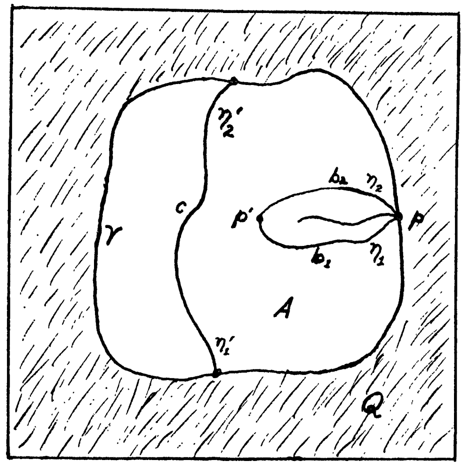

Figure 1

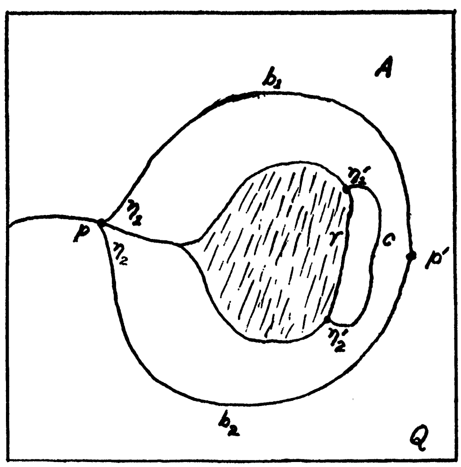

Figure 2

V-set. Since all contour components are distinct, no two such configurations can have points in common. Since the set of these components which contains triods is at most countable, we need only show that there can exist only countably many **V**-sets in Q.

Let V be a **V**-set, and assume first that $V \cap Q^*$ contains points other than p. Then either V contains a **V**-set V' such that $V' \cap Q^*$ is a single point, or a subarc of V lies in Q^*. Since Q^* can contain only countably many distinct subarcs, only countably many contour components can contain **V**-sets of this kind. Each of the remaining **V**-sets can then be assumed to intersect Q^* in only one point. Let l be one side of the square Q, $l = \{u, v | v = 0, 0 \leqq u \leqq 1\}$. Parallel to l let the sequence of line segments $\{l_n\}$, $n = 1, 2, \cdots$, be constructed where $l_n = \{p = (u, v) \in Q \mid 0 \leqq u \leqq 1, v = 1/n\}$, $n = 1, 2, \cdots$.

Then if V is a **V**-set with $V \cap Q^*$ a single point on l, one of the segments l_n must intersect both arcs τ_1, τ_2 of V. Thus the union $\tau_1 \cup \tau_2 \cup l_n$ will bound an open set $G(V) \subset Q$. However, since all **V**-sets are distinct, if $V_1 \neq V_2$,

$$G(V_1) \cap G(V_2) = 0.$$

Since there can be at most countably many disjoint open sets in the plane, only countably many **V**-sets of this type can have vertex point on l, and similarly for the other sides of Q. Thus under any circumstance only countably many contour components can fall under Case II.

Case III. As in Case I there exist two distinct ends $\eta_1' \neq \eta_2'$ with

$$\eta_1' < \eta_1 < \eta_2 < \eta_2'$$

satisfying (i) of Case I but not (ii); i.e., for every choice of η_1', η_2' there exists an end η_1'', $\eta_1' < \eta_1'' < \eta_1$, η_1'', η_2' satisfying (i) but $w_{\eta_1''} = w_{\eta_2'}$, or an end η_2'', $\eta_2 < \eta_2'' < \eta_2'$, η_1', η_1'' satisfying (i) and $w_{\eta_2''} = w_{\eta_1'}$.

We shall show first that under these conditions, exactly two ends end at p. Assume that three ends, η_1, η_2, η_3, $\eta_1 < \eta_2 < \eta_3$, have the property that $w_{\eta_1} = w_{\eta_2} = w_{\eta_3} = p$. Choose $\eta_1' < \eta_1$ but $\eta_1' \not< \eta_3$, $w_{\eta_1'} \neq p$. Choose η_2' such that $\eta_2 < \eta_2' < \eta_3$, $w_{\eta_2'} \neq p$. Let b_1, b_2, b_3 be defining arcs for η_1, η_2, η_3 respectively which intersect at p and also at a point $p_0 \in A(\alpha, \gamma)$. Thus $b_1 \cup b_3$ is a simple closed curve β', and all ends between η_1 and η_3 have defining arcs which lie inside β'. However, for $\eta_1'' < \eta_1$, $\eta_1'' \not< \eta_3$, $w_{\eta_1''} \neq p$, then $w_{\eta''}$ does not lie inside β'. Hence $w_{\eta_1''} \neq w_{\eta_2'}$ for any such η''. A similar argument shows that no η_2'' as described above can exist. Thus p is the end point for at most two ends.

Assume now that on γ Cases I, I′, II do not hold. Assume also that $\gamma \cap Q^* = 0$, and hence the ordering on γ is cyclic. Case III must hold, and any point of γ is the end point for at most two ends.

If each point of γ is the end point of exactly one end, then the contour is smooth between any two of its points and is hence an arc between any two of its points. Since the ordering on γ is cyclic, it can be seen from elementary considerations that, since Cases I and I′ do not hold, γ consists of two arcs joined only at their end points, and hence γ is a simple closed curve.

Assume that there is a point $p \in \gamma$ which is the end point of two ends η_1, η_2 with $\eta_1 < \eta_2$. Assume that there are two ends η_1', η_2', $\eta_1 < \eta_1' < \eta_2' < \eta_2$, for which there exist no ends η_1'', η_2'' with $w_{\eta_1''} = w_{\eta_1'}$, $w_{\eta_2''} = w_{\eta_2'}$. Since $\eta_1 \neq \eta_2$ and the order is cyclic, there exists an end η_0, $\eta_2 < \eta_0 < \eta_1$.

Assume as in Case I that γ is smoothed between η_1' and η_1, η_3' and η_1, η_1 and η_0. As in Case I this gives rise to a triod in γ. Similarly, the same argument holds if $\eta_2 < \eta_1' < \eta_2' < \eta_1$. Thus such cases can occur at most countably many times. If these cases are deleted, there can exist at most one end η_1', $\eta_1 < \eta_1' < \eta_2$, for which there is no end η_0 with $w_{\eta_0} = w_{\eta_1'}$. A similar situation must exist for the interval $\eta_2 < \eta < \eta_1$.

Assume that in the interval $\eta_1 \leqq \eta \leqq \eta_2$ the ends occur in distinct pairs with one possible exception where η, η' belong to the same pair if $w_\eta = w_{\eta'}$. Divide the ends into two classes. If η, η' is a pair as defined above, we place η in class C_1 if $\eta < \eta'$, and we place η' in class C_2, and similarly for the ends in the interval $\eta_2 \leqq \eta \leqq \eta_1$. Consider the classes C_1, C_2 for the interval $\eta_1 \leqq \eta \leqq \eta_2$. This includes all the ends of the interval with one possible exception, and every end in C_1 precedes every end of C_2 in the ordering. Hence the classes C_1, C_2 define a prime end by definition and hence an end η_0, since in this case all prime ends correspond to ends. Also η_0 is not a member of any pair since if η_0, η_0' were a pair, $\eta_0 < \eta_0'$, then there would exist an end between η_0 and η_0' and hence an end in C_1 which follows η_1, or an end in C_2 which precedes η_0', contrary to the fact that η_0 is defined by the classes C_1 and C_2. Hence η_0 is the unique end which is not paired to another in the interval $\eta_1 \leqq \eta_0 \leqq \eta_2$. Similarly there is a unique end $\bar{\eta}_0$ in the interval $\eta_2 \leqq \eta \leqq \eta_1$ with the same properties. However, the two intervals $\eta_1 \leqq \eta \leqq \eta_2$ and $\eta_2 \leqq \eta \leqq \eta_1$ include all ends ending on γ. Consider the interval $\bar{\eta}_0 \leqq \eta \leqq \eta_0$. The points w_η for this interval include all points of γ. However, in this interval each point of γ is the end point of exactly one end ending on γ. Hence γ is smooth between $\bar{\eta}_0$ and η_0 and is thus an arc.

If $\gamma \cap Q^* \neq 0$ and $\eta_1 < \eta_2$ is such that $w_{\eta_1} = w_{\eta_2} = p$, then if η_1 and η_2 are not respectively the first and last ends on γ, the same methods as above can be used to show that either there exist ends $\eta_1', \eta_2', \eta_1' < \eta_1 < \eta_2 < \eta_2'$, where $w_{\eta_1'} = w_{\eta_2'}$ and η_1', η_2' are respectively the first and last end on γ, or that γ contains a **V**-set or a triod. If η_1, η_2 are respectively the first and last ends on γ, then the methods used above show that either γ contains a triod, or that all ends with one possible exception η_1 can be paired as above, and the interval from η_1 to η_0 defines an arc with one end point on Q^*.

Thus in Case III, γ is an arc or contains a triod or a **V**-set. If we delete the components γ which contain triods or **V**-sets, we arrive at the conclusion that all but a countable family of the components γ in Case III are arcs.

The cases considered above exhaust all possible cases in which the contour γ is not smoothed between any two of its points. Thus with the exception of a countable number of cases and of the contour components in Case III which have already been shown to be arcs, all contour components γ with images of finite length are smooth between any two ends. If γ is a contour component having first and last ends ending on it, then V is smooth between the two end points and is hence an arc, since in the hyperspace Γ, a smoothed contour between two elements is an arc, and in this case $\Gamma = Q$. If the ordering of the ends on γ is cyclic, then given any two points of γ, γ consists of two arcs with these two points in common. Since all components containing triods have been deleted, the two arcs meet only at these points, and γ is a simple closed curve in Q. Finally, in the trivial case in which there is only one end ending on γ, γ is a point. Thus all contour components whose images are of finite length, with the exception of a countable family, are arcs, simple closed curves, or points.

Corollary. *Let $T:Q \rightarrow E_N$ be a nondegenerate mapping defining a surface S of finite Lebesgue area. Let $f:[S] \rightarrow$ Reals be a Lipschitzian function. Then there exists a representation T' of S for which the components of contours corresponding to almost all values of $f([S])$ are points, arcs, or simple closed curves.*

Proof. Let T' be the light representative of S. By the Cesari-Cavalieri inequality [1, p. 328], contours corresponding to almost all values of $f([S])$ have images in $[S]$ of finite length. By the theorem above, all but a countable family of these have the desired form. Hence almost all contours have the desired form.

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