CORRECTIONS TO THE PAPER "ON WITT VECTORS AND PERIODIC GROUP-VARIETIES"

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J-P. Serre has kindly called to the author's attention (in a letter of May 20, 1958) the presence of two mistakes in the paper mentioned in the title (Illinois J. Math., vol. 2 (1958) pp. 99–110; henceforth quoted as Wv). The first mistake is the presence of the word "inseparably" in the statement of Theorem (9) of Wv: as a counterexample, in characteristic 2 for shortness, we can consider the 2-dimensional periodic group-variety with n.h.g.p. $\{x, y\}$ and law of composition $(x, y) + (x', y') = (x + x', y + y' + xx' + x^2x'^2)$; it has period 4, but is not inseparably isogenous to a Witt variety since its maximal group-subvariety of period 2 is reducible; this counterexample is essentially due to Serre. The error entered the proof by way of the statement about inseparability contained in the last two lines of p. 105; this statement is gratuitous, and all other statements concerning inseparability which appear in the proof of (9) should be deleted. Corollary (10), and the remark which follows it, remain true for those group-varieties which are inseparably isogenous to Witt varieties. No other consequence of this error is found in Wv.

Theorem (9), as amended, is now true, but its last sentence is proved, in the first three lines of p. 106, only when G is a homomorphic image of $W_n(k)$ in a purely inseparable homomorphism; in order to prove it completely, it is sufficient to show that *if G, of dimension n, is a homomorphic image of $W_n(k)$ in a separable homomorphism σ, then $W_n(k)$ is a homomorphic image of G.* The proof of this statement can be reduced, in succession, first to the case in which the kernel of σ is generated by points of period p^n, and then to the case in which such kernel is generated by one point of period p^n; in this last case, the proof is as follows: if $X \in W_n(k)$ has period p^n, X can be considered as a unit of the ring R of Witt vectors of length n with elements in k, and therefore X^{-1} exists in R; if $Y = X^{-1}\pi X$, then the mapping $Z \to (Y - \pi)Z$ is an endomorphism of $W_n(k)$, whose kernel contains X; this endomorphism is onto $W_n(k)$, for otherwise its kernel should contain all Witt vectors of the type $(0, \cdots, 0, x)$, and it is readily seen that this is not the case.

The other mistake is of a more serious nature, and voids the claim, made in the footnote on p. 99 of Wv, of having improved on the result mentioned in that footnote; more precisely, Lemma (12) in the appendix of Wv is false (the error being contained in the 5th line on p. 108), and as a consequence (13) and (14) are false. A counterexample to (14) will be given at the end of this note. Something of the appendix of Wv can be rescued, and used to prove the statement mentioned in the footnote on the title page of Wv.

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We first remark that a group-variety G over k (algebraically closed) is a homomorphic image, in a homomorphism of positive degree, of a direct product $A \times B$, if and only if there exist irreducible group-subvarieties A', B' of G, homomorphic images of A, B respectively in homomorphisms of positive degree, such that (a) the product of any point (nondegenerate) of A' and any point of B' is commutative; (b) $G = (A', B')$ = *smallest group-subvariety of* G *containing* A' *and* B'; (c) $A' \cap B'$ contains only finitely many nondegenerate points. When this is the case, we shall say that G is a *complementary product of* A' *and* B'; the definition can be extended naturally to more than two *complementary factors*. We also remark that, by (9) of Wv, a group-variety is a complementary product of varieties of Witt type if and only if it is isogenous to a direct product of Witt varieties. Finally, if V, W are complementary products of varieties of Witt type, and $\gamma \in \Gamma(V, W)$, the following three statements are equivalent:

(A) $\{V, W, \gamma\}$ is a homomorphic image of $V \times W$ in a homomorphism which maps W onto W, and is therefore isogenous to $V \times W$;

(B) W is a complementary factor of $\{V, W, \gamma\}$;

(C) $\gamma(\alpha \times \alpha_1) \in \Gamma_0(V, W)$ for a suitable endomorphism α, of positive degree, of V onto V.

The γ's with property (C) form a group, denoted by $\Gamma_1(V, W)$. We can now prove, in place of (13) of Wv:

LEMMA. *Let V, W be varieties of Witt type, V being* 1-*dimensional; then either* $\{V, W, \gamma\}$ *is of Witt type, or* $\gamma \in \Gamma_1(V, W)$.

Proof. If $\{V, W, \gamma\}$ is not of Witt type, according to the first part of the proof of (13) of Wv we may assume γ to operate on $U = pW$; from this fact we can easily derive that the maximal vector subvariety of $\{V, W, \gamma\}$ has dimension 2; if we write it in the form (L, L'), with L' = *maximal vector subvariety of* W, and L a 1-dimensional vector variety, we conclude that $\{V, W, \gamma\} = (L, W)$ is a complementary product of L and W, Q.E.D.

The replacement of (14) of Wv will now be as follows:

THEOREM. *Let A be a periodic group-variety of period p^n; then A is isogenous to a direct product of Witt varieties, or, equivalently, A is a complementary product of varieties of Witt type; in particular, A possesses n-dimensional group-subvarieties of Witt type, and any one of these is a complementary factor of A.*

Proof. The proof, based on the previous lemma rather than on (13) of Wv, follows very closely the "proof" of (14) of Wv: the first statement is a consequence of the second; the two parts of the second statement will be proved by recurrence on dim A. If $X = pA$, X has period p^{n-1}; if X is not of Witt type, by the recurrence assumption X is a complementary product of Y and Z, where Y is of Witt type and dimension $n - 1$, while dim $Z > 0$; after setting W = *component of the identity in* $p^{-1}Y$, we have $W \subset A$; since W has period p^n, it possesses an n-dimensional group-subvariety of Witt type; it follows that A has the same property. If instead X is of Witt type, $L = A/X$ has period p, and is therefore isomorphic to a direct product $V_1 \times \cdots \times V_r$ of 1-dimensional vector varieties; but then

$$A \cong \{V_1 \times \cdots \times V_r, X, \gamma_1 \times \cdots \times \gamma_r\},$$

where $\gamma_i \in \Gamma(V_i, X)$, and $\gamma_i \notin \Gamma_1(V_i, X)$ for at least one value of i, say $i = 1$ (otherwise A would be isogenous to $L \times X$ and would have period p^{n-1}). Thus $\{V_1, X, \gamma_1\}$ is a group-subvariety of A, of dimension n, and it is of Witt type by the previous lemma.

Having now established that A possesses an n-dimensional group-subvariety W of Witt type, if $\dim A > n$ let B be an irreducible group-subvariety of A, containing W, and having dimension equal to $\dim A - 1$. Then, by the recurrence assumption, B is a complementary product (W, C), so that there is a homomorphism of A onto $\{V, W \times C, \delta_0 + \delta_1\}$ (mapping B onto $W \times C$ and W onto W), where V is a 1-dimensional vector variety, $\delta_0 \in \Gamma(V, W)$, and $\delta_1 \in \Gamma(V, C)$; now, $\{V, W, \delta_0\}$ is a homomorphic image of A/C, and has therefore period p^n, so that, by the lemma, $\delta_0 \in \Gamma_1(V, W)$; hence $\{V, W \times C, \delta_0 + \delta_1\}$ is a homomorphic image of a $\{V, C, \delta'\} \times W$, and is therefore isogenous to it; the same is then true of A, Q.E.D.

We shall now give a counterexample to Theorem (14) in its original form, again in characteristic 2 for brevity: let G be the 4-dimensional group-variety with n.h.g.p. $\{t, x, y, z\}$ and law of composition $(t, x, y, z) + (t', x', y', z') = (t + t', x + x', y + y' + tt', z + z' + xx' + y^2y'^2 + t^2t'^2(y^2 + y'^2 + t^4 + t'^4))$; this has period 8. Were it the direct product of varieties of Witt type, we would have, but for isomorphisms, $G = V \times W$, with V, W of Witt type and dimensions 1, 3, respectively. Then $H' = 2G$, which is 2-dimensional of equations $t = x = 0$ and period 4, should be of the type $E_V \times H$, with $H = 2W$; now, $K = V \times H$ coincides with (L, H'), L being the maximal vector subvariety of G; since L has equations $t = x + y^2 = 0$, K has equation $t = 0$, and should have H as a direct factor, so that E_K should be a simple intersection of $V \times E_H$ and H' on K. But this cannot be, since $t = x = y = 0$ is a double intersection, on K, of H' with L.

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