SOME EXTREME VALUE RESULTS FOR INDEFINITE HERMITIAN MATRICES II[1]

BY

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1. Introduction

Let H be a Hermitian transformation of unitary n-space U_n into itself with eigenvalues $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$; and let $E_r(a_1, \cdots, a_k)$ be the r^{th} elementary symmetric function of the numbers $a_1, \cdots, a_k$. If H is *nonnegative* Hermitian, and $f(x_1, \cdots, x_k) = E_r[(Hx_1, x_1), \cdots, (Hx_k, x_k)]$, $1 \leqq r \leqq k \leqq n$, it is known [4, pp. 527–8] that

$$\max f = \binom{k}{r} k^{-r} \left(\sum_{j=1}^{k} \lambda_j\right)^r,$$

$$\min f = E_r(\lambda_{n-k+1}, \cdots, \lambda_n),$$

where both max and min are taken over all sets of k orthonormal (o.n.) vectors $x_1, \cdots, x_k$ in U_n. When H is indefinite, the extreme values of f have been found for special cases of r and general k. The case $r = 1$ is due to Fan [1]; the cases $r = 2$, k are treated in [3]. In the present report we extend these results to the case of general r (Theorem 3).

For any set of k o.n. vectors $x_1, \cdots, x_k$, Fan's theorem requires that

$$\sum_{j=1}^{k} \lambda_j \geqq \sum_{j=1}^{k} (Hx_j, x_j) \geqq \sum_{j=1}^{k} \lambda_{n-j+1}.$$

Following Horn's notation [2]: if $a_1 \geqq \cdots \geqq a_m$ and $1 \leqq k \leqq m$, let $T^k(a_1, \cdots, a_m)$ be the set of real k-tuples $(b_1, \cdots, b_k)$ satisfying

$$\sum_{j=1}^{t} a_j \geqq \sum_{j=1}^{t} b_{i_j} \geqq \sum_{j=1}^{t} a_{m-j+1} \tag{1}$$

for $1 \leqq t \leqq k$ and all sequences $i_1, \cdots, i_t$ of positive integers satisfying $1 \leqq i_1 < \cdots < i_t \leqq k$. Fan's result implies that

$$((Hx_1, x_1), \cdots, (Hx_k, x_k)) \in T^k(\lambda),$$

where $\lambda = (\lambda_1, \cdots, \lambda_n)$.

We seek first the extreme values of $E_r(b)$ for $b = (b_1, \cdots, b_k) \in T^k(\lambda)$. Let β be such an extreme value. Theorem 1 shows that there exists $b \in T^k(\lambda)$ such that $E_r(b) = \beta$ and such that b is contained in some set $T^k(\lambda_{i_1}, \cdots, \lambda_{i_k})$, where $\lambda_{i_1}, \cdots, \lambda_{i_k}$ are the first s and last $k - s$ of the λ's, $0 \leqq s \leqq k$. Theorem 2 gives β in terms of the reduced set of k λ's. It is then a

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straightforward matter to construct o.n. vectors $y_1, \cdots, y_k$ for which $E_r[(Hy_1, y_1), \cdots, (Hy_k, y_k)] = \beta$.

In Theorem 5 we show that for any $b \in T^k(\lambda)$, there exist k o.n. vectors $y_1, \cdots, y_k$ such that $(Hy_j, y_j) = b_j$, $j = 1, \cdots, k$. Thus any question involving the field values (Hx_j, x_j) of o.n. vectors x_j can be discussed in terms of the elements of $T^k(\lambda)$. While this result does not turn out to be necessary for our discussion of the extreme values of E_r, it appears to be of interest in itself. As a corollary we give a formula for the maximum number of o.n. solutions of the equation $(Hx, x) = c$.

2. Extreme value results

By definition, $b \in T^k(\lambda)$, where $b = (b_1, \cdots, b_k)$ and $\lambda = (\lambda_1, \cdots, \lambda_n)$, $\lambda_1 \geqq \cdots \geqq \lambda_n$, if and only if

$$\sum_{j=1}^t \lambda_j \geqq \sum_{j=1}^t b_{i_j}, \tag{2}$$

and

$$\sum_{j=1}^t b_{i_j} \geqq \sum_{j=1}^t \lambda_{n-j+1} \tag{3}$$

for all $1 \leqq t \leqq k$ and all sequences $1 \leqq i_1 < \cdots < i_t \leqq k$. We collect some elementary facts concerning $T^k(\lambda)$ in

LEMMA 1. (i) *If $n = k$, $b \in T^k(\lambda)$ if and only if (2) or (3) holds and $\sum_{j=1}^k \lambda_j = \sum_{j=1}^k b_j$.*

(ii) *If $b_1 \geqq \cdots \geqq b_k$, $b \in T^k(\lambda)$ if and only if*

$$\sum_{j=1}^t \lambda_j \geqq \sum_{j=1}^t b_j, \tag{4}$$

and

$$\sum_{j=1}^t b_{k-j+1} \geqq \sum_{j=1}^t \lambda_{n-j+1} \tag{5}$$

for all $1 \leqq t \leqq k$.

(iii) *If $b_1 \geqq \cdots \geqq b_k$ and $n = k$, $b \in T^k(\lambda)$ if and only if (4) or (5) holds and $\sum_{j=1}^k \lambda_j = \sum_{j=1}^k b_j$.*

LEMMA 2. *Let $b_1, \cdots, b_k$ and $\lambda_1 \geqq \cdots \geqq \lambda_k$ be given. If $(b_1, \cdots, b_m) \in T^m(\lambda_1, \cdots, \lambda_m)$ and $(b_{m+1}, \cdots, b_k) \in T^{k-m}(\lambda_{m+1}, \cdots, \lambda_k)$, where $1 \leqq m < k$, then $(b_1, \cdots, b_k) \in T^k(\lambda_1, \cdots, \lambda_k)$.*

Proof. $\sum_{j=1}^m \lambda_j = \sum_{j=1}^m b_j$ and $\sum_{j=m+1}^k \lambda_j = \sum_{j=m+1}^k b_j$, by Lemma 1(i). Hence $\sum_{j=1}^k \lambda_j = \sum_{j=1}^k b_j$. For $1 = i_0 \leqq i_1 < \cdots < i_s \leqq m < i_{s+1} < \cdots < i_t \leqq i_{t+1} = k$,

$\sum_{j=1}^t b_{i_j} = \sum_{j=1}^s b_{i_j} + \sum_{j=s+1}^t b_{i_j} \leqq \sum_{j=1}^s \lambda_j + \sum_{j=m+1}^{m+t-s} \lambda_j \leqq \sum_{j=1}^t \lambda_j$.

Hence $(b_1, \cdots, b_k) \in T^k(\lambda_1, \cdots, \lambda_k)$.

THEOREM 1. *Let β be an extreme value of $E_r(a_1, \cdots, a_k)$ as $(a_1, \cdots, a_k)$ ranges over $T^k(\lambda_1, \cdots, \lambda_n)$, $1 \leqq r \leqq k \leqq n$. Then there exist*

$$b = (b_1, \cdots, b_k) \in T^k(\lambda_1, \cdots, \lambda_n)$$

and an integer s, $0 \leqq s \leqq k$,[2] *such that* $b \in T^k(\lambda_1, \cdots, \lambda_s, \lambda_{n-k+s+1}, \cdots, \lambda_n)$ *and* $E_r(b) = \beta$.

Proof. Let $b \in T^k(\lambda)$ such that $E_r(b) = \beta$. If $r = 1$, each b_j must appear in at least one *equality* (2) or (3); otherwise $E_1(b)$ could be increased or decreased by altering b_j, while keeping b in $T^k(\lambda)$. If $1 < r \leqq k$,

$$E_r(b) = b_1 E_{r-1}(b_2, \cdots, b_k) + E_r(b_2, \cdots, b_k). \tag{6}$$

If b_1 does not appear in an equality (2) or (3), $E_r(b) = \beta$ implies that $E_{r-1}(b_2, \cdots, b_k) = 0$, and $E_r(b)$ is independent of b_1. We can alter b_1 so as to bring about an equality involving b_1, while keeping b in $T^k(\lambda)$. Since this process does not disturb existing equalities, we can repeat it until the b_j's are exhausted. We may thus assume that each b_j is involved in an equality (2) or (3).

Assume now that $b_1 \geqq \cdots \geqq b_k$. We shall show that each b_j appearing in an equality (2) also appears in an equality (4). Suppose to the contrary that b_{i_l} is the largest b_j which is not in an equality (4), while $\sum_{j=1}^{t} b_{i_j} = \sum_{j=1}^{t} \lambda_j$, $1 \leqq l \leqq t$, $1 \leqq i_1 < \cdots < i_t \leqq k$. Then $\sum_{j=1}^{t} \lambda_j \geqq \sum_{j=1}^{t} b_j \geqq \sum_{j=1}^{t} b_{i_j} = \sum_{j=1}^{t} \lambda_j$; hence $i_l > t$. Since $\sum_{j=1}^{t} b_j = \sum_{j=1}^{t} b_{i_j}$ and $b_1 \geqq \cdots \geqq b_t \geqq b_{i_l} \geqq \cdots \geqq b_{i_t}$, it follows that $b_l = \cdots = b_t = \cdots b_{i_t} \geqq \lambda_t$, and $\sum_{j=1}^{i_t} b_j \geqq \sum_{j=1}^{t} \lambda_j + (i_t - t)\lambda_t \geqq \sum_{j=1}^{i_t} \lambda_j$. Hence b_{i_l} is contained in the equality $\sum_{j=1}^{i_t} b_j = \sum_{j=1}^{i_t} \lambda_j$, a contradiction. A similar argument shows that each b_j appearing in an equality (3) also appears in an equality (5).

If $\sum_{j=1}^{k} b_j = \sum_{j=1}^{k} \lambda_j$, then $b \in T^k(\lambda_1, \cdots, \lambda_k)$, by Lemma 1. Similarly, if $\sum_{j=1}^{k} b_j = \sum_{j=1}^{k} \lambda_{n-j+1}$, $b \in T^k(\lambda_{n-k+1}, \cdots, \lambda_n)$. Otherwise let s, $1 \leqq s \leqq k-1$, be the least integer such that b_{s+1} is not in an equality (4). Then

$$\sum_{j=1}^{s} b_j = \sum_{j=1}^{s} \lambda_j, \tag{7}$$

and

$$\sum_{j=q}^{k} b_j = \sum_{j=n-k+q}^{n} \lambda_j, \tag{8}$$

for some $q \leqq s + 1$, since b_{s+1} must be contained in an equality (5). By Lemma 1, $(b_1, \cdots, b_s) \in T^s(\lambda_1, \cdots, \lambda_s)$ and

$$(b_q, \cdots, b_k) \in T^{k-q+1}(\lambda_{n-k+q}, \cdots, \lambda_n).$$

If $q \leqq s$, $\sum_{j=q}^{s} b_j \geqq \sum_{j=q}^{s} \lambda_j \geqq \sum_{j=n-k+q}^{n-k+s} \lambda_j \geqq \sum_{j=q}^{s} b_j$. Hence $\sum_{j=q}^{s} b_j = \sum_{j=n-k+q}^{n-k+s} \lambda_j$; subtracting from (8), we get $\sum_{j=s+1}^{k} b_j = \sum_{j=n-k+s+1}^{n} \lambda_j$. Thus $(b_{s+1}, \cdots, b_k) \in T^{k-s}(\lambda_{n-k+s+1}, \cdots, \lambda_n)$, and by Lemma 2,

$$b \in T^k(\lambda_1, \cdots, \lambda_s, \lambda_{n-k+s+1}, \cdots, \lambda_n).$$

This completes the proof of the theorem.

If $b = (b_1, \cdots, b_k) \in T^k(\lambda_1, \cdots, \lambda_n)$, $n > k$, it is not necessarily true that $b \in T^k(\lambda_{i_1}, \cdots, \lambda_{i_k})$ for some selection of k λ's, even if $E_r(b)$ is an extreme value

[2] If $s = 0$ (or k), the initial (or terminal) segment is missing.

on $T^k(\lambda)$. It is not hard to show, however, that $b \in T^k(\lambda_{i_1}, \cdots, \lambda_{i_{k+1}})$ for some selection of the first s and last $k + 1 - s$ λ's. One can also find real numbers $b_{k+1}, \cdots, b_n$ such that $(b_1, \cdots, b_n) \in T^n(\lambda_1, \cdots, \lambda_n)$.

THEOREM 2. *For* $1 \leqq r \leqq k$ *the extreme values of* $E_r(a_1, \cdots, a_k)$ *as* $(a_1, \cdots, a_k)$ *ranges over* $T^k(\mu_1, \cdots, \mu_k)$ *are of the form*

$$E_r(b_1, \cdots, b_k), \tag{9a}$$

where

$$b_{k_j+1} = \cdots = b_{k_{j+1}} = \frac{\mu_{k_j+1} + \cdots + \mu_{k_{j+1}}}{k_{j+1} - k_j}, \qquad j = 0, 1, \cdots, q-1, \tag{9b}$$

and the k_j *are integers satisfying* $0 = k_0 < k_1 < \cdots < k_q = k$.

Proof. Let β be an extreme value of E_r over $T^k(\mu)$, where $\mu = (\mu_1, \cdots, \mu_k)$. Let B be the set of all $b = (b_1, \cdots, b_k)$ such that $E_r(b) = \beta$. Let S_t be the set of all $b \in T^k(\mu)$ such that $b_1 \geqq \cdots \geqq b_k$ and $\sum_{j=1}^t b_j = \sum_{j=1}^t \mu_j, 1 \leqq t \leqq k$. If $b \in S_t$, $1 \leqq t < k$, then $(b_1, \cdots, b_t) \in T^t(\mu_1, \cdots, \mu_t)$ and $(b_{t+1}, \cdots, b_k) \in T^{k-t}(\mu_{t+1}, \cdots, \mu_k)$. It will be convenient to define $S_0 = S_k$.

We shall show that if $b \in S_\rho \cap S_\tau \cap B = \mathfrak{J}, 0 \leqq \rho < \tau \leqq k$, and if $b_{\rho+1} > b_\tau$, then for some s, $\rho < s < \tau$, there exists $b' \in \mathfrak{J} \cap S_s$. Set

$$d = \min_{\rho < s < \tau} \left\{ \sum_{j=\rho+1}^{s} (\mu_j - b_j) \right\} = \sum_{j=\rho+1}^{s_0} (\mu_j - b_j), \qquad \rho < s_0 < \tau. \tag{10}$$

Since $b \in S_\rho$, $d = \sum_{j=1}^{s_0} (\mu_j - b_j) \geqq 0$. Set $b'_{\rho+1} = b_{\rho+1} + \varepsilon$, $b'_\tau = b_\tau - \varepsilon$, and $b'_j = b_j$ for $j \neq \rho + 1, \tau$. Then, for $|\varepsilon| \leqq d$, $\rho + 1 \leqq i_1 < \cdots < i_p \leqq \tau$, $p < \tau - \rho$, we have $\sum_{j=1}^p b'_{i_j} \leqq \sum_{j=1}^p b_{\rho+j} + |\varepsilon| \leqq \sum_{j=1}^p \mu_{\rho+j}$, by (10), while $\sum_{j=\rho+1}^\tau b'_j = \sum_{j=\rho+1}^\tau b_j = \sum_{j=\rho+1}^\tau \mu_j$; hence $(b'_{\rho+1}, \cdots, b'_\tau) \in T^{\tau-\rho}(\mu_{\rho+1}, \cdots, \mu_\tau)$. But $(b'_1, \cdots, b'_\rho) \in T^\rho(\mu_1, \cdots, \mu_\rho)$ if $\rho \neq 0$, and $(b'_{\tau+1}, \cdots, b'_k) \in T^{k-\tau}(\mu_{\tau+1}, \cdots, \mu_k)$ if $\tau \neq k$. By Lemma 2, $b' \in T^k(\mu)$ for $|\varepsilon| \leqq d$.

The next step is to show that $b' \in B$ for $|\varepsilon| \leqq d$. When $r = 1$, $E_r(b') = E_r(b) = \beta$. When $r = 2$, $E_r(b') = E_r(b) + (b_\tau - b_{\rho+1} - \varepsilon)\varepsilon$. Since $b_{\rho+1} > b_\tau$, $b \in B$, and $b' \in T^k(\mu)$ for $|\varepsilon| \leqq d$, it follows that $d = 0$, and $b' = b \in B$. When $r > 2$,

$$E_r(b') = E_r(b) + \varepsilon(b_\tau - b_{\rho+1} - \varepsilon)E_{r-2}(b_1, \cdots, \not{b}_{\rho+1}, \cdots, \not{b}_\tau, \cdots, b_k),$$

where $\not{b}_j$ indicates that b_j is deleted. Again, since $b_{\rho+1} > b_\tau$ and $b \in B$, it follows that $E_{r-2}(b_1, \cdots, \not{b}_{\rho+1}, \cdots, \not{b}_\tau, \cdots, b_k) = 0$ and $E_r(b') = E_r(b)$. Thus $b' \in B$ for $|\varepsilon| \leqq d$.

Set $\varepsilon = d$. Since $b_\rho \geqq \mu_\rho \geqq \mu_{\rho+1} \geqq b_{\rho+1} + d \geqq b_{\rho+2}$, and similarly $b_{\tau-1} \geqq b_\tau - d \geqq b_{\tau+1}$, $b'_1 \geqq \cdots \geqq b'_k$. Hence by (10) $b' \in \mathfrak{J} \cap S_{s_0}$.

For any $b \in T^k(\mu)$ such that $E_r(b) = \beta$ and $b_1 \geqq \cdots \geqq b_k$, we have $b \in S_0 \cap S_k \cap B$. If $b_1 = b_k$, then $b_j = (\mu_1 + \cdots + \mu_k)/k$ for $j = 1, \cdots, k$. Otherwise, by the preceding discussion, there exists s, $0 < s < k$, and b' such that $b' \in S_0 \cap S_s \cap S_k \cap B$. We may continue this refining process until we obtain some $b \in S_{k_0} \cap \cdots \cap S_{k_q} \cap B$, $0 = k_0 < k_1 < \cdots < k_q = k$, such that $b_{k_j+1} = \cdots = b_{k_{j+1}} = (\mu_{k_j+1} + \cdots + \mu_{k_{j+1}})/(k_{j+1} - k_j)$ for $j = 0, \cdots, q-1$.

THEOREM 3. *Let H be a Hermitian transformation on unitary n-space U_n with eigenvalues $\lambda_1 \geqq \cdots \geqq \lambda_n$. For $1 \leqq r \leqq k \leqq n$ set*

$$f(x_1, \cdots, x_k) = E_r[(Hx_1, x_1), \cdots, (Hx_k, x_k)].$$

The maximum and minimum values of f, as $x_1, \cdots, x_k$ *range over all sets of k o.n. vectors in U_n, have the form*

$$E_r(b_1, \cdots, b_k), \tag{9a}$$

where

$$b_{k_j+1} = \cdots = b_{k_{j+1}} = \frac{\mu_{k_j+1} + \cdots + \mu_{k_{j+1}}}{k_{j+1} - k_j}, \quad j = 0, 1, \cdots, q-1; \tag{9b}$$

the k_j are integers satisfying $0 = k_0 < k_1 < \cdots < k_q = k$, and $\mu_1, \cdots, \mu_k$ are the first s and last $k - s$ of the λ's for some s, $0 \leqq s \leqq k$.

Proof. By Fan's Theorem, $((Hx_1, x_1), \cdots, (Hx_k, x_k)) \in T^k(\lambda)$. Hence the extreme values of f are bounded by the extreme values of E_r on $T^k(\lambda)$. The latter are given by Theorems 1 and 2. The typical value (9) is taken on by f for the o.n. vectors

$$y_\alpha = \sum_{\gamma=k_j+1}^{k_{j+1}} \frac{\theta_j^{\gamma(\alpha-k_j)}}{\sqrt{k_{j+1} - k_j}} u_\gamma, \qquad \alpha = k_j + 1, \cdots, k_{j+1},$$

for $j = 0, \cdots, q-1$, where θ_j is a primitive $(k_{j+1} - k_j)^{\text{th}}$ root of unity, and $u_1, \cdots, u_k$ are o.n. eigenvectors of H corresponding to $\mu_1, \cdots, \mu_k$, respectively.

3. An existence theorem for orthonormal vectors

In Theorem 3 it was a relatively simple matter to pick out o.n. vectors $y_1, \cdots, y_k$ such that $(Hy_j, y_j) = b_j$ when the b_j's had the special form (9). One may ask whether it is always possible to find such vectors for any $b \in T^k(\mu)$, where $\mu = (\mu_1, \cdots, \mu_m)$ is a selection of the eigenvalues of the Hermitian matrix H. The answer, given in Theorem 5, is a consequence of the more general

THEOREM 4. *Let H be a Hermitian transformation on unitary n-space U_n. Let $x_1, \cdots, x_m$, $1 \leqq m \leqq n$, be m o.n. vectors in U_n, ordered so that $(Hx_j, x_j) = h_j \geqq h_{j+1}$, $1 \leqq j \leqq m-1$. Let $L(x_1, \cdots, x_m)$ denote the subspace spanned by $x_1, \cdots, x_m$. If $(b_1, \cdots, b_k) \in T^k(h_1, \cdots, h_m)$, then there exist o.n. vectors $y_1, \cdots, y_k$ in $L(x_1, \cdots, x_m)$ such that $(Hy_j, y_j) = b_j$, $j = 1, \cdots, k$.*

Proof. If $m = k = 1$, $y_1 = x_1$. Assume the theorem true when there are $m - 1$ x's. Since $h_1 \geqq b_1 \geqq h_m$, there exists an integer s such that $h_s \geqq b_1 \geqq h_{s+1}$. Hence there exist o.n. vectors y_1, $y_1^* \in L(x_s, x_{s+1})$ such that $(Hy_1, y_1) = b_1$. Furthermore, $(Hy_1^*, y_1^*) = h_s + h_{s+1} - b_1$, since if $z_1, \cdots, z_{n-2}$ is an o.n. basis for the complement of $L(x_s, x_{s+1})$ in U_n,

$$(Hy_1, y_1) + (Hy_1^*, y_1^*) = \operatorname{tr} H - \textstyle\sum_{j=1}^{n-2} (Hz_j, z_j)$$
$$= (Hx_s, x_s) + (Hx_{s+1}, x_{s+1}) = h_s + h_{s+1}.$$

We shall show that $(b_2, \cdots, b_k)$ lies in

$$T^{k-1}(h_1, \cdots, h_{s-1}, h_s + h_{s+1} - b_1, h_{s+2}, \cdots, h_m).$$

Note first that $h_{s-1} \geqq h_s \geqq h_s + h_{s+1} - b_1 \geqq h_{s+1} \geqq h_{s+2}$. Set $b_j' = b_j$, $j = 2, \cdots, k$. If $t \leqq s - 1$, then $\sum_{j=1}^{t} b_{i_j}' \leqq \sum_{j=1}^{t} h_j$ by hypothesis. If $s - 1 < t \leqq k - 1$, then $b_1 + \sum_{j=1}^{t} b_{i_j}' \leqq \sum_{j=1}^{t+1} h_j$; hence $\sum_{j=1}^{t} b_{i_j}' \leqq \sum_{j=1}^{s-1} h_j + (h_s + h_{s+1} - b_1) + \cdots + h_{t+1}$. Thus condition (2) holds; (3) is verified similarly.

By the induction hypothesis, there exist o.n. vectors $y_2, \cdots, y_k$ in $L(x_1, \cdots, x_{s-1}, y_1^*, x_{s+2}, \cdots, x_m)$ such that $(Hy_j, y_j) = b_j$, $j = 2, \cdots, k$. The vectors $y_1, \cdots, y_k$ are o.n. in $L(x_1, \cdots, x_m)$. This completes the proof of the theorem.

THEOREM 5. *Let H be a Hermitian transformation on U_n, and let $\mu_1 \geqq \cdots \geqq \mu_m$, $1 \leqq m \leqq n$, be a subset of the eigenvalues of H. If $(b_1, \cdots, b_k) \in T^k(\mu_1, \cdots, \mu_m)$, then there exist o.n. vectors $y_1, \cdots, y_k$ in U_n such that $(Hy_j, y_j) = b_j$, $j = 1, \cdots, k$.*

Proof. In Theorem 4, let $x_1, \cdots, x_m$ be o.n. eigenvectors corresponding to $\mu_1, \cdots, \mu_m$.

COROLLARY 1. *Let H be a Hermitian transformation with eigenvalues $\lambda_1 \geqq \cdots \geqq \lambda_n$. Let I_k, $1 \leqq k \leqq n$, denote the closed interval*

$$[(\textstyle\sum_{j=1}^{k} \lambda_j)/k, (\sum_{j=1}^{k} \lambda_{n-j+1})/k],$$

and let χ_k be the characteristic function of I_k. Let $M(c)$ be the largest number of o.n. solutions of the equation $(Hx, x) = c$. Then $M(c) = \sum_{k=1}^{n} \chi_k(c)$.

Proof.[3] Let I_0 be the real line, and let I_{n+1} be the null set. Suppose that $c \in I_q$, $c \notin I_{q+1}$. Since $I_j \supseteqq I_{j+1}$ for $j = 0, \cdots, n$, $\sum_{k=1}^{n} \chi_k(c) = q$. On the other hand, since $c \in I_q$, $\sum_{j=1}^{t} \lambda_j \geqq tc \geqq \sum_{j=1}^{t} \lambda_{n-j+1}$ for $1 \leqq t \leqq q$. Hence $(c, \cdots, c) \in T^q(\lambda)$, and by Theorem 5 there exist q o.n. vectors y_j such that $(Hy_j, y_j) = c$, $j = 1, \cdots, q$. Since $c \notin I_{q+1}$, $(c, \cdots, c) \notin T^{q+1}(\lambda)$, and the existence of $q + 1$ o.n. y's is denied by Fan's theorem. Hence $M(c) = q$.

Remark. Theorem 5 above may also be obtained from Theorem 4 of [2] and our remarks preceding Theorem 2.

[3] For another proof of this result cf. [5].

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