A GENERAL ERGODIC THEOREM

BY

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The purpose of this paper is to prove Theorem 1 which was conjectured by E. Hopf [4, p. 39].

The first operator theoretic generalization of the Birkhoff ergodic theorem was given by Doob [1] who proved that $\sum_{k=0}^{n} T^k f/n$ converges pointwise, where T is a Markov operator for which there is an invariant measure and f is the characteristic function of a set. It was noted by Kakutani [5] that Doob's method was applicable to give the same result for f merely a bounded function, and Doob [2] later applied the same method to obtain this convergence result for f in L_p, and more generally for $|f| \log^+ |f|$ in L_1 (relative to the invariant measure). Hopf [4] then proved the theorem assuming merely that f is integrable. Dunford & Schwartz [3] extended Hopf's result by dropping the assumption of positivity for T. More precisely, they dropped the restriction of assuming the operator to be a Markov operator and proved the theorem for an operator which does not increase the L_1 and L_∞ norms. Tsurumi [6] proved a result with the same conclusion as ours but with much stronger and somewhat complicated hypothesis. In our paper we drop the assumption that T does not increase the L_∞ norm, consider positive operators which do not increase the L_1 norm, and prove a theorem about ratios of sums of transforms of two functions. Our result implies the Hopf theorem. The Dunford and Schwartz theorem, although not implied by our theorem, can be obtained by modifying the method which we use in this paper.

THEOREM 1. *Let T be a positive linear operator on L_1 of a positive measure space $(S, \mathcal{G}, \mu)$, and let T have L_1 norm less than or equal to one. Then if f and p are functions in L_1, and if p is nonnegative, the limit*

$$\lim_{n\to\infty} \sum_{k=0}^{n} T^k f / \sum_{k=0}^{n} T^k p$$

exists and is finite almost everywhere on the set $A = \{s : T^k p > 0 \text{ for some } k \geqq 0\}$.

Let $D_n(f, p) = \sum_{k=0}^{n} T^k f / \sum_{k=0}^{n} T^k p$ and suppose in what follows that T satisfies the conditions stated in the theorem. All given functions in what follows will be supposed to be in L_1, and all functions which are constructed will be in L_1 by construction (this will be obvious).

LEMMA 1. *If $f = f^+ - f^-$, and if* $\sup \sum_{k=0}^{n} T^k f > 0$ *on a set B, then there exist sequences $\{d_k\}$ and $\{f_k\}$ of nonnegative functions such that*

(i) $\sum_{k=0}^{N} \int d_k + \int f_N \leqq \int f^+$,

(ii) $\sum_{k=0}^{\infty} d_k = f^-$ *on* B,

(iii) $T^N f^+ = \sum_{k=0}^{N} T^{N-k} d_k + f_N$.

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Proof. Define inductively

$$(1.1)\qquad \begin{aligned} d_0 = 0, \qquad f_0 &= f^+, \\ f_{i+1} = (Tf_i - f^- + d_0 + \cdots + d_i)^+, \qquad d_{i+1} &= Tf_i - f_{i+1}\,. \end{aligned}$$

First note that

$$(1.2)\qquad -f^- + d_0 + \cdots + d_i \leqq 0,$$

and that there is equality on the set where $f_i > 0$ since

$$\begin{aligned} f_i &= (Tf_{i-1} - f^- + d_0 + \cdots + d_{i-1})^+ \\ &= (Tf_{i-1} - f_i - f^- + d_0 + \cdots + d_{i-1} + f_i)^+ \\ &= (d_i - f^- + d_0 + \cdots + d_{i-1} + f_i)^+. \end{aligned}$$

It follows easily from (1.1) that

$$(1.3)\qquad T^j f^+ = \sum_{k=0}^{j} T^{j-k}\, d_k + f_j\,.$$

Note further that f_i is nonnegative by definition, and that so is d_i, by the last two equations of (1.1) and (1.2). From (1.3) we see that

$$(1.4)\qquad \sum_{j=0}^{n} T^j f^+ = \sum_{j=0}^{n} \sum_{k=0}^{j} T^{j-k} d_k + \sum_{j=0}^{n} f_j\,.$$

Now we prove

$$(1.5)\qquad \sum_{j=0}^{n} T^j f^+ \leqq \sum_{j=0}^{n} d_j + \sum_{j=1}^{n} T^j f^- + \sum_{j=0}^{n} f_j\,.$$

To see that (1.5) holds, note that

$$\sum_{j=0}^{n} \sum_{k=0}^{j} T^{j-k} d_k = \sum_{j=0}^{n} T^j\Big(\sum_{j=0}^{n-j} d_k\Big),$$

and that

$$(1.6)\qquad T^j f^- \geqq T^j\Big(\sum_{k=0}^{n-j} d_k\Big) \qquad \text{for} \quad 1 \leqq j \leqq n,$$

where (1.6) follows from (1.2).

Rewriting (1.5) we have

$$(1.7)\qquad \sum_{j=0}^{n} T^j(f^+ - f^-) \leqq \sum_{j=0}^{n} (d_j + f_j) - f^-.$$

We will now prove that

$$(1.8)\qquad \sum_{j=0}^{\infty} d_j - f^- \geqq 0 \qquad \text{almost everywhere on } B.$$

It is clear from the remark after (1.2) that (1.8) holds with equality almost everywhere on the set $C = \{s\colon f_i > 0 \text{ for some } i \geqq 0\}$. It remains to show that (1.8) holds in the set $B - C$. This follows on noting that (1.7) implies that on B (and on $B - C$ in particular)

$$\sum_{j=0}^{\infty}(d_j + f_j) - f^- \geqq 0.$$

Now we note that the condition (iii) is exactly (1.3) and that condition (ii) follows from formulas (1.2) and (1.8). To verify that (i) holds for the sequences defined in (1.1), note that we have

$$\int \left(\sum_{j=0}^{j} d_k + f_j\right) \geqq \int \left(\sum_{k=0}^{j} d_k + Tf_j\right) = \int \left(\sum_{k=0}^{j+1} d_k + f_{j+1}\right).$$

Now by induction on j we get (i).

LEMMA 2. *If* $p \geqq 0$, *then*

$$\lim_{n\to\infty} T^{n+j}p \Big/ \textstyle\sum_{k=0}^{n-1} T^k p = 0$$

for each fixed $j \geqq 0$ *almost everywhere on the set* $\{s : p > 0\}$.

Proof. Assume the contrary. Then there are a set D of nonzero measure and a constant $a > 0$ such that $T^{n+j}p > a\sum_{i=0}^{n-1} T^i p$ for infinitely many n at each point of D. Set

$$E_n = \{s : T^{n+j}p > a \textstyle\sum_{i=0}^{n-1} T^i p\}$$

$$e_n = \int_{E_n} [T^{n+j}p] - \left[a \sum_{i=0}^{n-1} \int_{E_n} T^i p\right].$$

We will now prove

$$e_n - e_{n+1} \geqq a \int_{E_{n+1}} p; \tag{2.1}$$

n will be fixed through the next paragraph.

Set

$$(T^i p)_1 = \text{ch } E_n\, T^i p, \qquad (T^i p)_2 = T^i p - (T^i p)_1$$

(where ch E_n means the characteristic function of E_n). Since the function $r = [(T^{n+j}p)_1] - [a\sum_{i=0}^{n-1} (T^i p)_1]$ is nonnegative and has integral equal to e_n, it follows that its transform Tr has integral over any set less than or equal to e_n. In particular,

$$\int_{E_{n+1}} Tr = \int_{E_{n+1}} [T(T^{n+j}p)_1] - \left[\sum_{i=0}^{n-1} \int_{E_{n+1}} T(T^i p)_1\right] \leqq e_n . \tag{2.2}$$

Similarly, since the function $w = [(T^{n+j}p)_2] - [a\sum_{i=0}^{n-1} (T^i p)_2] \leqq 0$, it follows that Tw has integral over any set less than or equal to zero. In particular

$$\int_{E_{n+1}} Tw = \int_{E_{n+1}} [T(T^{n+j}p)_2] - \left[a \sum_{i=0}^{n-1} \int_{E_{n+1}} T(T^i p)_2\right] \leqq 0. \tag{2.3}$$

Combining (2.2) and (2.3) we get

$$(2.4)\qquad \begin{aligned} e_{n+1} &= \int_{E_{n+1}} [T^{n+1+j}p] - \left[a \sum_{i=0}^{n} \int_{E_{n+1}} T^i p\right] \\ &= \int_{E_{n+1}} T(r+w) - ap \leqq e_n - a \int_{E_{n+1}} p. \end{aligned}$$

From (2.4) follows (2.1). In order to obtain a contradiction we note that $\sum_{n=0}^{\infty} \int_{E_n} p$ diverges, because every point of D is covered infinitely often by the E_n. This yields a contradiction since it follows from (2.1) that $\sum_{n=0}^{\infty} \int_{E_n} p$ converges.

LEMMA 3. *For $p \geqq 0$ and any g we have* $\sup D_n(g, p)$ *is finite almost everywhere on the set where* $\sum_{n=0}^{\infty} T^n p > 0$.

Proof. It is sufficient to prove the lemma under the hypothesis that $g \geqq 0$, and only to prove finiteness of the indicated supremum where $p > 0$. The first remark is obvious, and the second follows since we would also have then that $\sup D_n(T^k g, T^k p)$ is finite almost everywhere on the set where $T^k p > 0$, and this implies that $\sup D_n(g, p)$ is finite almost everywhere on the set where $T^k p > 0$.

Assume the contrary. Then the supremum of $D_n(g, p)$ is infinite almost everywhere on a set E of positive measure, and $p > b > 0$ on the set E for some positive constant b. We have, therefore, that for any positive constant a

$$\sup \textstyle\sum_{j=0}^{n} T^j((g - ap)^+ - (g - ap)^-) > 0$$

almost everywhere on E. Applying Lemma 1 with $f = g - ap$ we get from (i) and (ii) of Lemma 1 that

$$\int (g - ap)^+ \geqq \int_E \sum_{k=0}^{\infty} d_k = \int_E (g - ap)^-.$$

However, the quantity on the extreme right tends to infinity as a tends to infinity, and the one on the extreme left to zero, giving rise to a contradiction.

LEMMA 4. *If on a set F of finite measure, where F is contained in the set where $\sum_{k=0}^{\infty} T^k p = \infty$, we have that* $\sup D_n(f, p) > a$, *then given $\varepsilon > 0$, there exists a function g such that if we let $h = ap - (f - ap)^- + g$, then*

(i) *the limits superior and inferior of $D_n(f, p)$ are, respectively, the same as those of $D_n(h, p)$, almost everywhere on F,*

(ii) $\int (f - ap)^+ - \int (h - ap)^+ \geqq \int_F (f - ap)^- - \varepsilon$,

(iii) $(h - ap)^- \leqq \varepsilon$ *on* $F - F'$ *where* $\mu(F') < \varepsilon$,

(iv) $(h - ap)^- \leqq (f - ap)^-$.

Proof. The fact that the supremum of $D_n(f, p)$ is greater than the number a almost everywhere implies that the supremum of

$$\textstyle\sum_{k=0}^{n} T^k(f - ap)$$

is positive almost everywhere on F. We now apply Lemma 1 with f replaced by $f - ap$. We will let $g = \sum_{k=0}^{N} d_k + f_N$ for a sufficiently large N. We have by (ii) of Lemma 1 that

$$\int_F (f - ap)^- = \int_F \sum_{k=0}^{\infty} d_k ,$$

and hence if we choose N sufficiently large,

$$\int_F \sum_{k=0}^{N} d_k \geqq \int_F (f - ap)^- - \varepsilon. \tag{4.1}$$

Note that

$$\int (g - (f - ap)^-)^+ \leqq \int \left(g - \operatorname{ch} F \sum_{k=0}^{N} d_k\right) \tag{4.2}$$

since $\sum_{k=0}^{N} d_k - (f - ap)^- \leqq 0$ on F, by (ii) of Lemma 1. Now applying the definition of h we have that $(h - ap)^+ = (g - (f - ap)^-)^+$, and hence, by (4.2),

$$\int (f - ap)^+ - \int (h - ap)^+ \geqq \int (f - ap)^+ - \int g + \int_F \sum_{k=0}^{N} d_k . \tag{4.3}$$

Applying (i) of Lemma 1 with (4.1) gives that the N chosen is large enough to satisfy (ii). If N is taken large enough, we can also insure that $(\sum_{k=0}^{N} d_k - (f - ap)^-)^- \leqq \varepsilon$ on $F - F'$, where F' is as in (iii). Condition (iv) follows because $(g - (f - ap)^-)^- \leqq (f - ap)^-$ since $g \geqq 0$ and $(f - ap)^- \geqq 0$.

Note that for any function $q \geqq 0$

$$\lim_{n\to\infty} [D_n(q, q) - D_n(T^k q, q)] = 0 = [1 - \lim_{n\to\infty} D_n(T^k q, q)] \tag{4.4}$$

almost everywhere in the set where $\sum_{k=0}^{\infty} T^k q = \infty$ since the difference on the left side of (4.4) is equal to

$$(q + \cdots + T^{k-1}q - T^{n+1}q - \cdots - T^{k+n}q)/\textstyle\sum_{k=0}^{n} T^k q,$$

which tends to zero by Lemma 2 and the fact that the denominator tends to infinity. To see that (i) is satisfied we would like to show

$$\lim_{n\to\infty} [D_n(q, p) - D_n(T_k\, q, p)] = 0 \tag{4.5}$$

on the set G where $\sum_{k=0}^{\infty} T^k p = \infty$.

Now $D_n(T^k q, p)/D_n(q, p) = D_n(T^k q, q)$ which tends to 1 as n tends to infinity, by (4.4), on the set H where $\sum_{k=0}^{\infty} T^k q = \infty$. Since the supremum of $D_n(q, p)$ is finite almost everywhere on G by Lemma 3, we have that (4.5) holds almost everywhere on $H \cap G$. But on $G - H$, $\sum_{k=0}^{\infty} T^k p = \infty$ and $\sum_{k=0}^{\infty} T^k q < \infty$, and thus each of the terms in (4.5) tends to zero separately.

This establishes (4.5). That (i) holds follows from the fact that we can write the difference $D_n(h, p) - D_n(f, p)$ as the sum of terms exhibited in (4.5), i.e.,

$$D_n(f, p) - D_n(h, p) = D_n(f, p) - D_n(T^N f, p) + D_n(T^N f, p) - D_n(h, p)$$

and

$$\begin{aligned} D_n(T^N f, p) &- D_n(h, p) \\ &= D_n(T^N(ap - (f - ap)^-), p) - D_n(ap - (f - ap)^-, p) \\ &\quad + D_n(f_N + Td_{N-1} + \cdots + T^{N-1}d_1, p) \\ &\quad - D_n(f_N + d_{N-1} + \cdots + d_1, p). \end{aligned}$$

LEMMA 5. *If on a set J of finite measure, where J is contained in the set where $\sum_{k=0}^{\infty} T^k p = \infty$, we have that* $\inf D_n(f, p) < a$, *then given $\varepsilon > 0$ there exists a function g such that if we let $h = ap + (f - ap)^+ - g$, then*

(i) *the limits superior and inferior of $D_n(f, p)$ are, respectively, the same as those of $D_n(h, p)$, almost everywhere on J,*

(ii) $\int (f - ap)^- - \int (h - ap)^- \geqq \int_J (f_J - ap)^+ - \varepsilon$,

(iii) $(h - ap)^+ \leqq \varepsilon$ *on* $J - J'$ *where* $\mu(J') < \varepsilon$,

(iv) $(h - ap)^+ \leqq (f - ap)^+$.

The proof is the same as that of Lemma 4.

Proof of Theorem 1. We note that we may suppose without loss of generality that f is nonnegative. Note further that if the assertion of the theorem is proved on the set where $p > 0$, we can deduce from it the complete theorem, for if the limit of $D_n(f, p)$ did not exist on the set K where $T^k p > 0$, then $\sum_{j=0}^{\infty} T^j f = \sum_{j=0}^{\infty} T^j p = \infty$ on K. The limit of $D_n(T^k f, T^k p)$ exists on K by the part of the theorem we are supposing proved, and

$$\lim_{n\to\infty} D_n(T^k f, T^k p) = \lim_{n\to\infty} D_n(f, p)$$

on K because $\sum_{j=0}^{\infty} T^j f = \sum_{j=0}^{\infty} T^j p = \infty$ on K.

Assume there is a set L of nonzero measure where the limit of $D_n(f, g)$ does not exist. Keep in mind that $\sum_{j=0}^{\infty} T^j p = \infty$ on L. Then there exist positive constants a_1, a_2, and b, and a set of the form

$$M = \{s: \liminf_{n\to\infty} D_n(f, p) < a_1 < a_2 < \limsup_{n\to\infty} D_n(f, p), p > b\}$$

which has finite and positive measure. We will now derive a contradiction. First apply Lemma 4 to f, and let $r_1 = h$ (the h of Lemma 4). Because of (i) of Lemma 4 we can apply Lemma 5 to r_1, and we let r_2 be the h of that lemma. Because of (i) of Lemma 5 we can apply Lemma 4 to r_2 to obtain r_3, etc. Thus we obtain a sequence of functions $\{r_n\}$. Choose $\varepsilon > 0$ as follows:

$$\varepsilon < \min [\mu(M)/8, \quad (a_2 - a_1)b/8, \quad \sqrt{\tfrac{1}{8}\mu(M)b\cdot(a_2 - a_1)}, \quad \tfrac{1}{8}\mu(M)b\cdot(a_2 - a_1)].$$

We shall prove

$$(1^*)\qquad \int (r_i - a_1 p)^- \leqq \int (r_{i-1} - a_1 p)^- - \frac{1}{4}\mu(M)b(a_2 - a_1)$$

for i even and greater than or equal to 2. To prove (1*), we shall first prove

$$(2^*)\qquad \int_{M-M_i'} (r_{i-1} - a_1 p)^+ \geqq \int_{M-M_i'} (a_2 - a_1)p - \varepsilon\mu(M - M_i'),$$

where $\mu(M_i') \leqq \varepsilon$.

(2*) follows from Lemma 4 (iii), since (iii) implies that $(r_{i-1} - a_2 p)^- \leqq \varepsilon$ on $M - M_i'$, and hence that $r_{i-1} - a_2 p + \varepsilon \geqq 0$ on $M - M_i'$, and thus that the integral of $(r_{i-1} - a_2 p + (a_2 - a_1)p + \varepsilon)^+$ over $M - M_i'$ is greater than or equal to that of $(a_2 - a_1)p$ over the same set, from which follows (2*). Now (2*) together with Lemma 5 (ii) gives

$$\begin{aligned}\int (r_{i-1} - a_1 p)^- - \int (r_i - a_1 p)^- &\geqq \int_{M-M_i'} (a_2 - a_1)p - \varepsilon\mu(M - M_i') - \varepsilon \\ &\geqq (a_2 - a_1)b(\mu(M) - \varepsilon) - \varepsilon(\mu(M) - \varepsilon) - \varepsilon,\end{aligned}$$

from which we get easily (1*). Note further that, if $r_0 = f$,

$$(3^*)\qquad \int (r_i - a_1 p)^- \leqq \int (r_{i-1} - a_1 p)^- \qquad \text{for } i \text{ odd},$$

since by (iv) of Lemma 4 we have that $(r_i - a_2 p)^- \leqq (r_{i-1} - a_2 p)^-$, for i odd. Now we combine (1*) and (3*) to get that $\int (f - a_1 p)^- = \infty$, which is a contradiction.

The finiteness of the limit follows from Lemma 3. This concludes the proof of the theorem.

We conclude by pointing out that we can prove our theorem for ratios of sums of transforms of two measures (this was brought to our attention by J. L. Doob). We will call the variation of a completely additive set function its norm, and we will call the Radon-Nikodym derivative with respect to β of the absolutely continuous component of α, $d\alpha/d\beta$, where α and β are two such set functions.

THEOREM 2. *Let U be a positive linear operator from the space of finite completely additive set functions on a measurable space into itself with norm less than or equal to one. Let α and β be in the domain of U, with $\beta \geqq 0$. Then*

$$\lim_{n\to\infty} d\textstyle\sum_{k=0}^{n} U^k\alpha / d\sum_{k=0}^{n} U^k\beta$$

exists $U^k\beta$ almost everywhere, for every $k \geqq 0$.

Proof. Note first that if γ and δ are completely additive set functions, absolutely continuous with respect to some measure μ, then $d\gamma/d\delta = (d\gamma/d\mu)/(d\delta/d\mu)$. To see that Theorem 2 follows from Theorem 1, define the measure μ by

$$\mu = \sum_{k=0}^{\infty} c_k\, U^k[\beta + \alpha']$$

where α' is the sum of the positive and negative variations of α and $\sum_{k=0}^{\infty} c_k = 1$, $c_k > 0$. Then if f is any member of $L_1(\mu)$, define Tf as follows. Let ξ_f be the set function with

$$\xi_f(A) = \int_A f\,d\mu,$$

and let

$$Tf = dU\xi_f/d\mu.$$

Then T is linear, positive and has norm less than or equal to one, and so Theorem 1 is applicable and yields Theorem 2. This concludes the proof.

We remark that Theorem 1 may be obtained from Theorem 2 by defining $U\alpha(A) = \int_A T(d\alpha/d\mu)\,d\mu$, μ as in Theorem 1.

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