LINEAR SYSTEMS OF FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

BY

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Consider the system of linear differential equations

$$(1) \qquad \begin{cases} u'' + A_1(\lambda)u = \lambda f_1(u, v, w, u', v', t, \lambda), \\ v'' + A_2(\lambda)v = \lambda f_2(u, v, w, u', v', t, \lambda), \\ \qquad\quad w' = \lambda f_3(u, v, w, u', v', t, \lambda), \end{cases}$$

where λ is a real parameter,

$$u = (y_1, \cdots, y_\nu), \qquad v = (y_{\nu+1}, \cdots, y_\mu) \qquad w = (y_{\mu+1}, \cdots, y_n),$$

$$f_1 = (f_1^*, \cdots, f_\nu^*), \qquad f_2 = (f_{\nu+1}^*, \cdots, f_\mu^*), \qquad f_3 = (f_{\mu+1}^*, \cdots, f_n^*),$$

$A_1(\lambda) = \operatorname{diag}(\sigma_1^2, \cdots, \sigma_\nu^2)$, $A_2(\lambda) = \operatorname{diag}(\sigma_{\nu+1}^2, \cdots, \sigma_\mu^2)$, and the vector functions f_1, f_2, f_3 are *linear* functions of u, v, w, u', v'. The coefficients in these linear functions are real, periodic functions of t of period $T = 2\pi/\omega$, L-integrable in $[0, T]$, analytic in λ, and have mean value zero. Further, suppose that each $\sigma_j(\lambda)$, $j = 1, 2, \cdots, \mu$, is a real positive analytic function of λ with $\sigma_j(0) \pm \sigma_h(0) \neq m\omega$, $j \neq h$, $j, h = 1, 2, \cdots, \mu$, $\sigma_h(0) \neq m\omega$, $h = 1, 2, \cdots, \mu$, $m = 1, 2, \cdots$. Systems of type (1) for $|\lambda|$ small have recently been extensively investigated by a method which has been successively developed by L. Cesari, J. K. Hale and R. A. Gambill for both linear [1, 3, 4, 6, 8] and weakly nonlinear differential systems [2, 5, 7]. Most of the previous work has been concerned with systems of type (1) without the third vector equation, i.e., with systems of second order equations. The aim of the present paper is to prove a theorem concerning the boundedness of the AC (absolutely continuous) solutions of (1). By applying the same methods, the following theorem is proved:

THEOREM. *If*

(α) $f_1(u, -v, w, -u', v', -t, \lambda) = f_1(u, v, w, u', v', t, \lambda)$,

(β) $f_2(u, -v, w, -u', v', -t, \lambda) = -f_2(u, v, w, u', v', t, \lambda)$, *and*

(γ) $f_3(u, -v, w, -u', v', -t, \lambda) = -f_3(u, v, w, u', v', t, \lambda)$,

then for $|\lambda|$ *sufficiently small, all the AC solutions of* (1) *are bounded in* $(-\infty, +\infty)$.

This theorem generalizes some previous results of the author [8] for systems of linear equations of type (1) where the third vector equation did not

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appear. A different notation is used in this paper to simplify the presentation.

This theorem is proved by showing there is a fundamental system of AC solutions of (1) which are bounded for all values of t. More specifically, it is shown that the first 2μ characteristic exponents of (1) are purely imaginary and the remaining $n - \mu$ are zero.

By the transformation of variables

$$(2)\qquad \begin{aligned} y_j &= (1/2i\sigma_j)(z_{2j-1} + z_{2j}), \quad y_j' = \tfrac{1}{2}(z_{2j-1} - z_{2j}), \quad && j = 1, 2, \cdots, \mu, \\ y_k &= z_{\mu+k}, && k = \mu + 1, \cdots, n, \end{aligned}$$

system (1) is equivalent to the first order system

$$(3)\qquad z' = Az + \lambda g(z, t, \lambda),$$

where $z = (z_1, \cdots, z_{n+\mu})$, $A = \operatorname{diag}(\sigma_1^*, \cdots, \sigma_{n+\mu}^*)$, $\sigma_{2j-1}^* = i\sigma_j$, $\sigma_{2j}^* = -i\sigma_j$, $j = 1, 2, \cdots, \mu$, $\sigma_{2\mu+j}^* = 0$, $j = 1, 2, \cdots, n - \mu$, and $g = (g_1, \cdots, g_{n+\mu})$, $g_{2j-1} = f_j^*[(1/2i\sigma_1)(z_1 + z_2), \cdots, (1/2i\sigma_\mu)(z_{2\mu-1} + z_{2\mu}), z_{2\mu+1}, \cdots, z_{n+\mu}, \frac{1}{2}(z_1 - z_2), \cdots, \frac{1}{2}(z_{2\mu-1} - z_{2\mu}), t, \lambda]$, $g_{2j} = -g_{2j-1}$, $j = 1, 2, \cdots, \mu$; $g_{2\mu+j} = f_{\mu+j}^*$, $j = 1, 2, \cdots, n - \mu$.

By considering an auxiliary equation of (3),

$$(4)\qquad z' = Bz + \lambda g(z, t, \lambda),$$

transforming it into an integral equation, and employing the method mentioned above, we obtain AC solutions of the equation

$$(5)\qquad z' = (B - \lambda D)z + \lambda g(z, t, \lambda),$$

where D is a constant diagonal matrix which depends on B, g and λ. Then, by determining B so that

$$(6)\qquad B - \lambda D = A,$$

the obtained solutions of (5) become solutions of (3).

In the following, let C_ω denote the family of all functions which are finite sums of functions of the form $f(t) = e^{\alpha t}\phi(t)$, $-\infty < t < +\infty$, where α is any complex number and $\phi(t)$ is any complex-valued function of the real variable t, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$. If $\phi(t)$ has a Fourier series, $\phi(t) \sim \sum_{n=-\infty}^{+\infty} C_n e^{in\omega t}$, then the series

$$(7)\qquad f(t) = e^{\alpha t}\phi(t) \approx \sum_{n=-\infty}^{+\infty} C_n e^{(in\omega+\alpha)t}$$

is the series associated with $f(t)$. Moreover, in harmony with [1] and [6], the *mean value* $M[f]$ of $f(t)$ is the number $M[f] = 0$ if $in\omega + \alpha \neq 0$ for all n, $M[f] = C_n$ if $in\omega + \alpha = 0$ for some n. It is known [1, 6] that if $f(t) \in C_\omega$ and $M[f] = 0$, then there is one and only one primitive of $f(t)$, say $F(t)$, which belongs to C_ω and such that $M[F] = 0$.

Put $B = \operatorname{diag}(\rho_1, \cdots, \rho_{n+\mu})$ where $\rho_{2j-1}(\lambda) = i\tau_j(\lambda)$, $\rho_{2j}(\lambda) = -i\tau_j(\lambda)$, $j = 1, 2, \cdots, \mu$, $\rho_{2\mu+k} = 0$, $k = 1, 2, \cdots, n - \mu$; each τ_j is a real positive analytic function of λ with $\tau_j(0) \pm \tau_h(0) \neq m\omega$, $j \neq h$, $j, h = 1, 2, \cdots, \mu$, $\tau_h(0) \neq m\omega$, $h = 1, 2, \cdots, \mu$, $m = 1, 2, \cdots$. Let

$$z^{(m)} = x^{(0)} + \lambda x^{(1)} + \cdots + \lambda^m x^{(m)} \tag{8}$$

denote the m^{th} approximation to a solution of (4), and define the method of successive approximations as follows:

$$x^{(0)} = (a_1 e^{\rho_1 t}, \cdots, a_{n+\mu} e^{\rho_{n+\mu} t})$$

$$x^{(m)} = e^{Bt} \int e^{-B\alpha}[g(x^{(m-1)}, \alpha, \lambda) - (D^{(1)}x^{(m-1)} + \cdots + D^{(m)}x^{(0)})] \, d\alpha, \quad m = 1, 2, \cdots, \tag{9}$$

where $e^{Bt} = \operatorname{diag}(e^{\rho_1 t}, \cdots, e^{\rho_{n+\mu} t})$, $a_1, \cdots, a_{n+\mu}$ are complex numbers, and the matrix $D^{(r)}$ is defined by

$$\begin{aligned} a_j d_j^{(r)} &= M[e^{-\rho_j t} g_j(x^{(r-1)}, t, \lambda)], && \text{if } a_j \neq 0, \\ d_j^{(r)} &= 0 && \text{for any } a_j \text{ if } M[\cdots] = 0, \\ D^{(r)} &= \operatorname{diag}(d_1^{(r)}, \cdots, d_{n+\mu}^{(r)}), \end{aligned} \tag{10}$$

and the integrations are performed so as to obtain the unique primitive of mean value zero. In definition (10), it is to be understood that if $M[e^{-\rho_j t} g_j(x^{(r-1)}, t, \lambda)] \neq 0$ for any r, then the corresponding a_j is chosen $\neq 0$. It is clear that the integrand belongs to the class C_ω of functions and has mean value zero; consequently, there is a unique primitive of mean value zero. This method of successive approximations is exactly the same as the one defined by L. Cesari [1] except in his paper none of the ρ's were allowed to be zero. The proof of convergence of the method to a solution of an equation of the form (5) may be supplied in the same way as described in [1] or [6].

It is first shown that by a proper choice of the constants $a_1, \cdots, a_{n+\mu}$ the numbers $d_j^{(r)}$ are such that $d_{2h-1}^{(r)} = \bar{d}_{2h}^{(r)}$, $h = 1, 2, \cdots, \mu$ (the overbar denotes the complex conjugate), $d_h^{(r)} = \bar{d}_h^{(r)}$, $h = 2\mu + 1, \cdots, n + \mu$, $r = 1, 2, \cdots$ for every system of type (4). Under the conditions of the theorem and some additional restrictions on $a_1, \cdots, a_{n+\mu}$, it is then shown that $d_{2h-1}^{(r)} = -d_{2h}^{(r)}$, $h = 1, 2, \cdots, \mu$, $d_h^{(r)} = 0$, $h = 2\mu + 1, \cdots, n + \mu$, $r = 1, 2, \cdots$. Consequently, the system of $n + \mu$ equations (6) reduces to the μ equations

$$i\tau_h - \lambda d_{2h-1} = i\sigma_h, \qquad h = 1, 2, \cdots, \mu,$$

where each $d_{2h-1} = \sum_{r=1}^{\infty} \lambda^{r-1} d_{2h-1}^{(r)}$, $h = 1, 2, \cdots, \mu$, is purely imaginary. From the implicit function theorem, there exist real numbers $\tau_1, \cdots, \tau_\mu$ analytic in λ for $|\lambda|$ sufficiently small satisfying the above system of equations and $\tau_h = \sigma_h + O(\lambda)$. Consequently, there will be a solution of (3) with components z_j of the form $\sum_{m=1}^{\infty} \lambda^m x_j^{(m)}(t)$, where $x_j^{(m)}$ is given by (9). It is clear that such a solution is bounded in $(-\infty, +\infty)$ and AC. The final step in the proof of the theorem is to show that the above solutions yield $n + \mu$ linearly independent bounded AC solutions of (1).

By induction, it is very easy to prove the following (which assumes only that system (1) is real):

LEMMA 1. *If the algorithm* (9) *is applied to system* (4) *with* $a_{2j-1} = b_j$, $a_{2j} = -\bar{b}_j$, $j = 1, 2, \cdots, \mu$, $a_{2\mu+k} = b_{\mu+k}$, $b_{\mu+k}$ real, $k = 1, 2, \cdots, n - \mu$, *then*

$$x_{2j-1}^{(r)} = -\bar{x}_{2j}^{(r)}, \qquad d_{2j-1}^{(r)} = \bar{d}_{2j}^{(r)}, \qquad j = 1, 2, \cdots, \mu,$$

$$x_{2\mu+k}^{(r)} = \bar{x}_{2\mu+k}^{(r)}, \qquad d_{2\mu+k}^{(r)} = \bar{d}_{2\mu+k}^{(r)}, \qquad k = 1, 2, \cdots, n - \mu,$$

$$r = 1, 2, \cdots .$$

LEMMA 2. *If* f_1, f_2, f_3 *satisfy the conditions of the theorem and if the numbers* b_j *of Lemma* 1 *satisfy* $b_j = ic_j$, c_j *real,* $j = 1, 2, \cdots, \nu$, $b_j = c_j$, c_j *real,* $j = \nu + 1, \cdots, n + \mu$, *then*

$$x_{2j-1}^{(r)}(-t) = x_{2j}^{(r)}(t), \qquad d_{2j-1}^{(r)} = -d_{2j}^{(r)}, \qquad j = 1, 2, \cdots, \nu,$$

$$x_{2j-1}^{(r)}(-t) = -x_{2j}^{(r)}(t), \qquad d_{2j-1}^{(r)} = -d_{2j}^{(r)}, \qquad j = \nu + 1, \cdots, \mu,$$

$$x_j^{(r)}(-t) = x_j^{(r)}(t), \qquad d_j^{(r)} = 0, \qquad j = 2\mu + 1, \cdots, n + \mu,$$

$$r = 0, 1, 2, \cdots .$$

Proof. We first prove by induction that $x_{2j-1}^{(r)}(-t) = x_{2j}^{(r)}(t)$, $j = 1, 2, \cdots, \nu$, $x_{2j-1}^{(r)}(-t) = -x_{2j}^{(r)}(t)$, $j = \nu + 1, \cdots, \mu$, $x_j^{(r)}(-t) = x_j^{(r)}(t)$, $j = 2\mu + 1, \cdots, n$ and all r. From the choice of the numbers a_j, the assertion is true for $r = 0$. Assume the assertion true for $r = 0, 1, 2, \cdots, v - 1$ and all j. Then

$$x_{2j-1}^{(r)}(-t) + x_{2j}^{(r)}(-t) = x_{2j-1}^{(r)}(t) + x_{2j}^{(r)}(t),$$

$$x_{2j-1}^{(r)}(-t) - x_{2j}^{(r)}(-t) = -[x_{2j-1}^{(r)}(t) - x_{2j}^{(r)}(t)], \qquad j = 1, \ , \cdots, \nu;$$

$$x_{2j-1}^{(r)}(-t) + x_{2j}^{(r)}(-t) = -[x_{2j-1}^{(r)}(t) + x_{2j}^{(r)}(t)],$$

$$x_{2j-1}^{(r)}(-t) - x_{2j}^{(r)}(-t) = x_{2j-1}^{(r)}(t) - x_{2j}^{(r)}(t), \qquad j = \nu + 1, \cdots, \mu;$$

$$x_j^{(r)}(-t) = x_j^{(r)}(t), \qquad j = 2\mu + 1, \cdots, n + \mu \text{ and } r = 0, 1, \cdots, v - 1.$$

Consequently, from conditions (α), (β), and (γ) of the theorem and the definition of the g_j's in (3), it follows that

$$g_{2j-1}[x^{(r)}(-t), -t, \lambda] = g_{2j-1}[x^{(r)}(t), t, \lambda], \qquad j = 1, 2, \cdots, \nu,$$

$$(11) \quad g_{2j-1}[x^{(r)}(-t), -t, \lambda] = -g_{2j-1}[x^{(r)}(t), t, \lambda], \qquad j = \nu + 1, \cdots, \mu,$$

$$g_j[x^{(r)}(-t), -t, \lambda] = -g_j[x^{(r)}(t), t, \lambda], \qquad j = 2\mu + 1, \cdots, n + \mu,$$

$$r = 0, 1, 2, \cdots, v - 1.$$

Furthermore, since $q_{2j-1} = -q_{2j}$, $j = 1, 2, \cdots, \mu$ and $M[f(-t)] = M[f(t)]$, it follows from (11) that

$$M[e^{-i\tau_j t} g_{2j-1}(x^{(r)}(t), t, \lambda)] = M[e^{i\tau_j t} g_{2j-1}(x^{(r)}(-t), -t, \lambda)] =$$
$$M[e^{i\tau_j t} g_{2j-1}(x^{(r)}(t), t, \lambda)] = -M[e^{i\tau_j t} g_{2j}(x^{(r)}(t), t, \lambda)], \qquad j = 1, 2, \cdots, \nu.$$

Now, if $c_j \neq 0$, it follows from (10) that $d_{2j-1}^{(r)} = -d_{2j}^{(r)}$ and if $c_j = 0$, then from our assumptions on the σ_j concerning congruence, the above mean value will always be zero. Consequently, in any case, $d_{2j-1}^{(r)} = -d_{2j}^{(r)}$, $j = 1, 2, \cdots, \nu$ and $r = 1, 2, \cdots, v$. In the same way, $d_{2j-1}^{(r)} = -d_{2j}^{(r)}$, $j = \nu + 1, \cdots, \mu$ and $r = 1, 2, \cdots, v$. From (11),

$$M[g_j(x^{(r)}(t), t, \lambda)] = -M[g_j(x^{(r)}(t), t, \lambda)] \qquad \text{for } j = 2\mu + 1, \cdots, n + \mu,$$

and, therefore, $d_j^{(r)} = 0$ for $j = 2\mu + 1, \cdots, n + \mu$ and $r = 1, 2, \cdots, v$. From (9),

$$\begin{aligned} x_{2j-1}^{(v)}(-t) &= -e^{-i\tau_j t} \int e^{i\tau_j t} \\ &\cdot [g_{2j-1}(x^{(v-1)}(-t), -t, \lambda) - \{d_{2j-1}^{(1)} x_{2j-1}^{(v-1)}(-t) + \cdots + d_{2j-1}^{(v)} x_{2j-1}^{(0)}(-t)\}]\, dt \\ &= e^{-i\tau_j t} \int e^{i\tau_j t}[g_{2j}(x^{(v-1)}(t), t, \lambda) - \{d_{2j}^{(1)} x_{2j}^{(v-1)}(t) + \cdots + d_{2j}^{(v)} x_{2j}^{(0)}(t)\}]\, dt \\ &= x_{2j}^{(v)}(t), \qquad\qquad j = 1, 2, \cdots, \nu. \end{aligned}$$

In a similar manner, $x_{2j-1}^{(v)}(-t) = -x_{2j}^{(v)}(t)$, $j = \nu + 1, \cdots, \mu$, $x_j^{(v)}(-t) = x_j^{(v)}(t)$, $j = 2\mu + 1, \cdots, n + \mu$, and the induction on the $x_j^{(r)}$ is completed. If the assertion is true for $x_j^{(r)}$ for all r, then the other relations must hold for all r and the lemma is proved.

Thus, from the remarks preceding Lemma 1, it remains only to show that $n + \mu$ linearly independent bounded AC solutions of (1) can be obtained from these solutions. Suppose one of the c's of Lemma 2, say c_j, is chosen $\neq 0$ and all other c_k, $k \neq j$, are chosen $= 0$.

By taking j to be successively $1, 2, \cdots, n$ and using the transformation formulas (2), the above method of successive approximations leads to $n + \mu$ bounded solutions, $y^{(h)}(t, \lambda)$, $h = 1, 2, \cdots, n + \mu$, of the form

$$\begin{aligned} &y_k^{(2j-1)}(t, 0) = 0, \quad k \neq j, \qquad y_j^{(2j-1)}(t, 0) = (c_j/\sigma_j) \cos \tau_j t, \\ &y_k^{(2j)}(t, 0) = 0, \quad k \neq j, \qquad y_j^{(2j)}(t, 0) = -c_j \sin \tau_j t, \qquad j = 1, 2, \cdots, \nu; \\ &y_k^{(2j-1)}(t, 0) = 0, \quad k \neq j, \qquad y_j^{(2j-1)}(t, 0) = (c_j/\sigma_j) \sin \tau_j t, \\ &y_k^{(2j)}(t, 0) = 0, \quad k \neq j, \qquad y_j^{(2j)}(t, 0) = c_j \cos \tau_j t, \qquad j = \nu + 1, \cdots, \mu; \\ &y_k^{(2\mu+j)}(t, 0) = 0, \quad k \neq j, \qquad y_j^{(2\mu+j)}(t, 0) = c_{\mu+j}, \qquad j = 1, 2, \cdots, n - \mu. \end{aligned}$$

It is clear that these $n + \mu$ functions also form a fundamental system of AC solutions of (1) and the theorem is proved.

Example 1. Consider the system

$$y'' + \sigma^2 y = -\lambda \cos t \cdot w,$$
$$w' = \lambda \sin t \cdot y + \lambda \sin t \cdot w,$$

where $\sigma \not\equiv 0 \pmod 1$. This system satisfies the conditions of the preceding theorem with $\nu = 1$, $\mu = 0$, $n = 2$. Therefore, all the solutions of this equation are bounded for $|\lambda|$ sufficiently small. The following example illustrates how a change in only one of the periodic coefficients can lead to unbounded solutions.

Example 2. Consider the system

$$(12) \qquad \begin{cases} y'' + \sigma^2 y = -\lambda \cos t \cdot w, \\ w' = \lambda \cos t \cdot y + \lambda \sin t \cdot w. \end{cases}$$

By the transformation $y = (1/2i\sigma)(z_1 + z_2)$, $y' = \frac{1}{2}(z_1 - z_2)$, $w = z_3$, the above system is equivalent to the system

$$z_1' = i\sigma z_1 - \lambda \cos t \cdot z_3,$$
$$z_2' = -i\sigma z_2 + \lambda \cos t \cdot z_3,$$
$$z_3' = (\lambda \cos t/2i\sigma)(z_1 + z_2) + \lambda \sin t \cdot z_3.$$

The characteristic exponent τ which is close to zero may be obtained by applying the method of successive approximations (9) to the auxiliary system (4) with $B = \operatorname{diag}(i\sigma, -i\sigma, \tau)$ and the 0^{th} approximation as $z_1^{(0)} = z_2^{(0)} = 0$, $z_3^{(0)} = a$ (see [1] or [6]). Carrying out this procedure, one finds that $d_3^{(1)} = 0$, $d_3^{(2)} = 1/2(1 - \sigma^2)$. But τ must satisfy the equation

$$\tau - \lambda(d_3^{(1)} + \lambda d_3^{(2)} + \cdots) = 0,$$

and therefore $\tau = \lambda^2[2(1 - \sigma^2)]^{-1} + \cdots$, and if $|\sigma| < 1$, τ has a positive real part, and at least one solution of (12) is unbounded no matter how small $|\lambda|$.

The author has been able to obtain by a different method the above theorems (and slightly more general ones) for the case in which system (1) for $\lambda = 0$ has only one zero characteristic root. This method involves a discussion of the characteristic exponents as a function of λ without using successive approximations and, therefore, does not aid in the calculation of the solutions.

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