ALGEBRAIC LIE ALGEBRAS AND REPRESENTATIVE FUNCTIONS SUPPLEMENT

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The purpose of this note is to supplement the results obtained in [2] concerning the structure of the algebra of the representative functions on the universal enveloping algebra of a Lie algebra. Theorems 1, 2, and 3 are analogues of results obtained jointly with G. D. Mostow for Lie groups (to appear elsewhere), and the proofs are based on the same ideas, although the technicalities involved are rather different.

In order not to disrupt the continuity later, we begin with a simple fact concerning the universal enveloping algebra of a nilpotent Lie algebra.

LEMMA 1. *Let P be a nilpotent Lie algebra of finite dimension over an arbitrary field F, and choose a basis $x_1, \cdots, x_n$ for P such that each commutator $[x_i, x_j]$ is an F-linear combination of x_k's with $k < \min(i, j)$. Let U denote the universal enveloping algebra of P, and let $U^{[q]}$ stand for the subspace of U that is spanned by the ordered monomials $x_n^{e_n} \cdots x_1^{e_1}$ with $e_1 + \cdots + e_n \geqq q$. Then, for each q, there is an exponent q_1, such that $P^{q_1}U \subset U^{[q]}$.*

Proof. We define a weight function w on the set of the ordered monomials in our basis elements such that $w(1) = 0$, $w(x_i) = 2^{n-i}$, and the weight of an ordered monomial is the sum of the weights of its factors. Every element $u \in U$ can be written uniquely as an F-linear combination of ordered monomials. For $u \neq 0$, we define $w(u)$ to be the minimum taken by w on the set of ordered monomials occurring with a nonzero coefficient in the standard expression for u. Then we shall have $u \in U^{[q]}$ whenever $w(u) \geqq 2^{n-1}q$. Now we claim that, if $u \neq 0$, $w(x_i u) > w(u)$. Evidently, it suffices to establish this in the case where u is an ordered monomial. In that case, one easily shows by induction on the degree of u that $w(x_i u) \geqq w(x_i) + w(u)$. This establishes our claim, and the conclusion of Lemma 1 follows immediately.

Now we must recall some of the notation and results of [2, Section 6]. Let L be a finite-dimensional Lie algebra over a field F of characteristic 0. Let A denote the radical of L, and set $T = [L, A]$. Let S be a maximal semisimple subalgebra of L. $\mathbf{R}(L)$ denotes the algebra of the representative functions on the universal enveloping algebra $U(L)$ of L. We have defined a subalgebra $\mathbf{R}^S(L)$ of $\mathbf{R}(L)$ (as the canonical image of $\mathbf{R}(L/A)$ in $\mathbf{R}(L)$), such that the restriction to $U(S)$ maps $\mathbf{R}^S(L)$ isomorphically onto $\mathbf{R}(S)$, and a subalgebra $\mathbf{R}^A(L)$, such that the restriction to $U(A)$ maps $\mathbf{R}^A(L)$ isomorphically onto the restriction image $\mathbf{R}(L)_A$ of $\mathbf{R}(L)$ in $\mathbf{R}(A)$. Then

$$\mathbf{R}(L) = \mathbf{R}^S(L)\mathbf{R}^A(L) \approx \mathbf{R}^S(L) \otimes \mathbf{R}^A(L).$$

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We selected a basis $x_1, \cdots, x_m$ of T such that $[x_i, x_j]$ is a linear combination of x_k's with $k < \min(i, j)$, and we completed this to a basis $x_1, \cdots, x_n$ of A. The elements of $U(L)$ can then be written uniquely as sums of products $ux_n^{e_n} \cdots x_1^{e_1}$, with $u \in U(S)$. Let u_0 denote the component of u in F, in the standard decomposition $U(S) = F + SU(S)$. We defined a linear function g_i on $U(L)$ such that $g_i(ux_i) = u_0$, while g_i vanishes on all the other products of the above form. Then $g_{m+1}, \cdots, g_n$ span the space of the *elementary functions*, i.e., the functions vanishing on F and on $L^2U(L)$. Let F^* denote the algebraic closure of F. The *trigonometric functions* were defined as those F^*-linear combinations of the unitary homomorphisms of $U(L)$ into F^* which take their values in F. The unitary homomorphisms of $U(L)$ into F^* are actually the exponentials of the F^*-linear combinations of the elementary functions. The trigonometric functions constitute a subalgebra C of $\mathbf{R}(L)$. We have shown that the functions $g_1, \cdots, g_n$ are algebraically independent over C, and that $\mathbf{R}^A(L) = C[g_1, \cdots, g_n]$. Thus $\mathbf{R}^A(L) = C \otimes V$, where V is the algebra generated by the constants and $g_1, \cdots, g_n$.

It is seen directly from the definitions that

$$g_1^{f_1} \cdots g_n^{f_n}(ux_n^{e_n} \cdots x_1^{e_1}) = \delta_{e_1f_1} \cdots \delta_{e_nf_n} e_1! \cdots e_n! \, u_0,$$

where the δ's are the Kronecker symbols. Hence we see that a representative function belongs to V if and only if it vanishes on $SU(L)$ and on some $U(A)^{[q]}$, where the last is defined as in Lemma 1. We shall use this characterization in order to show that, with a suitable choice of the basis $x_1, \cdots, x_n$, the algebra V is stable under the left translations.

Let A^S denote the Lie algebra consisting of all $x \in A$ for which $[S, x] = (0)$. Since A is semisimple as an S-module, A is the sum of A^S and $[S, A]$. Since $[S, A] \subset T$, we have therefore $A = A^S + T$. For $x \in A^S$, denote by Z_x the subalgebra of A^S consisting of all elements of A^S that are annihilated by a power of the inner derivation effected by x. If we choose x so that Z_x is of the smallest possible dimension, then, by the elementary theory of Cartan subalgebras of a Lie algebra, Z_x is a nilpotent Lie algebra. By Fitting's Lemma, we have $A^S = Z_x + B$, where B is a subspace such that $[x, B] = B$. Hence $B \subset T$, so that $A = Z_x + T$. Thus we conclude that there is a nilpotent subalgebra P of A such that $[S, P] = (0)$ and $A = P + T$.

Now we choose our basis of A such that $x_{m+1}, \cdots, x_n$ lie in P. It is clear from our above characterization of V that, in order to conclude that V is stable under the left translations, it suffices to prove the following statement: given $x \in L$ and q, there is an r such that $U(A)^{[r]}x \subset U(S)U(A)^{[q]}$.

If we apply Lemma 1 to T we see immediately that this statement is true for every $x \in T$. Now let $x \in S$. Then we have

$$x_n^{e_n} \cdots x_1^{e_1}x = xx_n^{e_n} \cdots x_1^{e_1} + x_n^{e_n} \cdots x_{m+1}^{e_{m+1}}[x_m^{e_m} \cdots x_1^{e_1}, x].$$

Let $a = e_1 + \cdots + e_m$ and $b = e_{m+1} + \cdots + e_n$. Let a' be the largest exponent k such that $T^aU(T) \subset U(T)^{[k]}$. Then the second term on the right of the above equation belongs to $U(A)^{[b+a']}$. By Lemma 1, applied to T, a' becomes arbitrarily large with a. Hence it is clear that our statement is true also for every $x \in S$.

There remains only to verify our statement for $x \in P$. We have

$$x_n^{e_n} \cdots x_1^{e_1} x = x_n^{e_n} \cdots x_{m+1}^{e_{m+1}} x x_m^{e_m} \cdots x_1^{e_1} + x_n^{e_n} \cdots x_{m+1}^{e_{m+1}} [x_m^{e_m} \cdots x_1^{e_1}, x].$$

As in the last case, the second term on the right belongs to $U(A)^{[b+a']}$. The first term belongs to $P^{b+1}T^a$. For every c, let c^* denote the largest k such that $P^c U(P) \subset U(P)^{[k]}$. By Lemma 1, applied to P, c^* becomes arbitrarily large with c. Clearly, every element of $U(P)^{[q]}$ is a sum of products $u_s v_s$, where u_s is an ordered monomial of degree s in $x_n, \cdots, x_{m+1}$, and $v_s \in T^{q-s}U(T)$. We have

$$P^{b+1}T^a \subset U(P)^{[(b+1)^*]}T^a,$$

which is contained in the sum of the spaces $u_s T^{(b+1)^*+a-s}U(T)$. Hence we see that the first term on the right of the above equation belongs to the sum of the spaces $U(A)^{[s+((b+1)^*+a-s)']}$. For each s, this exponent is at least equal to the maximum of a and $((b+1)^*)'$. Hence it is clear that the statement we set out to prove is true also in the case where $x \in P$. Thus we may now conclude that V is stable under the left translations.

Since $\mathbf{R}^S(L)$ is stable under the left (and right) translations, the subalgebra $\mathbf{R}^S(L)V$ of $\mathbf{R}(L)$ is stable under the left translations. We have $\mathbf{R}(L) = C \otimes (\mathbf{R}^S(L)V)$. Let $u \to u^*$ denote the anti-automorphism of $U(L)$ such that $x^* = -x$, for every $x \in L$. This induces an involution $f \to f^*$ of $\mathbf{R}(L)$, where $f^*(u) = f(u^*)$. We have $f^* \cdot u = (u^* \cdot f)^*$. Clearly, the space of the elementary functions, the algebra C of the trigonometric functions, and $\mathbf{R}^S(L)$ are stable under this involution. Hence we have $\mathbf{R}(L) = C \otimes B$, where $B = \mathbf{R}^S(L)V^*$, so that B is finitely generated and stable under the right translations. Moreover, B contains the constants and the elementary functions. Thus we have the following result.

THEOREM 1. *Let L be a finite-dimensional Lie algebra over a field of characteristic 0. Then $\mathbf{R}(L)$ is a tensor product $C \otimes B$, where C is the algebra of the trigonometric functions, and B is a finitely generated algebra which contains the constants and the elementary functions, and which is stable under the right translations.*

Let δ be any differentiation of B into F. Since the elementary functions are determined by their restrictions to L, we can find an element $x \in L$ such that $\delta(f) = f(x)$, for every elementary function f. Let δ_1 denote the differentiation $h \to h(x)$ of C into F. Clearly, there is a differentiation of $\mathbf{R}(L)$ into F that coincides with δ on B and with δ_1 on C.

Now let D be any derivation of B into $\mathbf{R}(L)$ that annihilates the constants and commutes with the right translations. Let δ be the differentiation of B into F defined by $\delta(f) = D(f)(1)$, for every $f \in B$. Extend δ to a differentiation of $\mathbf{R}(L)$ into F by the method just explained. Denoting this extended differentiation still by δ, let D_1 be the proper derivation of $\mathbf{R}(L)$ defined by $D_1(f)(u) = \delta(f \cdot u)$, for every $f \in \mathbf{R}(L)$ and every $u \in U(L)$ (cf. [2, Proposition 1]). Then D_1 coincides with D on B. Moreover, our construction of D_1 amounted simply to enforcing the condition of [2, Theorem 6]. Hence it follows from that theorem that D_1 is the left translation by an element of L. Thus we have shown that every derivation of B into $\mathbf{R}(L)$ that annihilates the constants and commutes with the right translations is the restriction to B of the left translation by an element of L.

Let H denote the Lie algebra of all proper derivations of $\mathbf{R}(L)$, and let t denote the canonical monomorphism of L into H. We had already seen in [2, Section 6] that $t(L)$ is an ideal in H, and that $H/t(L)$ is abelian. Let J denote the subalgebra of all elements of H that annihilate B. Then our above extension result shows that $H = J + t(L)$, and this sum is evidently semidirect. Hence we have the following result.

THEOREM 2. *Let L be a finite-dimensional Lie algebra over a field of characteristic* 0. *Then the Lie algebra H of all proper derivations of* $\mathbf{R}(L)$ *is a semidirect sum $J + t(L)$, where $t(L)$ is the canonical image of L in H, which is an ideal of H, and J is an abelian Lie algebra which is isomorphic, by restriction, with the Lie algebra of all proper derivations of the algebra of the trigonometric functions.*

Next, we give a characterization, in terms of the structure of H, of the Lie algebras L that are faithfully representable as algebraic Lie algebras.

THEOREM 3. *The Lie algebra L has a faithful representation as an algebraic Lie algebra if and only if every proper derivation of* $\mathbf{R}(L)$ *can be written as the sum of a left translation and a proper derivation that commutes with every left translation.*

Proof. Let M be a finite-dimensional subspace of $\mathbf{R}(L)$ such that M is stable under the right and left translations and that the representation σ of L by left translations on M is faithful. Then M coincides with the space $\mathbf{R}(\sigma)$ of all representative functions associated with σ. Let M^* be the algebra generated by the constants and the elements of M. As we have shown in [2, Section 3], the algebraic Lie algebra hull of $\sigma(L)$ in the algebra $E(M)$ of all linear endomorphisms of M consists of the restrictions to M of the proper derivations of M^*. By [2, Theorem 4], every proper derivation of M^* extends to a proper derivation of $\mathbf{R}(L)$. Hence we conclude that the restriction image H_M of H in $E(M)$ is the algebraic Lie algebra hull of $\sigma(L)$.

Now let α denote the adjoint representation of L by derivations of L. We identify L with $\sigma(L)$. Accordingly, $\alpha(L)$ becomes identified with the image of $\sigma(L)$ in the algebra $E(\sigma(L))$ of all linear endomorphisms of $\sigma(L)$, i.e., for $x \in L$, $\alpha(x)$ is identified with the derivation $y \rightarrow \sigma(x)y - y\sigma(x)$ of $\sigma(L)$. Since H_M is the algebraic Lie algebra hull of $\sigma(L)$, $\sigma(L)$ is stable also under the commutations with the elements of H_M. Thus we have a representation of H_M in $E(\sigma(L))$. Since this is a subrepresentation of the adjoint representation of the algebraic Lie algebra H_M, it is the differential of a rational group representation. Hence we conclude that the image of H_M in $E(\sigma(L))$ is the algebraic Lie algebra hull of $\alpha(L)$.

Now suppose that L has a faithful representation as an algebraic Lie algebra. Then $\alpha(L)$ is an algebraic Lie algebra, so that $\alpha(L)$ coincides with the image of H_M in $E(\sigma(L))$. This means that, for every $D \in H$, there is an element $x \in L$ such that $(D - t(x))_M$ commutes with every left translation on M. Since $t(L)$ is an ideal in H, and since σ is faithful, this implies that $D - t(x)$ commutes with every left translation on $\mathbf{R}(L)$. Thus we have shown that the condition of Theorem 3 is necessary.

Conversely, suppose that the condition of Theorem 3 is satisfied. Then the image of H_M in $E(\sigma(L))$ coincides with $\alpha(L)$, and we conclude that $\alpha(L)$ is an algebraic Lie algebra. By [1, p. 156, Proposition 3], this implies that L has a faithful representation ρ such that $\rho(L)$ is an algebraic Lie algebra. This completes the proof of Theorem 3.

We have shown in [2, Section 7] that, if the radical A of L is nilpotent, then $\mathbf{R}(L) = C \otimes N$, where N is the algebra of all representative functions associated with representations that are nilpotent on A, and that, consequently, the Lie algebra H of all proper derivations is the direct sum $J + t(L)$, where J consists of the proper derivations annihilating N. Now we shall show that, conversely, if the algebra B of Theorem 1 can be chosen so as to be stable under the left translations as well as under the right translations, then the radical A of L is nilpotent.

Let F^* denote the algebraic closure of F, and consider the Lie algebra $L \otimes_F F^*$ over F^*. Evidently, $\mathbf{R}(L \otimes_F F^*) = \mathbf{R}(L) \otimes_F F^*$. Moreover, the algebra of the trigonometric functions for $L \otimes_F F^*$ is $C \otimes_F F^*$. Hence it is clear that we may assume, without loss of generality, that F is algebraically closed.

Now let M be a finite-dimensional subspace of $\mathbf{R}(L)$ such that M is stable under the right and left translations and that the constants and the elements of M generate B, where $\mathbf{R}(L) = C \otimes B$. Consider the representation σ of L by left translations on M. Our proof of Theorem 2 has shown that every proper derivation of B is a left translation by an element of L. Since B is the algebra generated by the constants and the representative functions associated with σ, we conclude from [2, Theorem 2] that $\sigma(L)$ is an algebraic Lie algebra. Since σ is faithful, we may identify L with $\sigma(L)$. Now it follows from [1, p. 144, Proposition 5] that L is a semidirect sum $Q + K$, where K is the kernel of the semisimple representation σ' associated with σ (defined as the representation in the direct sum of the factor spaces of a composition series for M with respect to L), and Q is a reductive subalgebra of L such that the representation of Q on M is semisimple. If P is the center of Q, we have therefore

$A = P + K$, and M is semisimple as a P-module. It suffices to show that $P = (0)$.

Let V denote the representation space for σ'. For every linear function γ on P, denote by V_γ the subspace of all $v \in V$ such that $p \cdot v = \gamma(p)v$, for all $p \in P$. Then, assuming F algebraically closed, V is a direct sum of V_γ's, and each V_γ is L-stable. Let W_γ denote the highest nonvanishing homogeneous component of the exterior algebra built over V_γ. Then W_γ is a 1-dimensional representation space for L and defines a representative function f_γ such that, for every $u \in U(L)$, the action of u on W_γ is the scalar multiplication by $f_\gamma(u)$. It is clear from this construction that f_γ belongs to the algebra generated by the representative functions associated with σ' and the constants. Since the representative functions associated with σ' are associated also with σ, it follows that $f_\gamma \in B$. On the other hand, f_γ is a unitary homomorphism of $U(L)$ into F, so that $f_\gamma \in C$. Since $C \cap B$ consists only of the constants, we conclude that f_γ is a constant. Thus we have $\gamma(p) = f_\gamma(p) = 0$, for every $p \in P$, so that $\gamma = 0$. Hence $\sigma'(P) = (0)$, which means that $P \subset K$, whence $P = (0)$. This completes the proof of our assertion.

We shall strengthen this result by examining the subalgebras S of $\mathbf{R}(L)$ such that $\mathbf{R}(L) = C \otimes S$. We know that there is a finitely generated algebra B such that the elements of B and C generate $\mathbf{R}(L)$. Let $b_1, \cdots, b_n$ be a set of generators of B, and write $b_i = \sum_j c_{ij} s_{ij}$, with $c_{ij} \in C$ and $s_{ij} \in S$. Let S_0 denote the subalgebra of S that is generated by these elements s_{ij} and the constants. Then we have $\mathbf{R}(L) = C \otimes S_0$, whence $S = S_0$. Thus S is necessarily finitely generated.

Now suppose that S is stable under the right translations. We shall show that then every elementary function belongs to S. In order to prove this, it evidently suffices to show that every elementary function belongs to $S \otimes_F F^*$, where F^* is the algebraic closure of F. Since

$$\mathbf{R}(L \otimes_F F^*) = (C \otimes_F F^*) \otimes_{F^*} (S \otimes_F F^*),$$

we may therefore assume without loss of generality that F is algebraically closed. Then C is spanned over F by the multiplicative group Q of the unitary homomorphisms of $U(L)$ into F, and the elements of Q are linearly independent over F, so that every element of $\mathbf{R}(L)$ is a unique S-linear combination of the elements of Q. Let f be an elementary function, and write $f = \sum_{q \in Q} s_q q$, with $s_q \in S$. Let $x \in L$ and translate from the right by x. This gives

$$f(x) = \sum_{q \in Q} ((s_q \cdot x)q + s_q q(x)q).$$

Equating the coefficients of the elements of Q on the two sides of this equation, we obtain

$$0 = s_q \cdot x + s_q q(x), \qquad \text{for all } 1 \neq q \in Q \text{ and all } x \in L.$$

Thus $s_q \cdot x = -q(x)s_q = q^{-1}(x)s_q$. This generalizes inductively to $s_q \cdot u = q^{-1}(u)s_q$, for every $u \in U(L)$. Evaluating this at $1 \in U(L)$, we obtain

$s_q = s_q(1)q^{-1}$. If $s_q(1) \neq 0$, this gives $q^{-1} \in S$, which is impossible, because $q \neq 1$ and $S \cap C$ consists only of the constants. Hence we must have $s_q(1) = 0$, and so $s_q = 0$, for all $1 \neq q \in Q$. Hence $f = s_1 \in S$.

In particular, we may now conclude that every right stable subalgebra B of $\mathbf{R}(L)$ such that $\mathbf{R}(L) = C \otimes B$ is finitely generated and contains the elementary functions. With what we have shown above, this gives the following result.

THEOREM 4. *We have* $\mathbf{R}(L) = C \otimes B$, *where* C *is the algebra of the trigonometric functions and* B *is a two-sidedly stable subalgebra containing the constants, (if and) only if the radical of* L *is nilpotent.*

The following example shows that $t(L)$ may be a direct ideal summand of the Lie algebra of all proper derivations even when $\mathbf{R}(L)$ cannot be factored as in Theorem 4.

Let F be an algebraically closed field of characteristic 0, and let L be a 2-dimensional Lie algebra over F, with a basis x, y such that $[x, y] = y$. The space of the elementary functions on $U(L)$ is spanned by a single function b_2 such that $b_2(x) = 1$. If we write the elements of $U(L)$ as linear combinations of the ordered monomials $y^q x^p$, then b_2 is actually given by $b_2(y^q x^p) = \delta_{q0}\,\delta_{p1}$. The algebra C of the trigonometric functions is generated by the unitary homomorphisms $\exp(\alpha b_2)$ of $U(L)$ into F, where α ranges over F. The homomorphism $\exp(\alpha b_2)$ vanishes on $yU(L)$ and sends x onto α. We define the linear function b_1 on $U(L)$ by setting $b_1(y^q x^p) = \delta_{q1}\,\delta_{p0}$. Let B denote the algebra generated by b_1, b_2 and the constants. Then we have $\mathbf{R}(L) = C \otimes B$, and B is as in Theorem 1. In fact, the translates of b_1 and b_2 are as follows:

$$(\alpha x + \beta y)\cdot b_2 = \alpha = b_2\cdot(\alpha x + \beta y);$$

$$(\alpha x + \beta y)\cdot b_1 = \beta \exp(b_2); \qquad b_1\cdot(\alpha x + \beta y) = \alpha b_1 + \beta.$$

Let H be the Lie algebra of all proper derivations of $\mathbf{R}(L)$, and let $H = J + t(L)$ be the semidirect sum decomposition of Theorem 2. As an F-space, J is isomorphic with the space of all additive endomorphisms of F. In fact, this isomorphism sends each additive endomorphism γ of F onto the element γ^* of J, where $\gamma^*(B) = (0)$, and

$$\gamma^*(\exp(\alpha b_2)) = \gamma(\alpha)\exp(\alpha b_2).$$

Now let us define a proper derivation γ' of $\mathbf{R}(L)$ as follows:

$$\gamma'(b_1) = 0, \qquad \gamma'(b_2) = -\gamma(1), \qquad \gamma'(\exp(\alpha b_2)) = (\gamma(\alpha) - \alpha\gamma(1))\exp(\alpha b_2).$$

Then $\gamma' = \gamma^* - \gamma(1)t(x)$. Hence, if K denotes the space of all the γ', we still have $H = K + t(L)$. Moreover, it is easy to check that K lies in the center of H and that $K \cap t(L) = (0)$. Thus $t(L)$ is a direct ideal summand in H. On the other hand, L is solvable and not nilpotent, so that it follows from Theorem 4 that $\mathbf{R}(L)$ cannot be factored as described in Theorem 4.

In view of the importance of the trigonometric functions with regard to the structure of $\mathbf{R}(L)$, it is of interest to characterize them intrinsically within $\mathbf{R}(L)$. In the case of an algebraically closed base field, this will be accomplished if we give an intrinsic characterization of the scalar multiples of the unitary homomorphisms of $U(L)$ into the base field. Hence the following result essentially answers our question.

THEOREM 5. *Let L be a finite-dimensional Lie algebra over a field F of characteristic 0. Then the set of the nonzero scalar multiples of the unitary homomorphisms of $U(L)$ into F coincides with the set of the units of the ring $\mathbf{R}(L)$.*

Proof. Evidently, an element f of $\mathbf{R}(L)$ is a nonzero multiple of a unitary homomorphism of $U(L)$ into F if and only if the space spanned by the left translates of f, which we denote $U(L)\cdot f$, is 1-dimensional. If $f \to f^*$ denotes the involution of $\mathbf{R}(L)$ defined just before stating Theorem 1, we have $ff^* = 1$, whenever f is a unitary homomorphism of $U(L)$ into F. Hence every nonzero scalar multiple of a unitary homomorphism of $U(L)$ into F is a unit of $\mathbf{R}(L)$. It suffices, therefore, to prove that if f is a unit of $\mathbf{R}(L)$ then $U(L)\cdot f$ is 1-dimensional.

First, let us consider the case where L is semisimple. In that case, the result to be proved amounts to saying that every unit of $\mathbf{R}(L)$ is a constant, because $[L, L] = L$. In proving this, we may evidently assume, without loss of generality, that F is algebraically closed. Then we can use the theory of weights for the representations of L, and we prove the result by adapting an argument due to B. Kostant (used by him in a forthcoming paper on the cohomology of homogeneous spaces). Choose a Cartan subalgebra C of L, and order the weights, with respect to C, of the representations of L in the usual way. This ordering is compatible with the addition of weights and is such that the highest weight of any nontrivial representation of L is greater than 0. Let f be a unit of $\mathbf{R}(L)$, so that there is an element $g \in \mathbf{R}(L)$ such that $fg = 1$. Let α be the highest weight occurring in the representation of L on $U(L)\cdot f$, and let β be the highest weight occurring in the representation of L on $U(L)\cdot g$. It suffices to show that $\alpha = 0$, because this means that the representation of L on $U(L)\cdot f$ is trivial.

We know (cf. [2, Section 5]) that $\mathbf{R}(L)$ is finitely generated. Hence there is a finite-dimensional two-sidedly stable subspace of $\mathbf{R}(L)$ whose elements generate $\mathbf{R}(L)$. We may identify L with its image under the representation by left translations on this subspace. Then L is an algebraic Lie algebra, because it is semisimple. Let G be the irreducible algebraic group whose Lie algebra is L. By [2, Theorem 3], every representation of L is the differential of a rational representation of G. In this way, G acts as a group of algebra automorphisms on $\mathbf{R}(L)$; see [2, Section 4].

We extend the base field from F to the field F^* (say) of the rational functions on G. Then the canonical extension of G over F^* contains a generic point t of G. If we extend the base field from F to F^* in a representation of

L, the weights for the extended representation (with respect to the extended Cartan subalgebra) are the canonical extensions of the weights of the original representation. Let ρ be the rational representation of G on $U(L)\cdot f$ whose differential $\rho^{\bullet}$ is the natural representation of L on $U(L)\cdot f$. By a standard result of the representation theory of algebraic groups, the algebra generated by the elements of $\rho^{\bullet}(L)$ is contained in the algebra generated by the elements of $\rho(G)$. It follows that every point of $U(L)\cdot f$ is a linear combination of specializations of $t(f)$. Hence it is clear that $t(f)$ must have a nonzero component in each weight subspace of the extension of $U(L)\cdot f$ over F^*. Similarly, $t(g)$ must have a nonzero component in each weight subspace of the extension of $U(L)\cdot g$. We still have $t(f)t(g) = 1$. Now let $t(f)_\rho$ denote the component of $t(f)$ in the weight space for the weight ρ, etc. We consider the canonical extension over F^* of the representation of L in the product space $(U(L)\cdot f)(U(L)\cdot g)$. The product $t(f)_\rho t(g)_\sigma$ evidently lies in the weight subspace of this space whose weight is $\rho + \sigma$. Now we have

$$\textstyle\sum_{\rho,\sigma} t(f)_\rho t(g)_\sigma = t(f)t(g) = 1.$$

If α and β are used also to denote the canonical extensions of the original weights α and β, the component of the highest weight on the left is $t(f)_\alpha t(g)_\beta \neq 0$. The right-hand side, 1, is in the component of weight 0. Hence we conclude that $\alpha + \beta = 0$. Since α and β are highest weights of representations of L, this implies that α and β are both 0. Thus we have shown that, if L is semisimple, every unit of $\mathbf{R}(L)$ is a constant.

Now let us consider the general case. We have $L = S + A$, where A is the radical of L, and S is a maximal semisimple subalgebra. In the notation we explained after Lemma 1, we have $\mathbf{R}(L) = C\mathbf{R}^S(L)[g_1, \cdots, g_n]$, and the set $g_1, \cdots, g_n$ is algebraically independent over the field of quotient of $C\mathbf{R}^S(L)$. Hence it is clear that every unit of $\mathbf{R}(L)$ must actually be a unit of $C\mathbf{R}^S(L)$.

Evidently, we may assume again that F is algebraically closed. Then the algebra C of the trigonometric functions consists of the linear combinations of the exponentials of the elementary functions. Now let f be a unit of $C\mathbf{R}^S(L)$, and take $g \in C\mathbf{R}^S(L)$ such that $fg = 1$. We can express f and g as finite $\mathbf{R}^S(L)$-linear combinations of functions $\exp(h)$, where h ranges over a finite set of elementary functions. Let $h_1, \cdots, h_p$ be a free basis for the additive group generated by this finite set of elementary functions, and put $y_i = \exp(h_i)$. Then the monomials in the y_i's, with negative exponents allowed, are easily seen to be linearly independent over F. Since $C\mathbf{R}^S(L) = C \otimes \mathbf{R}^S(L)$, these monomials are therefore linearly independent over $\mathbf{R}^S(L)$. The functions f and g can be written in the form

$$f = y_1^{a_1} \cdots y_p^{a_p} f_0(y_1, \cdots, y_p); \qquad g = y_1^{b_1} \cdots y_p^{b_p} g_0(y_1, \cdots, y_p),$$

where the a_i and the b_i are integers, and f_0, g_0 are polynomials with coefficients in $\mathbf{R}^S(L)$, such that $f_0(0, \cdots, 0) \neq 0$, and $g_0(0, \cdots, 0) \neq 0$. Since $fg = 1$, it follows from the independence of the monomials in the y_i's that we must have $f_0(y_1, \cdots, y_p) = f_0(0, \cdots, 0)$, and $f_0(0, \cdots, 0)g_0(0, \cdots, 0) = 1$. Thus $f_0(y_1, \cdots, y_p)$ is actually a unit of $\mathbf{R}^S(L)$. Since $\mathbf{R}^S(L)$ is isomorphic with $\mathbf{R}(S)$, it follows from the semisimple case that $f_0(y_1, \cdots, y_p)$ is actually a constant. Thus f is a scalar multiple of a monomial in the y_i's. Since the y_i's are unitary homomorphisms of $U(L)$ into F, this shows that f is a scalar multiple of a unitary homomorphism of $U(L)$ into F. The proof of Theorem 5 is now complete.

Observe that Theorem 5 immediately implies the analogous results for connected Lie groups and for irreducible algebraic groups: let G be a connected real or complex Lie group, or an algebraic linear group over a field of characteristic 0. Let $\mathbf{R}(G)$ denote the algebra of all analytic or rational, respectively, representative functions on G. Then *the units of* $\mathbf{R}(G)$ *are precisely the nonzero scalar multiples of the analytic or rational, respectively, homomorphisms of G into the multiplicative group of the base field.* This result follows immediately by considering the canonical monomorphism of $\mathbf{R}(G)$ into $\mathbf{R}(L)$, where L is the Lie algebra of G, and applying Theorem 5.

References

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