ON WEIGHTED DISTORTION IN CONFORMAL MAPPING

BY

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1. A few years ago Komatu and Nishimiya [4] raised the question of obtaining bounds for the distortion in the spherical metric of normalized univalent functions in the unit circle, that is, for the quantity $|f'(z)|\,(1 + |f(z)|^2)^{-1}$ for a given value of $|z|$. The explicit values they obtained were for the most part not sharp. Quite recently Oikawa [6] obtained the best possible upper and lower bounds for all values of $|z|$, $0 < |z| < 1$ using the variational method. None of the preceding authors seems to have realized that the lower bound has essentially been known for many years. Indeed by the standard distortion theorem, if $|z| = r$, $0 \leqq r < 1$,

$$|f'(z)|^{-1} \leqq (1 + r)^3/(1 - r),$$

while by a result of Löwner [5] in a form given by Robinson [7]

$$|f(z)|^2\,|f'(z)|^{-1} \leqq r^2/(1 - r^2).$$

By adding these,

$$(1 + |f(z)|^2)\,|f'(z)|^{-1} \leqq ((1 + r)^4 + r^2)/(1 - r^2),$$

which on inverting gives just the lower bound obtained by Oikawa. As is well known, equality occurs in this for the slit functions and for them only.

Of much more interest is the fact that the form of solution obtained by Oikawa has very little dependence on the explicit form of the spherical distortion. In fact an analogous result applies to expressions of the form $F(|f(z)|, |f'(z)|)$ whenever F satisfies certain fairly simple restrictions to be discussed in detail below.

2. As usual we denote by S the family of functions $f(z)$ regular and univalent for $|z| < 1$ with $f(0) = 0, f'(0) = 1$. We begin with a discussion of the functions which play the extremal role in our problems.

THEOREM 1. *Let $0 < r < 1$, $r/(1 + r)^2 \leqq q \leqq r/(1 - r)^2$. Then there exists a unique function $f(z, r, q)$ in S with $f(r, r, q) = q$ mapping $|z| < 1$ conformally onto an admissible domain* [2; 3, p. 49] *with respect to the quadratic differential for $q \neq r$,*

$$Q(w, a, q)dw^2 = \frac{q^2(w - a)dw^2}{aw^2(w - q)^2},$$

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where a is determined by

$$\log\left|\frac{((a-q)/a)^{1/2}-1}{((a-q)/a)^{1/2}+1}\right| + \left(\frac{a}{a-q}\right)^{1/2}\log 4\,|\,a\,| = \log r \tag{1}$$

and satisfies

$$a \geqq \tfrac{1}{4}, \qquad r/(1+r)^2 \leqq q < r,$$
$$a \leqq -\tfrac{1}{4}, \qquad r < q \leqq r/(1-r)^2,$$

or with respect to the quadratic differential for $q = r$,

$$Q(w, q)\,dw^2 = -\frac{q^2 dw^2}{w^2(w-q)^2}\,.$$

With $f(z, r, q)$ *is associated* (*except for* $q = r/(1+r)^2$, $r/(1-r)^2$) *a one-parameter family of functions* $f(z, r, q, \lambda) \in S$ *where* $f(z, r, q, \lambda)$ *is obtained from* $f(z, r, q)$ *by a translation of amount* λ *in the Q-metric along the trajectories of* $Q(w, a, q)\,dw^2$ ($Q(w, q)\,dw^2$) *combined with a rotation of the w-plane through the angle* $-\lambda$ *where*

$$\begin{aligned}|\,\lambda\,| \leqq \min\left(\pi, \left(\frac{a-q}{a}\right)^{1/2}\pi\right)& \\ - \left[\varphi + \left(\frac{a-q}{a}\right)^{1/2}\tan^{-1}\left[\frac{(r^{-1}-r)\sin\varphi}{2-(r^{-1}+r)\cos\varphi}\right]\right]&\end{aligned} \tag{2}$$

and

$$\cos\varphi = \frac{1}{2}\left[(r^{-1}+r) - (r^{-1}-r)\left(\frac{a-q}{a}\right)^{1/2}\right].$$

Of course $f(z, r, q, 0) \equiv f(z, r, q)$.

If $f(z) \in S$ *with* $|\,f(re^{i\theta})\,| = q$, *then*

$$|\,f'(re^{i\theta})\,| \leqq |\,f'(r, r, q)\,|, \tag{3}$$

equality occurring only if

$$f(z) = e^{i\theta} f(ze^{-i\theta}, r, q, \lambda).$$

In the circle $|\,z\,| < 1$ we regard the positive quadratic differential

$$\tilde{Q}(z, r, \varphi)\,dz^2 = -\frac{(z-e^{i\varphi})^2(z-e^{-i\varphi})^2}{z^2(z-r)^2(z-r^{-1})^2}\,dz^2$$

where $0 \leqq \varphi \leqq \pi$. The mapping

$$\zeta = \int (-\tilde{Q}(z, r, \varphi))^{1/2}\,dz,$$

where the radical is chosen to be positive on the segment $0 < z < r$, carries the upper unit semicircle onto a domain in the ζ-plane bounded by rectilinear segments of the following nature. Let A_1, A_2, A_3, A^*, A_4 be the images of $0, r, 1, e^{i\varphi}, -1$. Then for a suitable choice of the lower limit of integration,

$A_1 A_2$ is the real axis with both A_1 and A_2 at the point at infinity and $\Re\zeta$ increasing as we go from A_1 to A_2 ; $A_2 A_3$ is a half-infinite horizontal segment, the value of $\Im\zeta$ on it being positive with $\Re\zeta$ decreasing as we go from A_2 to A_3 ; $A_3 A^*$ is a vertical segment with $\Im\zeta$ decreasing as we go from A_3 to A^* and with $\Im\zeta$ positive at A^*; $A^* A_4$ is a vertical segment with $\Im\zeta$ increasing as we go from A^* to A_4 ; $A_4 A_1$ is a half-infinite horizontal segment with $\Re\zeta$ decreasing as we go from A_4 to A_1 . Exceptionally, two of A_3 , A^*, A_4 may coincide, in which case the corresponding segment reduces to a point.

Now let the domain bounded by the real axis and the segments $A_2 A_3$, $A_3 A_4$, $A_4 A_1$ (i.e., the domain obtained from the preceding by deleting the slit penetrating to A^*) be mapped conformally on the upper half w-plane with A_1 , A_2 , A_3 , A^*, A_4 going into w_1 , w_2 , w_3 , w^*, w_4 , where w_1 is the origin and w_3 or w_4 is the point at infinity according as $\Im\zeta$ is larger at A_3 or A_4 (these points coinciding at infinity when A_3 and A_4 coincide). Reflecting across the segments $-1 < z < 1$ and $w_4 w_1 w_3$ of the real axes in the z- and w-planes, we obtain a conformal mapping of $|z| < 1$ onto the w-plane minus a forked slit. The mapping will be assumed normalized so that $dw/dz = 1$ at $z = 0$. Then we denote w_2 by q (which is positive) and whichever of w_3 and w_4 is not the point at infinity by a (provided these points do not coincide). The corresponding mapping function is denoted provisionally by $g(z, r, \varphi)$.

It is clear that the function $g(z, r, \varphi)$ varies continuously with φ in the usual sense of convergence. Moreover

$$g(z, r, 0) \equiv z(1 + z)^{-2}, \qquad g(z, r, \pi) \equiv z(1 - z)^{-2};$$

thus q takes every value between $r(1 + r)^{-2}$ and $r(1 - r)^{-2}$ at least once. If we extend ζ as a (non-single-valued) function of w to the whole w-plane by reflection in the various segments $w_1 w_2$, $w_2 w_3$, $w_3 w_4$, $w_4 w_1$, we see at once that $-d\zeta^2$ is a quadratic differential on the w-sphere with double poles at 0 and q and, unless A_3 and A_4 coincide, a simple pole at the point at infinity and a simple zero at the point a. Thus the quadratic differential has either the form $(A_3 \neq A_4)$

$$\kappa_1 \frac{(w - a)dw^2}{w^2(w - q)^2}$$

or $(A_3 = A_4)$

$$\kappa_2 \frac{dw^2}{w^2(w - q)^2}.$$

Consideration of the fact that $g'(0, r, \varphi) = 1$ shows that $\kappa_1 = q^2/a$, $\kappa_2 = -q^2$.

Denoting this quadratic differential generically by $Q(w)dw^2$, we see that $g(z, r, \varphi)$ maps $|z| < 1$ onto an admissible domain with respect to it. Thus if $f \in S$ and has $f(re^{i\theta}) = qe^{i\chi}$, χ real (corresponding to this value of φ), we are in a position to apply the General Coefficient Theorem [2; 3, p. 51]. In the notation of that result we have now $\Re$ the w-sphere, Δ the domain $g(E, r, \varphi)$ where E denotes $|z| < 1$. Let $\Phi(w)$ be the inverse of $g(z, r, \varphi)$ defined in

$g(E, r, \varphi)$. Then the function $e^{-i\chi}f(e^{i\theta}\Phi(w))$ is an admissible function associated with Δ.

The quadratic differential $Q(w)dw^2$ has double poles P_1 at the origin and P_2 at the point q. The corresponding coefficients are

$$\alpha^{(1)} = -1, \qquad a^{(1)} = e^{i(\chi-\theta)},$$

$$\alpha^{(2)} = (q-a)/a \text{ or } -1, \qquad a^{(2)} = e^{i(\chi-\theta)}g'(r, r, \varphi)/f'(re^{i\theta}).$$

The fundamental inequality of the General Coefficient Theorem then gives

$$\Re\{-\log e^{i(\chi-\theta)} + \alpha^{(2)} \log [e^{i(\chi-\theta)}g'(r, r, \varphi)/f'(re^{i\theta})]\} \leqq 0.$$

Since in either case $\alpha^{(2)}$ is real and negative, we have that

$$|f'(re^{i\theta})| \leqq |g'(r, r, \varphi)|.$$

Moreover equality can occur only when the above admissible function constitutes a translation in the Q-metric (i.e., the metric $|Q(w)|^{1/2}|dw|$) along the trajectories of $Q(w)dw^2$.

From the latter remark we see that a given value of q can occur for at most one value of φ. Thus we may now replace the notation $g(z, r, \varphi)$ by the notation $f(z, r, q)$. Moreover an elementary calculation shows that the case where A_3 and A_4 coincide occurs precisely when $q = r$. Thus we have

$$a \geqq \tfrac{1}{4}, \qquad r/(1+r)^2 \leqq q < r,$$

$$a \leqq -\tfrac{1}{4}, \qquad r < q \leqq r/(1-r)^2.$$

In order to find the explicit relationship between a and q (for fixed r) we determine explicitly the extremal function. We have

$$\zeta = \int (z - e^{i\varphi})(z - e^{-i\varphi})[z(z-r)(z-r^{-1})]^{-1}\, dz,$$

and we choose the constant of integration so that A_3, A^*, A_4 lie on the imaginary axis. Then

$$\zeta = \log z + [((r + r^{-1}) - 2\cos\varphi)(r^{-1} - r)^{-1}] \log [(z - r^{-1})(z-r)^{-1}]$$
$$+ [((r + r^{-1}) - 2\cos\varphi)(r^{-1} - r)^{-1}] \log r.$$

Here determinations of the logarithms are principal on the segment $(0, r)$. We have for this the expansion about $z = 0$

$$\zeta = \log z - [((r + r^{-1}) - 2\cos\varphi)(r^{-1} - r)^{-1}] \log r + \text{positive powers of } z, \tag{4}$$

and about $z = r$

$$\begin{aligned}\zeta = &-[((r + r^{-1}) - 2\cos\varphi)(r^{-1} - r)^{-1}] \log (r - z) + \log r \\ &+[((r + r^{-1}) - 2\cos\varphi)(r^{-1} - r)^{-1}] \log (1 - r^2) \\ &+ \text{positive powers of } (r - z).\end{aligned} \tag{5}$$

We now get the similar formulas for w, treating for simplicity first the case where $a > 0$. Then

$$\zeta = \int qa^{-1/2}(a - w)^{1/2}[w(q - w)]^{-1}\, dw.$$

The choice of constant of integration corresponding to that made for z gives

$$\zeta = \log\left[\frac{a^{1/2} - (a - w)^{1/2}}{a^{1/2} + (a - w)^{1/2}}\right] - \left(\frac{a - q}{a}\right)^{1/2} \log\left[\frac{(a - w)^{1/2} - (a - q)^{1/2}}{(a - w)^{1/2} + (a - q)^{1/2}}\right],$$

where the radicals are to be positive and the determinations of the logarithms principal on the segment $(0, q)$. We have for this the expansion about $w = 0$

$$\zeta = \log w - \log 4a + \left(\frac{a - q}{a}\right)^{1/2} \log\left[\frac{a^{1/2} + (a - q)^{1/2}}{a^{1/2} - (a - q)^{1/2}}\right] + \text{positive powers of } w, \tag{6}$$

and about $w = q$

$$\begin{aligned}\zeta = -\left(\frac{a - q}{a}\right)^{1/2} \log (q - w) + \left(\frac{a - q}{a}\right)^{1/2} \log 4(a - q)\\ + \log\left[\frac{a^{1/2} - (a - q)^{1/2}}{a^{1/2} + (a - q)^{1/2}}\right] + \text{positive powers of } (q - w).\end{aligned} \tag{7}$$

Agreement of the expansions (5) and (7) shows first that

$$((a - q)/a)^{1/2} = [((r + r^{-1}) - 2 \cos \varphi)(r^{-1} - r)^{-1}]. \tag{8}$$

Then the fact that $dw/dz = 1$ at $z = 0$ gives us

$$\log r = \left(\frac{a}{a - q}\right)^{1/2} \log 4a + \log\left[\frac{a^{1/2} - (a - q)^{1/2}}{a^{1/2} + (a - q)^{1/2}}\right].$$

Explicit determination of the similar developments in the case where $a < 0$ shows that (8) is valid in any case, whereas we have in general

$$\log r = \left(\frac{a}{a - q}\right)^{1/2} \log 4 \mid a \mid + \log\left|\frac{((a - q)/a)^{1/2} - 1}{((a - q)/a)^{1/2} + 1}\right|.$$

It remains only to determine for what functions equality can occur in (3). As we have seen, this is possible only when the mapping given by $e^{-i\chi}f(e^{i\theta}\Phi(w))$ amounts to a translation in the Q-metric along the trajectories of $Q(w)dw^2$. Such a motion is equivalent to a vertical translation in the ζ-plane and may be made in either sense to the extent of the length of the slit penetrating to the point A^*. This length is found by direct calculation to be

$$\min\left(\pi, \left(\frac{a - q}{a}\right)^{1/2} \pi\right) - \left[\varphi + \left(\frac{a - q}{a}\right)^{1/2} \tan^{-1}\left[\frac{(r^{-1} - r) \sin \varphi}{2 - (r^{-1} + r) \cos \varphi}\right]\right],$$

where

$$\cos \varphi = \frac{1}{2}\left[(r^{-1} + r) - (r^{-1} - r)\left(\frac{a - q}{a}\right)^{1/2}\right].$$

THEOREM 2. *Let $0 < r < 1$, $q > 0$, and let $\varphi(z, r, q)$ denote the function $z[1 + (q^{-1} - (r^{-1} + r))z + z^2]^{-1}$. Let $f(z)$ be meromorphic and univalent for $|z| < 1$ with $f(0) = 0, f'(0) = 1$. If $|f(re^{i\theta})| = q$, θ real, then*

$$|f'(re^{i\theta})| \geqq |\varphi'(r, r, q)|,$$

equality occurring only if

$$f(z) = e^{i\theta}\varphi(ze^{-i\theta}, r, q).$$

This theorem is readily derived from a well known result on slit mappings, but for completeness we give its proof here on the same lines as that of Theorem 1.

Indeed $\varphi(z, r, q)$ maps $|z| < 1$ onto a domain $\Delta(r, q)$ admissible with respect to the quadratic differential

$$\hat{Q}(w, q)dw^2 = \frac{dw^2}{w^2(w - q)^2}.$$

Denoting by $\Psi(w)$ the inverse of $\varphi(z, r, q)$ defined in $\Delta(r, q)$, if $f(re^{i\theta}) = qe^{i\chi}$, χ real, we have $e^{-i\chi}f(e^{i\theta}\Psi(w))$ an admissible function associated with $\Delta(r, q)$. We can now apply the General Coefficient Theorem with $\mathfrak{R}$ the w-sphere, $Q(w)dw^2$ the quadratic differential $\hat{Q}(w, q)dw^2$, Δ the domain $\Delta(r, q)$, and the above admissible function.

The quadratic differential has double poles P_1 at the origin and P_2 at the point q. The corresponding coefficients are

$$\alpha^{(1)} = q^{-2}, \qquad a^{(1)} = e^{i(\chi-\theta)},$$
$$\alpha^{(2)} = q^{-2}, \qquad a^{(2)} = e^{i(\chi-\theta)}\varphi'(r, r, q)/f'(re^{i\theta}).$$

The fundamental inequality of the General Coefficient Theorem then gives

$$\mathfrak{R}\{q^{-2} \log e^{i(\chi-\theta)} + q^{-2} \log [e^{i(\chi-\theta)}\varphi'(r, r, q)/f'(re^{i\theta})]\} \leqq 0,$$

or

$$|f'(re^{i\theta})| \geqq |\varphi'(r, r, q)|.$$

The equality statement follows from the final statement in the General Coefficient Theorem [2, 3] together with the normalization $f'(0) = 1$.

3. We will now give the exact region of values of the pair $(|f(z)|, |f'(z)|)$ for $f(z) \in S$.

THEOREM 3. *The region of values of the pair $(|f(z)|, |f'(z)|)$ for $f(z) \in S$ is (where $|z| = r$, $0 < r < 1$) the closed region $L(r)$ of points (q, s) determined by the inequalities*

(9) $$r/(1 + r)^2 \leqq q \leqq r/(1 - r)^2,$$

$$(10) \qquad q^2\frac{1-r^2}{r^2} \leqq s \leqq \begin{cases} \dfrac{4\,|\,a-q\,|}{1-r^2}\,(4\,|\,a\,|)^{-a/(a-q)}, & q \neq r, \\ \dfrac{1}{1-r^2}, & q = r, \end{cases}$$

where a is determined by equation (1). *The boundary of $L(r)$ consists of two arcs* (*with common end points*) *$\Gamma_1(r)$ corresponding to the lower bound in* (10) *and $\Gamma_2(r)$ corresponding to the upper bound in* (10). *A boundary point (q, s) on $\Gamma_1(r)$ occurs only for the functions $e^{i\theta}\varphi(ze^{-i\theta}, r, q)$; a boundary point (q, s) on $\Gamma_2(r)$ occurs only for the functions $e^{i\theta}f(ze^{-i\theta}, r, q, \lambda)$ where λ satisfies the bound* (2).

Indeed we verify immediately that $\varphi(z, r, q)$ is in S for $r/(1+r)^2 \leqq q \leqq r/(1-r)^2$, and that

$$|\,\varphi'(r, r, q)\,| = q^2(1-r^2)/r^2.$$

On the other hand, comparing the expansions (5) and (7) and the corresponding ones for $a < 0$, we find for $q \neq r$

$$\log|\,f'(r, r, q)\,| = -\log(1-r^2) - (a/(a-q))\log 4\,|\,a\,| + \log 4\,|\,a-q\,|,$$

or

$$|\,f'(r, r, q)\,| = \frac{4\,|\,a-q\,|}{1-r^2}\,(4\,|\,a\,|)^{-a/(a-q)},$$

while directly for $q = r$

$$|\,f'(r, r, r)\,| = 1/(1-r^2).$$

Thus from Theorem 1 and Theorem 2 we obtain the bounds (10) together with the corresponding equality statements. It is verified immediately that $|\,f'(r, r, q)\,|$ is continuous at $q = r$; thus the boundary segments are arcs as stated.

Now it follows by a standard argument due to Grötzsch [1; 3, p. 94] that the region of values of the pair $(|\,f(re^{i\theta})\,|, |\,f'(re^{i\theta})\,|)$, θ real, is the closed region defined by the inequalities (9), (10).

The bounds (10) were given by Robinson [7], the lower bound as here, the upper bound in a different formal expression which necessitated the use of different formulae according as $q < r$ or $q > r$. He did not obtain the extremal functions or characterize the region of values, although, of course, as we have just seen and as is usual in most similar problems, the latter is straightforward once the inequalities are obtained.

4. The result of the preceding section reduces the determination of bounds for a quantity of the form $F(|\,f(z)\,|, |\,f'(z)\,|)$ to a problem in calculus. For the record we state this in the form of a theorem.

THEOREM 4. *For any function $F(q, s)$ defined on $L(r)$ we have for $|z| < 1$*

$$\min_{(q,s)\epsilon L(r)} F(q, s) \leqq F(|f(z)|, |f'(z)|) \leqq \max_{(q,s)\epsilon L(r)} F(q, s).$$

If the function F has more restrictive properties we can make more precise statements.

COROLLARY 1. *If the function $F(q, s)$ is defined on $L(r)$ and has no extreme value at an interior point of $L(r)$, then*

$$\min_{(q,s)\epsilon \Gamma_1(r) \cup \Gamma_2(r)} F(q, s) \leqq F(|f(z)|, |f'(z)|) \leqq \max_{(q,s)\epsilon \Gamma_1(r) \cup \Gamma_2(r)} F(q, s).$$

Equality can occur at most at points of $\Gamma_1(r) \cup \Gamma_2(r)$.

Further analysis of the occurrence of an extremum on $\Gamma_1(r)$ or $\Gamma_2(r)$ can be made on the basis of sharper properties of F. We will not elaborate this here.

As an illustration we will solve explicitly the problem treated by Komatu, Nishimiya and Oikawa. In it the function $F(q, s)$ is $s/(1 + q^2)$. It is clear that the minimum occurs on $\Gamma_1(r)$, the maximum on $\Gamma_2(r)$. For the minimum we have the minimum of $(1 - r^2)q^2/r^2(1 + q^2)$ on the interval $[r/(1 + r)^2, r/(1 - r)^2]$. Since this function of q is strictly increasing, the minimum value is evidently $(1 - r^2)[r^2 + (1 + r)^4]^{-1}$.

To deal with the maximum problem we observe that for $q \neq r$

$$\begin{aligned} &\frac{d}{dq} \log F(q, |f'(r, r, q)|) \\ &\quad = \frac{1}{q - a} + \left(\frac{1}{q - a}\right)^2 \left[q \frac{da}{dq} - a\right] \log 4 |a| - \frac{2q}{1 + q^2}, \end{aligned} \tag{11}$$

where da/dq is determined from the equation

$$\left(q \frac{da}{dq} - a\right) \log 4 |a| = \frac{2a(a - q)}{q} \tag{12}$$

obtained by differentiating (1). Substituting from (12) in (11) we obtain

$$\begin{aligned} &\frac{d}{dq} \log F(q, |f'(r, r, q)|) \\ &\quad = \frac{1}{q - a} + \frac{2a}{q(a - q)} - \frac{2q}{1 + q^2} = \frac{q - q^3 - 2a}{q(q - a)(1 + q^2)}. \end{aligned} \tag{13}$$

This last term is evidently positive if $q < r$, and also for $q > r$ but sufficiently close to r. From (12) we see that a/q increases with q. Writing (13) in the form

$$\frac{d}{dq} \log F(q, | f'(r, r, q) |) = \frac{1 - q^2 - 2a/q}{(q - a)(1 + q^2)},$$

we see that $F(q, |f'(r, r, q)|)$ increases with q until we possibly meet a point where $a = \frac{1}{2}(q - q^3)$, then decreases. Thus if this value of a is possible, the maximum occurs there, otherwise for $q = r/(1-r)^2$. Since as q increases, a/q increases and $1 - q^2$ decreases, such a solution exists only if r exceeds the smallest positive root r_0 of the equation

$$-\frac{1}{4} = \frac{1}{2}\left(\frac{r}{(1-r)^2} - \frac{r^3}{(1-r)^6}\right),$$

that is, of

$$r^6 - 4r^5 + 7r^4 - 10r^3 + 7r^2 - 4r + 1 = 0.$$

It is seen at once that this is the only such root in the interval $(0, 1)$.

Summarizing these statements we obtain Oikawa's result.

COROLLARY 2. *If r_0 denotes the smallest positive root of the equation*

$$r^6 - 4r^5 + 7r^4 - 10r^3 + 7r^2 - 4r + 1 = 0,$$

then

$$\frac{1-r^2}{r^2 + (1+r)^4} \leqq \frac{|f'(re^{i\theta})|}{1 + |f(re^{i\theta})|^2} \leqq \frac{1-r^2}{r^2+(1-r)^4}, \qquad 0 < r \leqq r_0,$$

$$\leqq M(r), \qquad r_0 \leqq r < 1,$$

where

$$\log M(r) = \log \frac{2q}{1-r^2} - \frac{q^2-1}{q^2+1} \log 2(q^3 - q),$$

and $q = q(r)$ is the function defined by

$$\log r = \left(\frac{q^2-1}{q^2+1}\right)^{1/2} \log 2(q^3 - q) + \log \frac{(q^2+1)^{1/2} - (q^2-1)^{1/2}}{(q^2+1)^{1/2} + (q^2-1)^{1/2}}. \tag{14}$$

For the lower bound, equality occurs only for the functions $z(1 + e^{-i\theta}z)^{-2}$ at the point $re^{i\theta}$, θ real. For the upper bound, $0 < r \leqq r_0$, equality occurs only for the functions $z(1 - e^{-i\theta}z)^{-2}$ at the point $re^{i\theta}$, θ real. For the upper bound, $r_0 < r < 1$, equality occurs only for the functions $e^{i\theta}f(ze^{-i\theta}, r, q, \lambda)$ at the point $re^{i\theta}$, θ real, q determined from (14), and λ satisfying the inequality (2).

5. The following remark must certainly be known although the author cannot recall its occurrence in the literature.

Remark. In most explicit extremal problems treated by the method of the extremal metric and the variational method, one is led to a solution determined by a quadratic differential. In the present instance we see, however, that the function F may readily be chosen in Theorem 4 so that maximizing or minimizing functions f do not correspond to quadratic differentials. This correspondence occurs only when in some sense the problem has a measure of linearity. It is just to such problems that the General Coefficient Theorem applies.

BIBLIOGRAPHY

1. H. GRÖTZSCH, *Über die Verschiebung bei schlichter konformer Abbildung schlichter Bereiche*, Berichte über die Verhandlungen der Sächsischen Akademie der

Wissenschaften zu Leipzig, Mathematisch-Physische Klasse, vol. 83 (1931), pp. 254–279.

2. JAMES A. JENKINS, *A general coefficient theorem*, Trans. Amer. Math. Soc., vol. 77 (1954), pp. 262–280.
3. ———, *Univalent functions and conformal mapping*, Berlin-Göttingen-Heidelberg, Springer-Verlag, 1958.
4. Y. KOMATU AND H. NISHIMIYA, *On distortion in schlicht mappings*, Kōdai Math. Sem. Rep., vol. 2 (1950), pp. 47–50.
5. K. LÖWNER, *Über Extremumsätze bei der konformen Abbildung des Äusseren des Einheitskreises*, Math. Zeit., vol. 3 (1919), pp. 65–77.
6. K. OIKAWA, *A distortion theorem on schlicht functions*, Kōdai Math. Sem. Rep., vol. 9 (1957), pp. 140–144.
7. R. M. ROBINSON, *Bounded univalent functions*, Trans. Amer. Math. Soc., vol. 52 (1942), pp. 426–449.

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