A NOTE ON ALGEBRAIC GEOMETRY OVER GROUND RINGS

The invariance of Hilbert characteristic functions under the specialization process

BY

HEISUKE HIRONAKA

We know the following two results on the invariance of Hilbert characteristic functions of projective varieties or positive divisors on a projective variety under the specialization process:

(1) J. Igusa has shown in [2] that normal varieties in an algebraic family have the same Hilbert characteristic function.

(2) T. Matsusaka has shown in [4] that the virtual arithmetic genera of algebraically equivalent divisors on a nonsingular variety are the same, which implies that all members of an algebraic family of positive divisors on a nonsingular variety have the same Hilbert characteristic function.[1]

On the other hand the notion and the theory of the specialization were generalized by G. Shimura, [9], to the reduction of algebraic varieties and cycles on an algebraic variety with respect to an arbitrary discrete valuation ring in the ground field. The idea of taking a discrete valuation ring, with respect to which varieties or cycles on a variety are specialized, is not only imperative in the case of 'reduction modulo p' but is also helpful in the equi-characteristic case, or the case of 'specialization over a field' in the algebraic geometry over a fixed universal domain. When we want to discuss the local properties of the specialization of a given projective variety V and V-cycles with respect to a discrete valuation ring v in a ground field k, it is convenient to assume that v is a discrete valuation ring of rank 1 which satisfies the finiteness condition for integral extensions,[2] and in fact we suffer no loss of generality in this paper by doing so. Furthermore it will be convenient to fix a generic point, P, of V over k and to take *a model of the function field $k(P)$ over the ground ring v having V as its underlying projective variety*, i.e., the set of the specialization rings of all specializations of P over v as well as over k. This model will be denoted by $[V]_v$. The general theory of algebraic models over Dedekind domains was established by M. Nagata, [6].

In this paper we shall study some local properties of the specialization

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[1] The Hilbert characteristic function of a positive divisor on a projective variety will be defined in Part II. As for the relation between the Hilbert characteristic function and the virtual arithemetic genus of a positive divisor on a projective variety we refer to [11], Part III (10, 11).

[2] The finiteness condition for integral extensions is said to be satisfied by a ring v if every integral extension of v' of v is a finite v-module whenever the field of quotients of v' is a finite algebraic extension of that of v. (See the introduction of [6].)

process; more precisely we shall observe how the prime ideal of the ground ring v behaves in the specialization rings of the model $[V]_v$ (Part I). In those terms we shall give a necessary and sufficient condition for a projective variety (or a positive divisor on a projective variety) and its specialization to have the same Hilbert characteristic function (Part II, Theorems 3 and 4). Those theorems will include results (1) and (2) above. All of our results will be applicable to reduction modulo p, and result (2) will be generalized in the sense that the simultaneously specialized ambient variety may have singularities outside the specialized positive divisors.

Part I

First of all we fix an arbitrary universal domain $\mathbf{K}$. Let k_0 be a field in $\mathbf{K}$, and let v_0 be a *local ground ring* in k_0, that is, either k_0 itself or a discrete valuation ring of rank 1 which satisfies the finiteness condition.[3] For convenience we consider the residue field $\bar{v}_0$ of v_0 as a field in another universal domain $\bar{\mathbf{K}}$; in the case $v_0 = k_0$, $\bar{v}_0$ is simply an isomorphic image of k_0. Let $(x) = (x_1, \cdots, x_m)$ be a set of elements in $\mathbf{K}$ and $(\bar{x}) = (\bar{x}_1, \cdots, \bar{x}_m)$ a set of elements in $\bar{\mathbf{K}}$. We say that $(\bar{x})$ is a specialization of (x) over v_0 and denote it by $(x) \xrightarrow{v_0} (\bar{x})$ if for every $F(X) = F(X_1, \cdots, X_m)$ in $v_0[X]$ such that $F(x) = 0$ we have $\bar{F}(\bar{x}) = 0$ where $\bar{F}(X)$ denotes the residue class of $F(X)$ in $\bar{v}_0[X]$. We denote by $[(x) \xrightarrow{v_0} (\bar{x})]$ the specialization ring of $(x) \xrightarrow{v_0} (\bar{x})$, that is,

$$\{G(x)/F(x) \mid G(X) \in v_0[X],\ F(X) \in v_0[X] \text{ and } \bar{F}(\bar{x}) \neq 0\}.$$

Throughout this paper we shall fix two projective n-spaces S^n and $\bar{S}^n$ over the universal domains $\mathbf{K}$ and $\bar{\mathbf{K}}$ respectively. Let Z be a positive S^n-cycle and $\bar{Z}$ a positive $\bar{S}^n$-cycle. If Z and $\bar{Z}$ have the same dimension and the same degree and if the Chow point $C(\bar{Z})$ of $\bar{Z}$ is a specialization of the Chow point $C(Z)$ of Z over v_0, we say that $\bar{Z}$ is a specialization of Z over v_0 and denote this by $Z \xrightarrow{v_0} \bar{Z}$.[4] Let k be an extension of k_0, that is, a field in $\mathbf{K}$ which is finitely generated over k_0, such that Z is k-rational. Then we always have a local ground ring v in k which is an extension of v_0 and over which the same $\bar{Z}$ is a specialization of Z. The residue field $\bar{v}$ of v will be considered as an extension of $\bar{v}_0(C(\bar{Z}))$ in $\bar{\mathbf{K}}$. To construct such an extension v of v_0 we may adopt, for instance, the following method: we may assume that $C(Z) = (1, y_1, \cdots, y_t)$, $C(\bar{Z}) = (1, \bar{y}_1, \cdots, \bar{y}_t)$, and $(1, y_1, \cdots, y_t) \xrightarrow{v_0} (1, \bar{y}_1, \cdots, \bar{y}_t)$. Let $f_1, \cdots, f_s$ be generators of the ideal of those elements in $v_0[y]$ which are specialized into zero over $(y) \xrightarrow{v_0} (\bar{y})$. By choosing a suitable index i, $(f_1/f_i, \cdots, f_s/f_i)$ has a (finite) specialization in $\bar{\mathbf{K}}$ over $(y) \xrightarrow{v_0} (\bar{y})$, that is, f_i is a nonunit in $v_0[y, f/f_i]$ where $(y, f/f_i)$ denotes $(y_1, \cdots, y_t, f_1/f_i, \cdots, f_s/f_i)$.

[3] See footnote 2.

[4] See [3] and [9].

Since Z is k-rational, $k_0(y)$ is contained in k. If $k \neq k_0(y)$, we take a finite number of elements in k, say $(z) = (z_1, \cdots, z_r)$, which generate k over $k(y)$ such that f_i is still a nonunit in $v_0[y, f/f_i, z]$. Let R be the derived normal ring of $v_0[y, f/f_i, z]$, and let p be one of the minimal prime ideals of $f_i R$. It follows from Theorem 3, [6], that $v = R_p$ is a local ground ring (satisfying the finiteness condition). Since p contains all f's, v is an extension of v_0 in k, and we have $Z \xrightarrow{v} \bar{Z}$.

LEMMA 1. *Let v be a local ground ring in a field k. Let Z be a k-rational positive S^n-cycle and $\bar{Z}$ an $\bar{S}^n$-cycle such that $Z \xrightarrow{v} \bar{Z}$. Then the point set $|\bar{Z}|$ of $\bar{Z}$ coincides with $\{\bar{P} \in \bar{S}^n \mid P \xrightarrow{v} \bar{P}$ for some $P \in |Z|\}$.*

This lemma is true for an arbitrary valuation ring v in k such that Z is k-rational and $Z \xrightarrow{v} \bar{Z}$. In [9] it was introduced as definition of specialization and verified in an implicit way, and it is not hard to give a proof directly by means of Chow forms associated to the cycles.

LEMMA 2. *Let Z be a k-prime rational S^n-cycle and P a generic point of Z over k. Let $\bar{Z}'$ be a $\bar{v}$-prime rational $\bar{S}^n$-cycle which is a component of the specialization $\bar{Z}$ of Z over a local ground ring v in k, and let $\bar{P}'$ be a generic point of $\bar{Z}'$ over $\bar{v}$. Then $\bar{Z}$ is $\bar{v}$-rational, and the coefficient of $\bar{Z}'$ in the expression of $\bar{Z}$ as a linear combination of $\bar{v}$-prime rational $\bar{S}^n$-cycles is equal to the multiplicity $e(p\mathfrak{O})$ of the primary ideal $p\mathfrak{O}$, where $\mathfrak{O} = [(P) \xrightarrow{v} (\bar{P}')]$ and p is the prime ideal of v.*

Proof. Let $\bar{Z}''$ be the locus of $\bar{P}'$ over the algebraic closure of $\bar{v}$. Let μ be the coefficient of $\bar{Z}''$ in the expression of $\bar{Z}$ as a linear combination of absolutely irreducible $\bar{S}^n$-cycles. We want to prove $\mu = [\bar{v}(\bar{P}'), \bar{v}]_\iota\, e(p\mathfrak{O})$ where $[\bar{v}(\bar{P}'), \bar{v}]_\iota$ denotes the order of inseparability of $\bar{v}(\bar{P}')$ over $\bar{v}$. This will obviously suffice to prove Lemma 2.

We first consider the case when Z has no multiple component, that is, when $k(P)$ is separably generated over k. We may assume $P = (1, x_1, \cdots, x_n)$ and $\bar{P}' = (1, \bar{x}_1, \cdots, \bar{x}_n)$. Let u_i^j $(1 \leqq j \leqq n, 0 \leqq i \leqq r)$, $r = \dim Z$, be independent variables over $k(P)$. Put $u_i^0 = -\sum_{j=1}^n u_i^j x_j$ $(0 \leqq i \leqq r)$. Similarly let $\bar{u}_i^j$ $(1 \leqq j \leqq n, 0 \leqq i \leqq r)$ be independent variables over $\bar{v}(\bar{P}')$, and put $\bar{u}_i^0 = -\sum_{j=1}^n \bar{u}_i^j \bar{x}_j$ $(0 \leqq i \leqq r)$. Denote $[(u, x) \xrightarrow{v} (\bar{u}, \bar{x})]$ by $\tilde{\mathfrak{R}}$ and $[(u) \xrightarrow{v} (\bar{u})]$ by $\mathfrak{R}$, where

$$(u) = (u_j^i : 0 \leqq j \leqq n, 0 \leqq i \leqq r) \quad \text{and} \quad (\bar{u}) = (\bar{u}_j^i : 0 \leqq j \leqq n, 0 \leqq i \leqq r).$$

Since the hyperplanes $\sum_{j=0}^n \bar{u}_i^j Y_j = 0$ $(0 \leqq i \leqq r)$ meet in $\bar{Z}$ at only one point $\bar{P}'$, $\tilde{\mathfrak{R}}$ is integral over $\mathfrak{R}$. Hence $\tilde{\mathfrak{R}}$ is a finite $\mathfrak{R}$-module by the corollary to Theorem 2, [6]. Therefore, applying the extension formula (Theorem 2, [7]) to the overring $\tilde{\mathfrak{R}}$ of $\mathfrak{R}$, we have

$$[\bar{v}(\bar{P}', \bar{u}):\bar{v}(\bar{u})]e(p\tilde{\mathfrak{R}}) = [k(P, u):k(u)]e(p\mathfrak{R}).$$

Since $k(P)$ is separably generated over k, $[k(P, u), k(u)]$ is equal to one. $\bar{v}(\bar{P}', \bar{u})$ is purely inseparable over $\bar{v}(\bar{u})$ and $[\bar{v}(\bar{P}', \bar{u}):\bar{v}(\bar{u})][\bar{v}(\bar{u}), \bar{v}]_\iota = [\bar{v}(\bar{P}'):\bar{v}]_\iota$. Moreover we can prove that $\tilde{\mathfrak{R}}$ is the ring of quotients of

$$\mathfrak{O}[u_j^i : 1 \leqq j \leqq n, 0 \leqq i \leqq r]$$

with respect to the prime ideal generated by the maximal ideal of $\mathfrak{O}$, hence that we have $e(p\mathfrak{O}) = e(p\tilde{\mathfrak{R}})$.

Thus we obtain

$$\text{(a)} \qquad [\bar{v}(\bar{P}'):\bar{v}]_\iota\, e(p\mathfrak{O}) = [\bar{v}(\bar{u}):\bar{v}]_\iota\, e(p\mathfrak{R}).$$

Let U_i^j $(0 \leqq j \leqq n, 0 \leqq i \leqq r)$ be independent variables over $k(P)$, and denote $[(U) \xrightarrow{v} (\bar{u})]$ by $\mathfrak{T}$. Similarly let $\bar{U}_i^j$ $(0 \leqq j \leqq n, 0 \leqq i \leqq r)$ be independent variables over $\bar{v}(\bar{P}')$, and denote $[(\bar{U}) \xrightarrow{\bar{v}} (\bar{u})]$ by $\bar{\mathfrak{T}}$. Obviously $\mathfrak{T}/p\mathfrak{T} = \bar{\mathfrak{T}}$.

Let F be the Chow form associated to Z, whose coefficients may be chosen from v and not all from p. Obviously $\mathfrak{T}/F\mathfrak{T} = \mathfrak{R}$. Therefore we have $e(p\mathfrak{R}) = e((p, F)\mathfrak{T}) = e(\bar{F}\bar{\mathfrak{T}})$ where the residue class $\bar{F}$ of F modulo p is the Chow form associated to $\bar{Z}$. It is easily verified that $[\bar{v}(\bar{u}):\bar{v}]_\iota\, e(\bar{F}\bar{\mathfrak{T}})$ is equal to μ. The equality (a) implies the lemma.

Next we consider the case when Z has multiple components. The multiplicity is equal to $[k(P):k]_\iota$. Let h be a purely inseparable extension of k such that $h(P)$ is separably generated over h. The extension of v in h is unique. We denote this extension by w, and its prime ideal by q. Put $[(x) \xrightarrow{w} (\bar{x})] = \mathfrak{Q}$. By the above results we have $[\bar{w}(\bar{x}):\bar{w}]_\iota\, e(q\mathfrak{Q}) = \mu/p^\nu$, where $p^\nu = [k(P):k]_\iota$ and $\bar{w}$ is the residue field of w. We have to prove $[\bar{w}(\bar{x}):\bar{w}]_\iota\, e(q\mathfrak{Q}) = [\bar{v}(\bar{x}):\bar{v}]_\iota\, e(p\mathfrak{O})/p^\nu$. Since h is purely inseparable over k, $\mathfrak{Q}$ is integral over $\mathfrak{O}$ and therefore a finite $\mathfrak{O}$-module. We can apply the extension formula to the overring $\mathfrak{Q}$ of $\mathfrak{O}$ and obtain

$$\text{(b)} \qquad [\bar{w}(\bar{x}):\bar{v}(\bar{x})]e(p\mathfrak{Q}) = [h(x):k(x)]e(p\mathfrak{O}).$$

Let $q^r w = pw$ and $s = [\bar{w}:\bar{v}]$. Then we have $e(p\mathfrak{Q}) = re(q\mathfrak{Q})$ and

$$[\bar{w}(\bar{x}):\bar{v}(\bar{x})] = s[\bar{w}(\bar{x}):\bar{w}]_\iota/[\bar{v}(\bar{x}):\bar{v}]_\iota\,.$$

Therefore the left hand side of (b) is equal to

$$\frac{[\bar{w}(\bar{x}):\bar{w}]_\iota e(q\mathfrak{O})}{[\bar{v}(\bar{x}):\bar{v}]_\iota}\, s \cdot r.$$

On the other hand the right hand side of (b) is equal to

$$\frac{[h:k]\, e(p\mathfrak{O})}{[k(x):k]_\iota} = \frac{sr}{p^\nu}\, e(p\mathfrak{O})\,.$$

Thus (b) implies the required equality

$$[\bar{w}(\bar{x}):\bar{w}]_\iota\, e(q\mathfrak{Q}) = [\bar{v}(\bar{x}):\bar{v}]_\iota\, e(p\mathfrak{O})/p^{\nu},$$

and the proof is completed.

Let V be a *variety* in S^n, that is, a positive S^n-cycle with no multiple components. Assume that V is k-rational and let v be a local ground ring in k. We denote by $I_v(V)$ the homogeneous ideal of V in $v[Y] = v[Y_0, Y_1, \cdots, Y_n]$; the homogeneous ideal $I_k(V)$ of V in $k[Y]$ is equal to $I_v(V)k[Y]$, and we have $I_k(V) \cap v[Y] = I_v(V)$. Put $v[Y]/I_v(V) = v[y]$ where each y_i is the residue class of Y_i and the same notation v is used for the residue ring of v. (The natural homomorphism restricted to v is an isomorphism.) Let $\mathfrak{S}$ be the total ring of quotients of $v[y]$. Let $\mathfrak{P}$ be a prime ideal in $v[y]$ which is homogeneous and does not contain all y_i $(0 \leqq i \leqq n)$. The ring of quotients $v[y]_{\mathfrak{P}}$ can be imbedded isomorphically in $\mathfrak{S}$. Namely let $\mathfrak{n}_1, \mathfrak{n}_2, \cdots$, and $\mathfrak{n}_t$ be the prime ideals of zero in $v[y]$. We may assume that there exists a certain index j such that $\mathfrak{n}_i \subseteqq \mathfrak{P}$ if and only if $i \leqq j$. Then there exists a system of idempotent elements in $\mathfrak{S}$, $e_1, e_2, \cdots, e_n$, such that $e_i e_j = 0$ for $i \neq j$, $e_i^2 = e_i$ and that $\mathfrak{n}_i = (e_1, \cdots, \hat{e}_i, \cdots, e_t)$ for $1 \leqq i \leqq t$. The kernel of the homomorphism of $v[y]$ onto the subring $v[y](e_1 + e_2 + \cdots + e_j)$ of $\mathfrak{S}$ is equal to $\bigcap_{i=1}^{j} \mathfrak{n}_i$, so that the total ring of quotients of $v[y]_{\mathfrak{P}}$ coincides with $\mathfrak{S}(e_1 + e_2 + \cdots + e_j)$.

Those elements in $v[y]_{\mathfrak{P}}$ which are homogeneous and of degree zero form a subring of $v[y]_{\mathfrak{P}}$, hence, a subring of $\mathfrak{S}$. We denote this subring of $\mathfrak{S}$ by $\mathfrak{O}_{\mathfrak{P}}$. We denote by $[V]_v$ the set of the subrings $\mathfrak{O}_{\mathfrak{P}}$ of $\mathfrak{S}$ which are obtained as above. Now we establish some notation and give a lemma which will be useful later on. Assume $y_0 \notin \mathfrak{P}$. We use the same notation y_0 for its image in $v[y]_{\mathfrak{P}}$. Then y_0 is a unit in $v[y]_{\mathfrak{P}}$ and does not satisfy any algebraic relation with coefficients in $\mathfrak{O}_{\mathfrak{P}}$. We denote by $\mathfrak{O}_{\mathfrak{P}}(y_0)$ the ring of quotients of $\mathfrak{O}_{\mathfrak{P}}[y_0]$ with respect to the prime ideal generated by the maximal ideal of $\mathfrak{O}_{\mathfrak{P}}$. Let $\mathfrak{A}$ be an arbitrary homogeneous ideal in $v[y]$ such that $\mathfrak{A} \subseteqq \mathfrak{P}$. Then we denote $\mathfrak{A}v[y]_{\mathfrak{P}} \cap \mathfrak{O}_{\mathfrak{P}}$ by $\mathfrak{A}\mathfrak{O}_{\mathfrak{P}}$. The following lemma can be easily verified.

LEMMA 3. $\mathfrak{O}_{\mathfrak{P}}(y_0) = v[y]_{\mathfrak{P}}$ *and* $\mathfrak{A}\mathfrak{O}_{\mathfrak{P}}(y_0) = \mathfrak{A}v[y]_{\mathfrak{P}}$, *where* $\mathfrak{A}\mathfrak{O}_{\mathfrak{P}}(y_0)$ *denotes the ideal in* $\mathfrak{O}_{\mathfrak{P}}(y_0)$ *generated by* $\mathfrak{A}\mathfrak{O}_{\mathfrak{P}}$ *and* $\mathfrak{A}v[Y]_{\mathfrak{P}}$ *the ideal in* $v[Y]_{\mathfrak{P}}$ *generated by* $\mathfrak{A}$.

Moreover we note that the local ring $\mathfrak{O}_{\mathfrak{P}}$ can be obtained as the ring of quotients of $v[y/y_0] = v[y_1/y_0, y_2/y_0, \cdots y_n/y_0]$ with respect to the prime ideal $\mathfrak{P}v[y]_{\mathfrak{P}} \cap v[y/y_0]$, where the same notation y_i is used for its image in $v[y]_{\mathfrak{P}}$. Such a local ring obtained as a ring of quotients of a finitely generated ring over v will be called *a spot over the ground ring v*.

Let C be a k-prime rational S^n-cycle which is contained in V. In the case that $\mathfrak{P} = I_v(C)/I_v(V)$ we shall use either $\mathfrak{O}_C$ or $\mathfrak{O}_{\mathfrak{P}}$ to denote the ring $\mathfrak{O}_{\mathfrak{P}}$. In this way those spots of $[V]_v$ in which the prime ideal of v is a unit correspond in a one-to-one way to the k-prime rational S^n-cycles which are contained in V.

We denote by $[V]_k$ the set of spots each of which is obtained as a subring of the total ring of quotients of $k[y] = k[Y]/I_k(V)$ by taking all homogeneous elements of degree 0 in $k[y]_{\mathfrak{P}}$ for some homogeneous prime ideal

$$\mathfrak{P} \not\supseteq (y_0, y_1, \cdots, y_n).$$

$[V]_k$ is the subset of $[V]_v$ which consists of all spots of $[V]_v$ in which the prime ideal of v is a unit.

Let $\bar{V}$ be a variety in $\bar{S}^n$ such that $V \xrightarrow{v} \bar{V}$. We denote by $I_v(\bar{V})$ the homogeneous ideal of $\bar{V}$ in $v[Y]$, that is, the ideal in $v[Y]$ generated by all forms $F(Y)$ whose residue classes $\bar{F}(Y)$ belong to the ideal $I_{\bar{v}}(\bar{V})$ in $\bar{v}[Y]$. Obviously $I_v(\bar{V}) \supseteqq p[Y]$ and $I_v(\bar{V})/p[Y] = I_{\bar{v}}(\bar{V})$. We define $[\bar{V}]_{\bar{v}}$ in the same way as we did $[V]_k$. (Note that $\bar{V}$ is $\bar{v}$-rational by Lemma 1.)

Let $\bar{C}'$ be a $\bar{v}$-prime rational $\bar{S}^n$-cycle which is contained in $\bar{V}$. We shall use the notation $\mathfrak{O}_{\bar{C}'}$ for $\mathfrak{O}_{\mathfrak{P}}$ when $\mathfrak{P} = I_v(\bar{C}')/I_v(V)$. In this way the $\bar{v}$-prime rational $\bar{S}^n$-cycles which are contained in $\bar{V}$ correspond in a one-to-one way to those spots of $[V]_v$ in which the prime ideal of v is a nonunit.

LEMMA 4. *Let $\mathfrak{O}$ be a spot which has no zero-divisors. Let π be a nonunit of $\mathfrak{O}$. Suppose that*

(1) *$\pi\mathfrak{O}$ has only one minimal prime ideal $\mathfrak{m}$ and $\pi\mathfrak{O}_{\mathfrak{m}}$ is the maximal ideal of $\mathfrak{O}_{\mathfrak{m}}$;*

(2) *$\mathfrak{O}/\mathfrak{m}$ is a normal local ring.*

Then $\mathfrak{m} = \pi\mathfrak{O}$, and $\mathfrak{O}$ is normal itself.

Proof. Put $\mathfrak{O}' = \mathfrak{O}_{\mathfrak{m}} \cap \mathfrak{O}[1/\pi]$. Since $\mathfrak{O}'$ is contained in $\mathfrak{O}_{\mathfrak{n}}$ for all minimal prime ideals $\mathfrak{n}$ in $\mathfrak{O}$, $\mathfrak{O}'$ is integrally dependent on $\mathfrak{O}$. Therefore $\mathfrak{O}'$ is a finite $\mathfrak{O}$-module by the corollary to Theorem 2, [6]. Let $\mathfrak{m}'$ be a minimal prime ideal of $\pi\mathfrak{O}'$. We have $\mathfrak{O}'_{m'} = \mathfrak{O}_m$ by (1). Since $\pi^{-1}\mathfrak{m}' \subseteqq \mathfrak{O}'_{m'} \cap \mathfrak{O}'[1/\pi] = \mathfrak{O}_m \cap \mathfrak{O}[1/\pi] = \mathfrak{O}'$, we conclude that $\mathfrak{m}' = \pi\mathfrak{O}'$ and $\mathfrak{O}'/\pi\mathfrak{O}' = \mathfrak{O}/\mathfrak{m}$. On the other hand $\mathfrak{O}' \subseteqq \mathfrak{O}[1/\pi]$, and hence there exists an integer $e \geqq 0$ such that $\pi^e\mathfrak{O}' \subseteqq \mathfrak{O}$. Let e be the smallest integer with this property. We want to prove $e = 0$, that is, $\mathfrak{O}' = \mathfrak{O}$. Suppose $e > 0$. $\mathfrak{O}'/\pi\mathfrak{O}' = \mathfrak{O}/\mathfrak{m}$ implies $\mathfrak{O}' = \pi\mathfrak{O}' + \mathfrak{O}$. Therefore $\pi^{e-1}\mathfrak{O}' = \pi^e\mathfrak{O}' + \pi^{e-1}\mathfrak{O} \subseteqq \mathfrak{O}$, which contradicts the minimality of the integer e. Thus $\mathfrak{O}' = \mathfrak{O}$ and $\mathfrak{m} = \pi\mathfrak{O}$. Now let $\tilde{\mathfrak{O}}$ be the derived normal ring of $\mathfrak{O}$. Then $\tilde{\mathfrak{O}}$ is a finite $\mathfrak{O}$-module and, by the above results, $\tilde{\mathfrak{O}}/\pi\tilde{\mathfrak{O}} = \mathfrak{O}/\pi\mathfrak{O}$. Therefore we conclude $\tilde{\mathfrak{O}} = \mathfrak{O}$ by Azumaya's lemma (see §6, [5]).

We say that V (resp. $\bar{V}$) is *k-normal* (resp. *$\bar{v}$-normal*) if every spot of $[V]_k$ (resp. $[\bar{V}]_{\bar{v}}$) is normal. We say that V (resp. $\bar{V}$) is *k-nonsingular* (resp. *$\bar{v}$-nonsingular*) if every spot of $[V]_k$ (resp. $[\bar{V}]_{\bar{v}}$) is regular. A regular spot over a ground ring v in k is said to be *v-unramified regular* if either it contains k or it has a regular system of parameters which includes a prime element of v.

THEOREM 1. *Let V be a k-rational variety in S^n and $\bar{V}$ a $\bar{v}$-rational variety in $\bar{S}^n$ such that $V \xrightarrow{v} \bar{V}$. Let $\bar{C}'$ be a $\bar{v}$-prime rational $\bar{S}^n$-cycle which is contained in $\bar{V}$. Then*

(1) *The ideal of* $\bar{V}$ *in the spot* $\mathfrak{O}_{\bar{C}'}$ *of* $[V]_v$, *that is,* $(I_v(\bar{V})/I_v(V))\mathfrak{O}_{\bar{C}'}$, *always coincides with the radical of* $p\mathfrak{O}_{\bar{C}'}$ *where* p *is the prime ideal of* v, *and it is equal to* $p\mathfrak{O}_{\bar{C}'}$ *if and only if* $p\mathfrak{O}_{\bar{C}'}$ *has no imbedded prime ideals.*

(2) *If the spot of* $[\bar{V}]_{\bar{v}}$ *which corresponds to* $\bar{C}'$ *is normal, then the spot* $\mathfrak{O}_{\bar{C}'}$ *of* $[V]_v$ *is also normal, and* $p\mathfrak{O}_{\bar{C}'}$ *is a prime ideal. If* $\bar{V}$ *is* $\bar{v}$*-normal, then every spot of* $[V]_v$ *is normal, and in particular* V *is* k*-normal.*

(3) *If the spot of* $[\bar{V}]_{\bar{v}}$ *which corresponds to* $\bar{C}'$ *is regular, then the spot* $\mathfrak{O}_{\bar{C}'}$ *is* v*-unramified regular. If* $\bar{V}$ *is* $\bar{v}$*-nonsingular, then every spot of* $[V]_v$ *is* v*-unramified regular, and in particular* V *is* k*-nonsingular.*

Proof. (1) Let $\bar{V}_i'$ $(1 \leqq i \leqq \lambda)$ be the distinct $\bar{v}$-prime rational components of $\bar{V}$. Then $I_v(\bar{V}) = \bigcap_{i=1}^{\lambda} I_v(\bar{V}_i')$. By Lemma 1, $I_v(\bar{V}_i')\,(1 \leqq i \leqq \lambda)$ are also the minimal prime ideals of $(I_v(V), p)v[Y]$. Take an arbitrary component of $\bar{V}$ containing $\bar{C}'$, say $\bar{V}_1'$. Since the coefficient of $\bar{V}_1'$ in the expression of $\bar{V}$ is equal to one, there exists only one k-prime rational component of V, say V_1, such that the specialization $\bar{V}_1$ of V_1 over v contains $\bar{V}_1'$, and, by Lemma 2, p generates the maximal ideal of $\mathfrak{O}_{\bar{V}_1'}$. (Observe that if P_1 is a generic point of V_1 over k and $\bar{P}_1'$ a generic point of $\bar{V}_1'$ over $\bar{v}$, then $\mathfrak{O}_{\bar{V}'} \cong [P_1 \xrightarrow{v} \bar{P}_1']$.) This implies that p generates the maximal ideal in the ring of quotients of $\mathfrak{O}_{\bar{C}'}$ with respect to each minimal prime ideal of $p\mathfrak{O}_{\bar{C}'}$, that is, with respect to every prime ideal of $(I_v(\bar{V})/I_v(V))\mathfrak{O}_{\bar{C}'}$). The assertions in (1) follow immediately.

(2) By (1) the spot of $[\bar{V}]_{\bar{v}}$ which corresponds to $\bar{C}'$ coincides with $\mathfrak{O}_{\bar{C}'}/\mathfrak{m}$, where $\mathfrak{m}$ denotes the radical of $p\mathfrak{O}_{\bar{C}'}$. Since it is normal, $\mathfrak{m}$ is necessarily a prime ideal, and p generates the maximal ideal of the ring of quotients of $\mathfrak{O}_{\bar{C}'}$ with respect to $\mathfrak{m}$. By Lemma 4, therefore, $\mathfrak{O}_{\bar{C}'}$ is normal and $p\mathfrak{O}_{\bar{C}'} = \mathfrak{m}$. If $\bar{V}$ is $\bar{v}$-normal, every spot of $[V]_v$ which corresponds to a $\bar{v}$-prime rational $\bar{S}^n$-cycle is normal. But for every k-prime rational S^n-cycle C which is contained in V, the spot $\mathfrak{O}_C$ of $[V]_v$ is a ring of quotients of $\mathfrak{O}_{\bar{C}'}$ of $[V]_v$ which corresponds to a $\bar{v}$-prime rational component of the specialization $\bar{C}$ of C over v; hence, $\mathfrak{O}_C$ is also normal. Therefore every spot of $[V]_v$ is normal, and V is k-normal.

(3) By assumption $\mathfrak{O}_{\bar{C}'}/\mathfrak{m}$ is regular, hence, normal. By (2), $\mathfrak{m} = p\mathfrak{O}_{C'}$. Therefore, if the residue classes of elements $u_1, u_2, \cdots$, and u_r of $\mathfrak{O}_{\bar{C}'}$ form a regular system of parameters of $\mathfrak{O}_{\bar{C}'}/\mathfrak{m}$, then a prime element of v and u_1, $u_2, \cdots$, and u_r form a regular system of parameters of $\mathfrak{O}_{\bar{C}'}$, i.e., $\mathfrak{O}_{\bar{C}'}$ is v-unramified regular. The last assertion can be proved in the same way as in (2). We note that a ring of quotients of a regular local ring with respect to a prime ideal is always regular, [8].

In this paper we need the factorization theorem in regular local rings, but the author does not know any general proof of this theorem even in the case of v-unramified regular spots.[5] Here we shall give a proof of this theorem under a certain restriction.

[5] In the case of equicharacteristic ground ring v every regular (not necessarily v-unramified) spot over v is factorizable. (See [1].)

LEMMA 5. *Let notations be the same as in Theorem 1. If $\bar{C}'$ contains an absolutely simple point*[6] *of $\bar{V}$, then every minimal prime ideal in $\mathfrak{O}_{\bar{C}'}$ is principal.*

Proof. Let $\bar{P}'$ be a generic point of $\bar{C}'$ over $\bar{v}$. Let w be an extension of v in an algebraic extension of k such that $\bar{w}(\bar{P}')$ is separably generated over $\bar{w}$ ($\bar{w}$ = the residue field of w). Let $\bar{C}''$ be the locus of $\bar{P}'$ over $\bar{w}$. We write $\mathfrak{O}$ instead of $\mathfrak{O}_{\bar{C}'}$ and $\tilde{\mathfrak{O}}$ instead of the spot $\mathfrak{O}_{\bar{C}''}$ of $[V]_w$. By Theorem 1 (3) $\tilde{\mathfrak{O}}$ is w-unramified regular. Since the residue field of $\tilde{\mathfrak{O}}$ is separably generated over $\bar{w}$, the completion $\tilde{\mathfrak{O}}^*$ of $\tilde{\mathfrak{O}}$ is isomorphic to a formal power series over a complete discrete valuation ring; hence, every minimal prime ideal in $\tilde{\mathfrak{O}}^*$ is principal (Corollary to Theorem 2 [1]). The adherence of $\mathfrak{O}$ in $\tilde{\mathfrak{O}}^*$ can be identified with the completion $\mathfrak{O}^*$ of $\mathfrak{O}$. Then $\tilde{\mathfrak{O}}^*$ is a free $\mathfrak{O}^*$-module of finite type. In fact, let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{O}^*$, and let $w_1, w_2, \cdots,$ any w_γ be a set of elements of $\tilde{\mathfrak{O}}^*$ such that their residue classes modulo $\mathfrak{m}\tilde{\mathfrak{O}}^*$ form a base of $\tilde{\mathfrak{O}}^*/\mathfrak{m}\tilde{\mathfrak{O}}^*$ as a vector space over $\mathfrak{O}^*/\mathfrak{m}$. Then $w_1, w_2, \cdots$, and w_γ generate $\tilde{\mathfrak{O}}^*$ over $\mathfrak{O}^*$. Since $\gamma = [\tilde{\mathfrak{O}}^*/\tilde{\mathfrak{m}} : \mathfrak{O}^*/\mathfrak{m}]e(\mathfrak{m}\tilde{\mathfrak{O}}^*) = [\tilde{\mathfrak{O}}^* : \mathfrak{O}^*]e(\mathfrak{m}) = [\tilde{\mathfrak{O}}^* : \mathfrak{O}^*]$ where $\tilde{\mathfrak{m}}$ denotes the maximal ideal of $\tilde{\mathfrak{O}}^*$, we conclude that they form a free base of $\tilde{\mathfrak{O}}^*$ as an $\mathfrak{O}^*$-module.

Now let $\mathfrak{P}$ be an arbitrary minimal prime ideal of $\mathfrak{O}$. We want to prove that $\mathfrak{P}$ is principal. Since $\mathfrak{P}\mathfrak{O}_{\mathfrak{P}}$ is principal, $\mathfrak{P}\tilde{\mathfrak{O}}_S^*$, $S = \mathfrak{O} - \mathfrak{P}$, is also principal, hence, unmixed of rank 1. Let us prove that $\mathfrak{P}\tilde{\mathfrak{O}}^*$ is unmixed of rank 1. Suppose $\mathfrak{P}\tilde{\mathfrak{O}}^*$ has a prime ideal of rank greater than 1. Then such a prime ideal contains an element $a \in S$, and there exists an element b^* of $\tilde{\mathfrak{O}}^*$ such that $b^* \notin \mathfrak{P}\tilde{\mathfrak{O}}^*$ but $ab^* \in \mathfrak{P}\tilde{\mathfrak{O}}^*$. $b^* = \sum_{i=1}^{\gamma} w_i b_i$ with $b_i \in \mathfrak{O}^*$. Since the w_i are free, $ab^* \in \mathfrak{P}\tilde{\mathfrak{O}}^*$ implies $ab_i \in \mathfrak{P}\mathfrak{O}^*$ for all i. This says that all $b_i \in \mathfrak{P}\mathfrak{O}^* : a\mathfrak{O}^* = (\mathfrak{P} : a\mathfrak{O})\mathfrak{O}^* = \mathfrak{P}\mathfrak{O}^*$. This contradicts the assumption that $b^* \notin \mathfrak{P}\tilde{\mathfrak{O}}^*$. By this we conclude that $\mathfrak{P}\tilde{\mathfrak{O}}^*$ is unmixed of rank 1, and therefore that $\mathfrak{P}\tilde{\mathfrak{O}}^*$ is principal. Furthermore we can take a generator f of $\mathfrak{P}\tilde{\mathfrak{O}}^*$ out of $\mathfrak{P}$, for, if $f\tilde{\mathfrak{O}}^* = \mathfrak{P}\tilde{\mathfrak{O}}^*$, then $f = \sum_i f_i c_i^*$ with $f_i \in \mathfrak{P}$ and $c_i^* \in \tilde{\mathfrak{O}}^*$ and each $f_i = d_i^* f$ with $d_i^* \in \tilde{\mathfrak{O}}^*$. Hence $\sum_i d_i^* c_i^* = 1$. Since $\tilde{\mathfrak{O}}^*$ is a local ring, some d_i^* must be a unit in $\tilde{\mathfrak{O}}^*$, and then we may replace f by f_i. Again by means of the free base of $\tilde{\mathfrak{O}}^*$ over $\mathfrak{O}^*$ we can prove that if $f \in \mathfrak{P}$ and $f\tilde{\mathfrak{O}}^* = \mathfrak{P}\tilde{\mathfrak{O}}^*$ then $f\mathfrak{O}^* = \mathfrak{P}\mathfrak{O}^*$ and $f\mathfrak{O} = f\mathfrak{O}^* \cap \mathfrak{O} = \mathfrak{P}\mathfrak{O}^* \cap \mathfrak{O} = \mathfrak{P}$. This completes the proof.

Let Z be a positive k-rational S^n-cycle which is contained in V and of dimension $= \dim V - 1$. We say that Z is a (positive) V-divisor if every k-prime rational component of Z contains absolutely simple points of V. We define and denote by $I_v(Z, V)$ the ideal of (Z, V) in $v[Y]$ in the following way: Let $Z = \sum_i \gamma_i Z_i$ where Z_i are the distinct k-prime rational components of Z. Let $I_v(\gamma_i Z_i, V)$ denote the primary ideal of $(I_v(Z_i)^{\gamma_i}, I_v(V))$ belonging to the prime ideal $I_v(Z_i)$ for each i. Then $I_v(Z, V) = \bigcap_i I_v(\gamma_i Z_i, V)$.

The ideal $I_k(Z, V)$ of (Z, V) in $k[Y]$ is defined in a similar way; we have $I_k(Z, V) = I_v(Z, V)k[Y]$ and $I_k(Z, V) \cap v[Y] = I_v(Z, V)$.

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[6] See [10].

Let $\bar{Z}$ be a positive $\bar{S}^n$-cycle such that $(Z, V) \xrightarrow{v} (\bar{Z}, \bar{V})$. Hereafter we assume that $\bar{Z}$ is a positive $\bar{V}$-divisor. Then we define and denote by $I_v(\bar{Z}, \bar{V})$ the ideal of $(\bar{Z}, \bar{V})$ in $v[Y]$ in the same way as $I_v(Z, V)$, that is,

$$I_v(\bar{Z}, \bar{V}) = \cap_j I_v(\gamma_j' \bar{Z}_j', \bar{V})$$

where $\bar{Z}_j'$ are the distinct $\bar{v}$-prime rational components of $\bar{Z}$, $\bar{Z} = \sum \gamma_j' \bar{Z}_j'$ and $I_v(\gamma_j' \bar{Z}_j', \bar{V}) =$ the primary ideal of $(I_v(\bar{Z}_j')^{\gamma_j'}, I_v(\bar{V}))\, v[Y]$ belonging to the prime ideal $I_v(\bar{Z}_j')$ for each j. It is easily verified that $I_v(\bar{Z}, \bar{V})/p[Y]$ coincides with the ideal $I_{\bar{v}}(\bar{Z}, \bar{V})$ of $(\bar{Z}, \bar{V})$ in $\bar{v}[Y]$.

THEOREM 2. *Let V be a k-rational variety in S^n and $\bar{V}$ a $\bar{v}$-rational variety in $\bar{S}^n$. Let Z be a k-rational positive V-divisor and $\bar{Z}$ a positive $\bar{V}$-divisor such that $(Z, V) \xrightarrow{v} (\bar{Z}, \bar{V})$. Let $\bar{C}'$ be a $\bar{v}$-prime rational $\bar{S}^n$-cycle which is contained in $\bar{Z}$.*

Let us consider two ideals in the spot $\mathfrak{O}_{\bar{C}'}$ of $[V]_v$ as follows: one, denoted by $\mathfrak{A}$, is the ideal of $\bar{Z}$ in $\mathfrak{O}_{\bar{C}'}$, that is, $(I_v(\bar{Z}, \bar{V})/I_v(V))\mathfrak{O}_{\bar{C}'}$; and the other, denoted by $\mathfrak{B}$, is the ideal which is generated by both the prime ideal p of v and the ideal of Z in $\mathfrak{O}_{\bar{C}'}$, that is, $(I_v(Z, V)/I_v(V))\mathfrak{O}_{\bar{C}'}$.

Then $\mathfrak{A}$ and $\mathfrak{B}$ have the same minimal prime ideals and the same primary ideals belonging to the minimal prime ideals, and $\mathfrak{A}$ coincides with $\mathfrak{B}$ if and only if $\mathfrak{B}$ has no imbedded prime ideals. Moreover if $\bar{C}'$ contains absolutely simple points of $\bar{V}$, then $\mathfrak{A}$ coincides with $\mathfrak{B}$.

Proof. First let us consider the case when $\bar{C}'$ contains absolutely simple points of $\bar{V}$. By Theorem 1 (3) and Lemma 5, $\mathfrak{O}_{\bar{C}'}$ is v-unramified regular and factorizable. Therefore the prime ideal of Z_i in $\mathfrak{O}_{\bar{C}'}$, that is,

$$(I_v(Z_i)/I_v(V))\mathfrak{O}_{\bar{C}'},$$

is principal; let f_i be a generator of this ideal. Then it is proved easily that the ideal of Z in $\mathfrak{O}_{\bar{C}'}$, that is, $(I_v(Z, V)/I_v(V))\mathfrak{O}_{\bar{C}'}$, is equal to $(\prod_i f_i^{\gamma_i})\mathfrak{O}_{\bar{C}'}$. Therefore the ideal $\mathfrak{B}$ is equal to $(\prod_i f_i^{\gamma_i}, p)\mathfrak{O}_{\bar{C}'}$ and hence unmixed. Thus we have proved that the last statement of the theorem follows from the first one. Let us consider the case when $\bar{C}'$ is a $\bar{v}$-prime rational component of $\bar{Z}$. We note that every component of $\bar{Z}$ contains absolutely simple points of $\bar{V}$. Some f_i in the above reasoning may be units in $\mathfrak{O}_{\bar{C}'}$; in fact f_i is a unit in $\mathfrak{O}_{\bar{C}'}$ if and only if the specialization $\bar{Z}_i$ of Z_i over v does not contain $\bar{C}'$. Hereafter we omit such f_i. By Lemma 2 the coefficient of $\bar{C}'$ in the expression of $\bar{Z}_i$ is equal to the multiplicity $e((p, f_i)\mathfrak{O}_{\bar{C}'}/f_i\mathfrak{O}_{\bar{C}'})$ for each i. Therefore by the associativity formula (Theorem 8, [7]) we have $e((p, \prod_i f_i^{\gamma_i})\mathfrak{O}_{\bar{C}'})$ is equal to the coefficient of $\bar{C}'$ in the expression of $\bar{Z}$. Let γ' be this integer. Since the ideal of $\bar{V}$ in $\mathfrak{O}_{\bar{C}'}$ is generated by p (Theorem 1 (1)) and the maximal ideal $(I_v(\bar{C}')/I_v(V))\mathfrak{O}_{\bar{C}'}$ is generated by two elements one of which belongs to p, we can see that the ideal $(I_v(\gamma'\bar{C}', \bar{V})/I_v(V))\mathfrak{O}_{\bar{C}'}$ coincides with $(p, \prod_i f_i^{\gamma_i})\mathfrak{O}_{\bar{C}'}$. Therefore it is easily verified that $\mathfrak{A}$ and $\mathfrak{B}$ coincide for the $\bar{v}$-prime rational component $\bar{C}'$ of $\bar{Z}$. This result shows that for general $\bar{C}'$ in the theorem the two ideals $\mathfrak{A}$ and $\mathfrak{B}$ have the same primary ideals belonging to the minimal prime ideals of $\mathfrak{A}$, but the fact that $\mathfrak{A}$ and $\mathfrak{B}$ have the same minimal prime ideals follows directly from Lemma 1. Thus the proof of the theorem is completed.

Part II

In this part we shall discuss what condition should be imposed on the spots of $[V]_v$ in order that the variety V (resp. the V-divisor, Z) and the variety $\bar{V}$ (resp. the $\bar{V}$-divisor, $\bar{Z}$) have the same Hilbert characteristic function.

The Hilbert characteristic function $\chi_V(m)$ of V is a polynomial in m, whose value for m sufficiently large is equal to $\dim_k[k[Y]/I_k(V)]_m$,[7] where $[k[Y]/I_k(V)]_m$ denotes the k-module of all homogeneous elements of degree m in $k[Y]/I_k(V)$. We can prove that

$$\chi_V(m) = \dim_{\bar{v}}[v[Y]/(I_v(V),\, p)v[Y]]_m \tag{1}$$

for m sufficiently large. In fact the v-submodule $I_v(V)_m$ of $v[Y]_m$ is of finite type and torsion free, hence, a free v-module. On the other hand, since $I_v(V):p = I_v(V)$ as is easily seen, $v[Y]_m/I_v(V)_m$ is also torsion free as well as of finite type, and therefore it is a free v-module. By choosing a set of elements in $v[Y]_m$ which forms a free base of $v[Y]_m/I_v(V)_m$ modulo $I_v(V)_m$ and by joining them to a free base of $I_v(V)_m$, we can make a free base of $v[Y]_m$. Let $\{M_1, M_2, \cdots, M_\mu, N_1, N_2, \cdots, N_\nu\}$ be such a free base of $v[Y]_m$ where $\{N_1, \cdots, N_\nu\}$ is a free base of $I_v(V)_m$. Then it is easily verified that the set $\{M_1, M_2, \cdots, M_\mu\}$ forms a free base of k-module $[k[Y]/I_k(V)]_m$ modulo $I_k(V)$ and also of $\bar{v}$-module $[v[Y]/(I_v(V), p)v[Y]]_m$ modulo $(I_v(V), p)v[Y]$, and the equality (1) follows.

On the other hand the Hilbert characteristic function $\chi_{\bar{V}}(m)$ of $\bar{V}$ is equal to $\dim_{\bar{v}}[\bar{v}[Y]/I_{\bar{v}}(\bar{V})]_m$ for m sufficiently large, by definition. Therefore we have

$$\chi_{\bar{V}}(m) = \dim_{\bar{v}}[v[Y]/I_v(\bar{V})]_m \tag{2}$$

for m sufficiently large.

Next let us consider the case of the specialization $(Z, V)\xrightarrow{v}(\bar{Z}, \bar{V})$. The *Hilbert characteristic function of the positive V-divisor Z* is defined and denoted by $\chi_{Z,V}(m)$ as a polynomial in m whose values are equal to

$$\dim_k[k[Y]/I_k(Z, V)]_m$$

for m sufficiently large.[8] The same reasoning as that used in proving the equalities (1) and (2) is applicable to verify the following equalities:

$$\chi_{Z,V}(m) = \dim_{\bar{v}}[v[Y]/(I_v(Z, V),\, p)v[Y]]_m \tag{1)*}$$

for m sufficiently large, and

$$\chi_{\bar{Z},\bar{V}}(m) = \dim_{\bar{v}}[v[Y]/I_v(\bar{Z}, \bar{V})]_m \tag{2)*}$$

for m sufficiently large.

[7] This value is independent of k if V is k-rational.

[8] This value is independent of k if both V and Z are k-rational.

These results arouse our interest in the relation between the two ideals $(I_v(V), p)v[Y]$ and $I_v(\bar{V})$ in the former case, and that between $(I_v(Z, V), p)v[Y]$ and $I_v(\bar{Z}, \bar{V})$ in the latter case.

As for the former case Theorem 1 (2) implies that the two ideals $(I_v(V), p)v[Y]$ and $I_v(\bar{V})$ have the same minimal prime ideals and that for every minimal prime ideal I of them we have $(I_v(\bar{V})/I_v(V))\mathfrak{O}_{\mathfrak{P}} = p\mathfrak{O}_{\mathfrak{P}}$, where $\mathfrak{O}_{\mathfrak{P}}$ is the spot of $[V]_v$ which corresponds to the prime ideal $\mathfrak{P} = I/I_v(V)$ in $v[Y]/I_v(V)$. Therefore by means of Lemma 3 we have

$$I_v(\bar{V})v[Y]_I = (I_v(V), p)v[Y]_I,$$

and we can express the general relation between the two ideals by writing

$$(I_v(V), p)v[Y] = I_v(\bar{V}) \cap J$$

with a homogeneous ideal J in $v[Y]$ every prime ideal of which is properly imbedded in some of the prime ideals of $I_v(\bar{V})$. The equalities (1) and (2) imply that we have $\chi_V(m) = \chi_{\bar{V}}(m)$ if and only if $[(I_v(V), p)v[Y]]_m = I_v(\bar{V})_m$ for m sufficiently large, or equivalently (as is easily verified) if and only if the ideal J is irrelevant in the sense that J contains all monomials in Y's of sufficiently large degree.

THEOREM 3. *Varieties V in S^n and $\bar{V}$ in $\bar{S}^n$, such that $V \xrightarrow{v} \bar{V}$, have the same Hilbert characteristic function if and only if the prime ideal p of v generates an unmixed ideal in every spot of $[V]_v$. Moreover if $\bar{V}$ is $\bar{v}$-normal, then V and $\bar{V}$ have the same Hilbert characteristic function, and V is k-normal.*

Proof. Put $v[y] = v[Y]/I_v(V)$, and let us take an arbitrary homogeneous prime ideal $\mathfrak{P}$ in $v[y]$ which does not contain all the y's. Then, by means of Lemma 3, $p\mathfrak{O}_{\mathfrak{P}}$ is unmixed if and only if $pv[y]_{\mathfrak{P}}$ is unmixed. Therefore p generates an unmixed ideal in every spot of $[V]_v$ if and only if $pv[y]_{\mathfrak{P}}$ is unmixed for all such $\mathfrak{P}$. It is now easily seen that the previous result implies the first assertion of the theorem. The last half of the theorem follows directly from Theorem 1 (2).

Now let us consider the latter case, that is, the case of the specialization $(Z, V) \xrightarrow{v} (\bar{Z}, \bar{V})$. Theorem 2 implies that the two ideals $(I_v(Z, V), p)v[Y]$ and $I_v(\bar{Z}, \bar{V})$ have the same minimal prime ideals and that for every minimal prime ideal I of them we have $(I_v(\bar{Z}, \bar{V})/I_v(V))\mathfrak{O}_{\mathfrak{P}} = (I_v(Z, V)/I_v(V), p)\mathfrak{O}_{\mathfrak{P}}$, where $\mathfrak{O}_{\mathfrak{P}}$ is the spot of $[V]_v$ which corresponds to the prime ideal $\mathfrak{P} = I/I_v(V)$ in $v[Y]/I_v(V)$. Therefore by means of Lemma 3 we have

$$I_v(\bar{Z}, \bar{V})v[Y]_I = (I_v(Z, V), p)v[Y]_I,$$

and we have the general relation

$$(I_v(Z, V), p)v[Y] = I_v(\bar{Z}, \bar{V}) \cap J^*$$

with a homogeneous ideal J^* in $v[Y]$ each of whose prime ideals is properly imbedded in at least one of the prime ideals of $I_v(\bar{Z}, \bar{V})$. The equalities

(1)* and (2)* imply that we have $\chi_{Z,V}(m) = \chi_{\bar{Z},\bar{V}}(m)$ if and only if the ideal J^* is irrelevant.

THEOREM 4. *A positive V-divisor Z in S^n and a positive $\bar{V}$-divisor $\bar{Z}$ in $\bar{S}^n$, such that $(Z, V) \xrightarrow{v} (\bar{Z}, \bar{V})$, have the same characteristic function if and only if the prime ideal p of v and the ideal of Z generate an unmixed ideal in every spot of $[V]_v$. Moreover if $\bar{Z}$ does not contain any singular point of $\bar{V}$ (in the absolute sense), then they have the same Hilbert characteristic function, and Z does not contain any singular point of V.*

Proof. The same reasoning as that used in the proof of Theorem 3 is applicable to prove this theorem. Namely, for every homogeneous prime ideal $\mathfrak{P}$ of $v[y]$ which contains $I_v(\bar{Z}, \bar{V})/I_v(V)$ but does not contain all the y's, p and the ideal of Z generate an unmixed ideal in the spot $\mathfrak{O}_{\mathfrak{P}}$ of $[V]_v$ if and only if $(I_v(Z, V)/I_v(V), p)v[y]_{\mathfrak{P}}$ is unmixed. Therefore the previous result implies the first assertion of the theorem. The other assertions of the theorem follow from Theorem 2 and Theorem 1 (3).

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KYOTO UNIVERSITY
KYOTO, JAPAN
HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS