SOLUTION OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS BY SERIES OF EXPONENTIAL FUNCTIONS[1]

BY

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1. Introduction

The aim of this paper is to construct almost periodic solutions of systems of nonlinear differential equations of the form

$$y' = f(y) + \sum_{k=1}^{m} g_k e^{i\omega_k x}. \tag{1.1}$$

Here y is an n-dimensional vector; the dash indicates differentiation with respect to x; the g_k are constant vectors, and the ω_k are real, not necessarily commensurable, numbers. The components of the vector $f(y)$ are assumed to be analytic functions of the components of y.

It will be proved that there exists, under certain assumptions, a particular solution of (1.1) of the form

$$y = a_0 + \sum_{r=1}^{\infty} a_r e^{i\mu_r x}. \tag{1.2}$$

The vectors a_r, $r \geqq 1$, can be determined successively by solving n^{th} order systems of linear algebraic equations. The numbers μ_r are linear combinations of the ω_k with nonnegative integral coefficients. In the language of the theory of nonlinear vibrations this means that the solution is a superposition of "combination oscillations."

In Appendix 2 of [1] Friedrichs indicated a method for constructing an almost periodic solution of differential equations that differ from (1.1) in that $f(y)$ must be a polynomial, while the almost periodic term is allowed to be an infinite series.

The series in (1.2) will be shown to converge if the g_k are sufficiently small. In this sense the differential equation (1.1) can be regarded, in this paper, as a perturbation of the equation $y' = f(y)$. In [2] Biryuk has solved a somewhat similar perturbation problem. His perturbation term is more general than the one in (1.1), but his unperturbed equation is linear. Also, Biryuk's method does not lead to an expression for the solution as explicit as (1.2). The method of this paper has also some relation to a procedure suggested by E. Weber [3].

In an as yet unpublished article M. Golomb has generalized the results of this paper considerably. Among other things he has given an extension to

Received June 19, 1957.

[1] This work was sponsored in part by the Office of Naval Research and in part by the United States Army under Contract No. DA-11-022-ORD-2059.

the case that the almost periodic term is an infinite series of exponential functions. He also permits the function $f(y)$ to depend on x, and he proves, furthermore, that superharmonic solutions exist in certain cases.

2. The formal procedure

We begin by stating some hypotheses.

ASSUMPTION 1. *The equation* $f(y) = 0$ *possesses a solution* $y = a_0$.

If v is any vector with components $v_1, v_2, \cdots, v_n$, the symbol $\| v \|$ will denote the norm

$$(2.1) \qquad \| v \| = \max_j | v_j | .$$

ASSUMPTION 2. *The components* $f_j(y)$ *of the vector functions* $f(y)$ *are holomorphic functions of the n components* $y_1, y_2, \cdots, y_n$ *of the vector y in the domain*

$$(2.2) \qquad \| y - a_0 \| \leqq \rho \qquad (\rho \textit{ a positive constant}).$$

Let the vector u be defined by

$$(2.3) \qquad y = a_0 + u.$$

If $f(y) = f(a_0 + u)$ is expanded into a power series in the components $u_1, u_2, \cdots, u_n$ of u and use is made of Assumption 1, the differential equation (1.1) takes on the form

$$(2.4) \qquad u' = Au + \phi(u) + \sum_{k=1}^{m} g_k e^{i\omega_k x}.$$

Here A is a square constant matrix, viz. the Jacobian of the components f_j of f with respect to the variables $y_1, y_2, \cdots, y_n$ at the point $y = a_0$. The components ϕ_j, $j = 1, 2, \cdots, n$, of $\phi(u)$ possess series expansions in powers of $u_1, u_2, \cdots, u_n$ without constant or linear terms. These series converge in the domain

$$(2.5) \qquad \| u \| \leqq \rho,$$

thanks to Assumption 2.

ASSUMPTION 3. *No eigenvalue of A is purely imaginary.*

Consider the set of all ordered m-tuples $(n_1, n_2, \cdots, n_m)$ of nonnegative integers. This set may be ordered according to the following rule. If $\sum_{j=1}^m n_j^{(1)} < \sum_{j=1}^m n_j^{(2)}$, the m-tuple $(n_1^{(1)}, n_2^{(1)}, \cdots, n_m^{(1)})$ comes before $(n_1^{(2)}, n_2^{(2)}, \cdots, n_m^{(2)})$. If $\sum_{j=1}^m n_j^{(1)} = \sum_{j=1}^m n_j^{(2)}$, consider the first component that is not the same for the two m-tuples and place first the m-tuple for which this component is larger. Let the $(r + 1)^{\text{st}}$ of these m-tuples be denoted by N_r, and let Ω be an abbreviation for the m-tuple $(\omega_1, \omega_2, \cdots, \omega_m)$. Define the sequence of real numbers μ_r by

$$(2.6) \qquad \mu_r = N_r \cdot \Omega = n_1^{(r)}\omega_1 + n_2^{(r)}\omega_2 + \cdots + n_m^{(r)}\omega_m , \qquad r = 1, 2, \cdots .$$

The numbers μ_r are not necessarily all distinct, but the functions $\mu_r = \mu_r(\omega_1, \cdots, \omega_m)$ are. In fact, the same number may occur infinitely often in this sequence. Observe that

$$\mu_k = N_k \cdot \Omega = \omega_k, \qquad k = 1, 2, \cdots, m. \tag{2.7}$$

The differential equation (2.4) can be formally satisfied by a series of the form

$$u = \sum_{r=1}^{\infty} a_r e^{iN_r \cdot \Omega x}. \tag{2.8}$$

In order to see this, insert the series into (2.4), expand, and reorder according to the exponential factors $e^{iN_s \cdot \Omega x}$. In this reordering the ω_k are to be treated as parameters rather than as given numerical values, so that

$$e^{iN_\alpha \cdot \Omega x} e^{iN_\beta \cdot \Omega x} = e^{iN_\gamma \cdot \Omega x},$$

where $N_\gamma = N_\alpha + N_\beta$ in the sense of vector addition. In view of the ordering principle for the N_r one has always $\gamma \geqq \alpha + \beta$. (As mentioned before, it is possible to have $N_\alpha \cdot \Omega = N_\beta \cdot \Omega$, $\alpha \neq \beta$, for certain choices of Ω, so that substitution of numerical values for the ω_k might make indistinguishable exponential factors that have to be treated differently in the scheme to be described.) Since $\phi(u)$ has no constant or linear terms, the full expansion of $\phi(u)$ after insertion of (2.8) for u contains only a finite number of terms having a given exponential factor $e^{iN_s \cdot \Omega x}$, and the coefficient of each such term is a monomial in the components of $a_1, a_2, \cdots, a_{s-1}$.

After having reordered the terms in the right member of (2.4) in the manner described, we require that for every $r \geqq 1$ the coefficients of $e^{iN_r \cdot \Omega x}$ in the two members be the same. This leads to a recursive sequence of equations for the a_r of the form

$$(A - i\omega_r I)a_r = g_r, \qquad 1 \leqq r \leqq m \tag{2.9}$$

$$(A - iN_r \cdot \Omega I)a_r = h_r(a_1, a_2, \cdots, a_{r-1}), \qquad r > m. \tag{2.10}$$

The components of the vector functions h_r are polynomials in the components of $a_1, a_2, \cdots, a_{r-1}$ without constant or linear terms. Thanks to Assumption 3 the matrices $A - iN_r \cdot \Omega I$, $r \geqq 1$, are all nonsingular, so that the a_r can be computed successively.

3. The convergence of of the series

The convergence of the series (2.8) with coefficients a_r that are determined from (2.9) and (2.10) will now be studied by the method of dominating series. To that end it will first be shown that *there exists a number c, independent of r, such that, for a solution of* (2.9), (2.10),

$$\| a_r \| \leqq c \| g_r \|, \qquad r \leqq m \tag{3.1}$$

$$\| a_r \| \leqq c \| h_r(a_1, a_2, \cdots, a_{r-1}) \|, \qquad r > m. \tag{3.2}$$

The equations (2.9) and (2.10) are of the form $(A - \lambda I)v = a$, where v and a are vectors and λ a parameter. Since $v = (A - \lambda I)^{-1}a$, it follows that $\| v \| \leqq b(\lambda) \| a \|$, where $b(\lambda)$ is the "row-sum" norm of $(A - \lambda I)^{-1}$, i.e., the largest sum of the absolute values of the elements in any one row of $(A - \lambda I)^{-1}$. The elements of $(A - \lambda I)^{-1}$ are rational functions vanishing at infinity. Hence, $\lim_{\lambda\to\infty} b(\lambda) = 0$. The values of λ occurring in (2.9) and (2.10) are $\lambda = N_r \cdot \Omega$, $r = 1, 2, \cdots$. By Assumption 3 they are bounded away from the eigenvalues of A, which are the poles of the elements of $(A - \lambda I)^{-1}$. Therefore, there exists a number c such that $b(N_r \cdot \Omega) \leqq c$. This proves (3.1) and (3.2).

Next, a problem will be constructed that dominates (2.4) in the sense of the method of dominating series. Let M be some upper bound for $\| \phi \|$ in the domain defined by the inequality (2.5). It is a well-known fact from the theory of analytic functions that the coefficients of the terms of degree k in the power series for a component ϕ_j of ϕ are numerically not greater than $M\rho^{-k}$. Hence, the series

$$M \sum_{s_1+s_2+\cdots+s_n>1} \rho^{-(s_1+s_2+\cdots+s_n)} u_1^{s_1} u_2^{s_2} \cdots u_n^{s_n}$$

represents a function that dominates all ϕ_j, $j = 1, 2, \cdots, n$. The series

$$M \sum_{s=2}^{\infty} \left(\frac{u_1 + u_2 + \cdots + u_n}{\rho} \right)^s, \tag{3.3}$$

when expanded by the multinomial theorem, has still larger positive coefficients. Hence the scalar function

$$\hat{\phi}(u) = M \left[\left(1 - \rho^{-1} \sum_{j=1}^{n} u_j \right)^{-1} - 1 - \rho^{-1} \sum_{j=1}^{n} u_j \right], \tag{3.4}$$

whose power series expansion is (3.3), dominates all ϕ_j.

The last mentioned fact has the following consequence. Replace the vector u in (3.4) formally by the vectorial series

$$\sum_{r=1}^{\infty} b_r e^{iN_r \cdot \Omega x}, \tag{3.5}$$

expand, and collect terms with the same exponential factor, as was done before with $\phi(u)$. There results a scalar series of the form

$$\sum_{r=1}^{\infty} \hat{h}_r(b_1, \cdots, b_{r-1}) e^{iN_r \cdot \Omega x},$$

whose coefficients $\hat{h}_r$ are polynomials in the components of $b_1, b_2, \cdots, b_{r-1}$ without constant or linear terms. These polynomials dominate the n components of the vector function $h_r(b_1, \cdots, b_{r-1})$ in the sense that the coefficients of $\hat{h}_r$ are nonnegative and not less than the moduli of the corresponding coefficients of the components of h_r.

If we denote, furthermore, by g a number such that

$$\|g_k\| \leqq g, \qquad k = 1, 2, \cdots, m, \tag{3.6}$$

then the n scalar equations

$$v_j - cg(e^{i\omega_1 x} + e^{i\omega_2 x} + \cdots + e^{i\omega_m x}) - c\hat{\phi}(v) = 0, \qquad j = 1, 2, \cdots, n, \tag{3.7}$$

for the components v_j of the vector v can be shown to constitute the desired dominating problem. Observe that, since $\hat{\phi}(v)$ depends on $\sum_{j=1}^n v_j$ only, (3.7) is equivalent to one equation for $\sum_{j=1}^n v_j$, and all v_j are equal. Now we attempt to solve (3.7) by a series of the form (3.5), i.e., we insert

$$v = \sum_{r=1}^{\infty} b_r e^{iN_r \cdot \Omega_r x} \tag{3.8}$$

into (3.7). If b_{r_j}, $j = 1, 2, \cdots, n$, are the components of b_r, we obtain the recursion formulas

$$b_{1_j} = b_{2_j} = \cdots = b_{m_j} = cg, \qquad j = 1, 2, \cdots, n. \tag{3.9}$$

$$b_{r_j} = c\hat{h}_r(b_1, b_2, \cdots, b_{r-1}), \qquad r > m, \quad j = 1, 2, \cdots, n. \tag{3.10}$$

Because of (3.1), (3.6), and (3.9),

$$\|a_r\| \leqq gc = b_{r_j} = \|b_r\|, \qquad r = 1, 2, \cdots, m.$$

From (3.2), (3.9), and (3.10) and the dominating property of $\hat{h}_r$ one concludes that

$$\|a_{m+1}\| \leqq c\hat{h}_{m+1}(b_1, b_2, \cdots, b_m) = b_{m+1,j} = \|b_{m+1}\|,$$

and, by induction,

$$\|a_r\| \leqq \|b_r\|, \qquad r = 1, 2, \cdots. \tag{3.11}$$

In view of the last inequality it suffices to prove the uniform and absolute convergence of the series in (3.8) in order to deduce that the series in (2.8) converges in a similar manner. For the study of the convergence of (3.8) replace $e^{i\omega_k x}$, $k = 1, 2, \cdots, m$, in (3.7) by

$$z_k = e^{i\omega_k x}, \qquad k = 1, 2, \cdots, m, \tag{3.12}$$

and sum over j, which leads to the scalar equation

$$\sum_{j=1}^n v_j - ncg \sum_{k=1}^m z_k - nc\hat{\phi}(v) = 0. \tag{3.13}$$

This can be regarded as an equation for $w = \sum_{j=1}^n v_j$ in terms of $\zeta = ncg \sum_{k=1}^m z_k$. Since $\hat{\phi}(v)$ has no constant or linear terms in its expansion (cf. (3.3)), the equation (3.13) is satisfied for $\zeta = w = 0$, and the partial derivative of the left member with respect to w is not zero at that point. Hence, (3.13) defines w as a holomorphic function of ζ in a certain circle $|\zeta| \leqq \zeta_0$, i.e., for

$$\left| ncg \sum_{k=1}^m z_k \right| \leqq \zeta_0. \tag{3.14}$$

Since the v_j are all equal, every v_j is representable as a uniformly and absolutely convergent power series

$$v_j = \sum_{n_1+n_2+\cdots+n_m>0} c_{n_1 n_2 \cdots n_m} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}, \tag{3.15}$$

as long as (3.14) holds.

The terms of (3.15) may be assumed to be arranged according to the same principle as the numbers $N_r \cdot \Omega$ before. The coefficients $c_{n_1 n_2 \cdots n_m}$ could be calculated recursively by insertion into

$$v_j = cg \sum_{k=1}^{m} z_k - c\hat{\phi}(v), \qquad j = 1, \cdots, n$$

and identification of corresponding terms right and left. But these recursion formulas become identical with (3.9), (3.10) if $c_{n_1 n_2 \cdots n_m}$ is replaced by b_{rj}, r being the order number of the vector $N_r = (n_1, n_2, \cdots, n_m)$ and j being arbitrary, since all b_{rj}, $j = 1, 2, \cdots, n$, are equal. This proves the uniform and absolute convergence of the series in (3.8) whenever the condition

$$cg \left| \sum_{k=1}^{m} e^{i\omega_k x} \right| \leqq \zeta_0 \tag{3.16}$$

is satisfied. The series (2.8) converges therefore in like manner.

It remains to be shown that (2.8) satisfies the differential equation. Since the polynomials h_r have no constant terms, the b_{rj}, as determined from (3.9) and (3.10), tend to zero with g, while the domain of uniform convergence of the series, as defined by (3.16), expands as g becomes smaller. Hence, the function v of (3.8) shrinks to zero with g, and the same is true of the function u of (2.8) which is dominated by v. For sufficiently small g the function u will therefore satisfy the inequality (2.5). This implies that the series obtained by inserting (2.8) into the right member of (2.4) and collecting terms according to exponential factors $e^{iN_r \cdot \Omega x}$ converges uniformly in every smaller domain. The series so obtained is the termwise derivative of the series in (2.8) since this is precisely the content of equations (2.9) and (2.10). Hence, the function defined by (2.8) satisfies the differential equation. Thus the following theorem has been proved.

THEOREM. *A differential equation of the form* (1.1) *satisfying Assumptions* 1, 2, *and* 3, *as well as the inequalities*

$$\| g_k \| \leqq g_0, \qquad k = 1, 2, \cdots, m,$$

where $g_0 > 0$ is a constant depending on $f(y)$ only, possesses a particular solution admitting a series expansion of the form

$$y = a_0 + \sum_{r=1}^{\infty} a_r e^{i\mu_r x}$$

that converges uniformly and absolutely for $-\infty < x < \infty$. *The sequence* μ_r *is formed of the numbers* $\sum_{k=1}^{m} n_k\omega_k$ *(not necessarily all different), where the* n_k *are nonnegative integers not all zero. For a suitable ordering of the terms the coefficients* a_r, $r > 0$, *can be recursively calculated by solving systems of linear equations.*

If all ω_k are integral multiples of the same number ω, this result and its proof remain valid if A has some purely imaginary eigenvalues $i\nu_j$, provided ν_j is not a multiple of ω.

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