MARKOFF PROCESSES AND POTENTIALS III

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The numbering of the sections in this part continues the numbering in the first two installments, pages 44–93 and 316–369 of the first volume of this journal.

Consider again the situation treated in the introduction to the first installment. Given a measure ξ that is invariant under the transition measures.

$$\xi(B) = \int_{\mathcal{K}} \xi(dr) P_\tau(r, B), \qquad B \subset \mathcal{K},$$

one can construct a stationary Markoff process defined for all values of time, positive or negative. The process obtained by reversing the direction of time is well known to be a stationary Markoff process also. If the original transition measures are absolutely continuous with respect to ξ,

$$P_\tau(r, ds) = p_\tau(r, s)\xi(ds),$$

then the reversed process has for transition measures

$$Q_\tau(r, ds) \equiv p_\tau(s, r)\xi(ds).$$

Let us suppose that matters stand so, and that the $Q_\tau(r, ds)$ satisfy the conditions of regularity imposed upon the $P_\tau(r, ds)$. There exists then a one-to-one correspondence between measures excessive for $P_\tau(r, ds)$ and functions excessive for $Q_\tau(r, ds)$, as well as a dual correspondence in which the two families of transition measures exchange roles. The results of the first two installments may now be stated in terms of one or the other class of excessive functions, with a consequent sharpening of certain statements. One obtains in this way, for example, the representation of an excessive function as the sum of a potential and an excessive function having certain additional properties; this particular result is unfortunately somewhat misleading, for a similar representation can be proved to hold in the setting of the second installment. One can also define naturally a capacity of sets that behaves like the Newtonian capacity and has the same interpretation by means of processes.

It is rather too stringent to suppose the invariance of ξ or the absolute continuity of $P_\tau(r, ds)$ relative to ξ, as we have done above, for either hypothesis rules out the potential theory of the heat equation on a bounded domain. We shall treat instead the relative theory, with ξ an excessive measure and certain

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averages of the transition measures absolutely continuous. The two classes of processes do not then arise from a stationary Markoff process with time running one way or the other, and the proofs are somewhat more complicated than for the situation described above. The details are set forth in §§17–19 and §21.

In §20 we discuss the statement that some point must be regular for a compact set unless the set is negligible. The statement is true of Brownian motion in space, but not in space-time, and the discrepancy accounts for much of the behavior that distinguishes superharmonic functions from superparabolic functions. There are several equivalent formulations of the statement in terms of excessive functions. The statement is proved to hold if the functions $p_\tau(r, s)$ appearing above are symmetric in r and s, that is to say, if the transition measures of the stationary Markoff process remain the same when the direction of time is reversed.

17. Dual processes

Let $\mathcal{H}$ be a separable locally compact Hausdorff space, $\mathcal{C}(\mathcal{H})$ the Banach space of functions continuous on $\mathcal{H}$ and vanishing at infinity. We suppose given a semigroup of linear transformations H_τ of $\mathcal{C}(\mathcal{H})$ satisfying the relations

$$0 \leqq H_\tau f \leqq \max f, \qquad \lim_{\tau \to 0} H_\tau f = f,$$

for every positive function f in $\mathcal{C}(\mathcal{H})$. This semigroup will be held fast in the remainder of the paper.

In order to apply the results of preceding sections it is necessary to extend the semigroup as in §15. The transformation H_τ can be written

$$H_\tau f.(r) \equiv \int H_\tau(r, ds) f(s), \tag{17.1}$$

with $H_\tau(r, ds)$ a measure on $\mathcal{H}$ of mass not greater than 1. Adjoin a point w to $\mathcal{H}$, and define Markoff transition measures on the enlarged space by setting

$$\begin{aligned} P_\tau(r, B) &= H_\tau(r, B), && r \in \mathcal{H}, \quad B \subset \mathcal{H}, \\ P_\tau(r, w) &= 1 - H_\tau(r, \mathcal{H}), && r \in \mathcal{H}, \\ P_\tau(w, w) &= 1. \end{aligned}$$

These transition measures were shown in §15 to satisfy hypotheses (A) and (C). The corresponding processes will be denoted by X as before.

The $H_\tau(r, ds)$ are of course the transition measures relative to the time a process reaches the point w. This time differs with probability 1—even certainly—from the time the process first hits a given subset of $\mathcal{H}$, and a sample path remains at w once it arrives there. So the terminal time seldom need be mentioned, for the sets we shall speak of all lie in $\mathcal{H}$.

The field $\mathfrak{A}(\mathcal{H})$ is defined as in §1. A function on $\mathcal{H}$ is understood to be measurable over $\mathfrak{A}(\mathcal{K})$ and to be extended to the enlarged space by giving it the value 0 at w. This convention permits one to write

$$H_\tau f.(r) = \int_\Omega f(X(\tau))\, d\omega,$$

for example, instead of restricting the integral to a certain subset of Ω. The function $H_\tau f$ is understood to be defined by the right member of (17.1), which makes sense whenever f is positive or bounded.

The notation now differs a little from that of the first two installments; we have reserved the notation introduced in the second installment for the situation treated in the latter part of this section.

Let α be a positive Lebesgue measurable function on J, the interval $0 < \tau < \infty$. The kernel $H(\alpha, r, ds)$ is defined to be

$$H(\alpha, r, B) \equiv \int_J \alpha(\tau) H_\tau(r, B)\, d\tau, \qquad r \in \mathcal{K}, \quad B \subset \mathcal{K}.$$

It will sometimes be written simply $H(\alpha)$ or H_α ; these expressions are used also to denote the associated transformation of functions or of measures. The reader should keep in mind the following facts: The integral of α bounds $H(\alpha, r, \mathcal{K})$. The kernel H_α increases to H_β as α increases to β; a similar statement holds as α decreases, if the integral of α is finite. The kernel is linear in α in the sense that

$$H_{\alpha+\beta} = H_\alpha + H_\beta\,, \qquad H_{\lambda\alpha} = \lambda H_\alpha\,,$$

with λ a positive number. Let us denote by α_σ , for σ a positive number, the function defined on J as $\alpha(\tau - \sigma)$ for τ greater than σ and as 0 otherwise; then $H(\alpha_\sigma)$ is precisely the composition $H_\alpha H_\sigma$. Consequently, $H_\alpha H_\beta$ coincides with $H(\alpha * \beta)$, where

$$\alpha * \beta.(\tau) \equiv \int_0^\tau \alpha(\tau - \sigma)\beta(\sigma)\, d\sigma.$$

Suppose for the moment that the integral of α is 1. There is then a positive random variable U having α as density function. If X is a process starting at the point r and if U is independent of X, one has the interpretation

$$H(\alpha, r, B) \equiv \mathcal{P}\{X(U) \in B\}, \qquad B \subset \mathcal{K}.$$

A good many of the later proofs are most naturally carried out in two steps. The first argues in terms of X and U to establish some relation for almost all points; the second uses continuity of the analytical expressions to show there are no exceptional points. Such a course complicates the notation; I have chosen rather to avoid the stochastic arguments.

We shall ordinarily be concerned with functions in Γ, the class of positive continuous functions on J with compact support, or in the subclass Γ_1 comprising the functions having unit integrals. We shall write $\gamma \prec \delta$, for γ and δ

in Γ_1, if there is a number ρ such that γ vanishes in the interval (ρ, ∞) and δ vanishes in the interval $(0, \rho)$. A fundamental sequence is a sequence of functions γ_n in Γ_1 such that $\gamma_{n+1} \prec \gamma_n$ and such that, for every strictly positive number ρ, the function γ_n vanishes in the interval (ρ, ∞), provided n is sufficiently large.

Let α increase to β, all functions being in Γ. By Dini's theorem, the difference $\beta - \alpha$ is bounded by $\varepsilon\gamma$, with γ a fixed element of Γ and ε a variable number decreasing to 0. This remark is useful in several passages to the limit.

The next hypothesis is assumed in the remainder of the paper:

(*F*) *The measure* ξ *is excessive relative to the transition measures* $H_\tau(r, ds)$. *For every* γ *in* Γ *there is a function* $h(\gamma, r, s)$ *defined on* $\mathcal{K} \times \mathcal{K}$ *which satisfies the relation*

$$H(\gamma, r, B) = \int_B h(\gamma, r, s)\xi(ds), \qquad r \in \mathcal{K}, \quad B \subset \mathcal{K},$$

and which belongs to $\mathcal{C}(\mathcal{K})$ *when considered as a function of* r *or of* s. *An integral* $\int \xi(dr)f(r)h(\gamma, r, s)$ *belongs to* $\mathcal{C}(\mathcal{K})$ *as function of* s, *provided* f *is continuous and has compact support in* $\mathcal{K}$. *As* γ *runs through some fundamental sequence,*

$$\limsup \int_G \xi(dr)h(\gamma, r, s) \geqq 1, \qquad s \in G,$$

for every open set G *in* $\mathcal{K}$.

It was proved in §15 that special sets may be replaced by open sets having compact closures, in a situation as regular as the present one; so the measure ξ is finite on compact sets. The absolute continuity of the measures $H(\gamma, r, ds)$ with respect to ξ implies that $\xi(G)$ is strictly positive if G is open and not empty; this fact, of course, follows also from the last assertion in the hypothesis. A set is therefore dense in $\mathcal{K}$ if its complement is a null set for ξ. A null set for ξ is obviously approximately null; the converse follows from Proposition 14.2.

For fixed γ and r, the function $h(\gamma, r, s)$ of s is positive except on a set of ξ-measure null; it is therefore positive everywhere. A similar argument shows it to be linear and increasing in γ. Thus, by the remark preceding the hypothesis, $h(\gamma, r, s)$ increases to $h(\gamma', r, s)$ as γ increases to γ'.

The integral $\int \xi(dr)h(\gamma, r, s)$ is lower semicontinuous in s, by Fatou's lemma. If γ belongs to Γ_1, then ξ, since it is excessive, dominates the measure ξH_γ; thus $\int \xi(dr)h(\gamma, r, s)$ is bounded by 1 for s outside a null set relative to ξ, so for all s without exceptions. Consequently, the upper limit mentioned in (*F*) is in fact a limit, the value being the constant 1 on G. The further relation

$$\lim \int_{\mathcal{K}-G} \xi(dr)h(\gamma, r, s) = 0, \qquad s \in G,$$

must then be true as γ runs through the fundamental sequence mentioned in the hypothesis. It is now clear that

$$\int \xi(dr)f(r)h(\gamma, r, s) \rightarrow f(s), \qquad s \in \mathcal{K}, \tag{17.2}$$

if f belongs to $\mathcal{C}(\mathcal{K})$ and γ runs through the same fundamental sequence. The integral itself belongs to $\mathcal{C}(\mathcal{K})$ as a function of s; for f can be written as the sum of two continuous functions, one with compact support and one uniformly small, so that the integral differs by an arbitrarily small amount from a function in $\mathcal{C}(\mathcal{K})$, according to hypothesis (F) and the second sentence of this paragraph.

For r, γ, γ' fixed, the equation

$$\int h(\gamma, r, t)h(\gamma', t, s)\xi(dt) = h(\gamma * \gamma', r, s) \tag{17.3}$$

holds for all s outside a null set for ξ, hence on a set dense in $\mathcal{K}$. The integral is continuous in s, according to what has just been proved, because $h(\gamma, r, t)$ belongs to $\mathcal{C}(\mathcal{K})$ as a function of t. So the equation is true for all r and s.

Let us define the linear transformation $f \rightarrow f\hat{H}_\gamma$ of $\mathcal{C}(\mathcal{K})$ into itself by the formula

$$f\hat{H}_\gamma.(s) \equiv \int f(r)\xi(dr)h(\gamma, r, s). \tag{17.4}$$

(The unusual arrangement of the symbols is not necessary at the moment, but it is in keeping with later usage.) The transformation is linear in γ; it has a bound not exceeding the integral of γ; and $\hat{H}_\gamma \hat{H}_{\gamma'}$ coincides with $\hat{H}_{\gamma * \gamma'}$ according to (17.3). Consider a bounded measure μ, not necessarily positive, and a function f in $\mathcal{C}(\mathcal{K})$; according to (17.2), the relation

$$\int f\hat{H}_\gamma.(s)\mu(ds) \rightarrow \int f(s)\mu(ds)$$

must hold, as γ runs through the fundamental sequence mentioned in hypothesis (F). So the collection of functions $f\hat{H}_\gamma$, with f and γ variable, is dense in $\mathcal{C}(\mathcal{K})$.

The correspondence $\gamma \rightarrow \hat{H}_\gamma$ can be extended by continuity to a bounded representation, by linear transformations on $\mathcal{C}(\mathcal{K})$, of the convolution algebra of integrable functions on the additive semigroup of the positive reals. Since the ranges of the transformations span all of $\mathcal{C}(\mathcal{K})$, there is a semigroup of bounded linear transformations $\hat{H}_\tau$ of $\mathcal{C}(\mathcal{K})$ such that $f\hat{H}_\tau$ converges strongly to f as $\tau \rightarrow 0$ and such that

$$f\hat{H}_\gamma = \int \gamma(\tau)f\hat{H}_\tau \, d\tau, \qquad \gamma \in \Gamma, \quad f \in \mathcal{C}(\mathcal{K}).$$

The proof is to be found, for example, in §32C of the book, *Abstract Harmonic Analysis*, by L. H. Loomis. The transformations $\hat{H}_\tau$ clearly have all the properties of the transformations H_τ, sending positive functions into positive ones and not increasing the norm. We can therefore write

$$f\hat{H}_\tau.(s) \equiv \int f(r)\hat{H}_\tau(dr, s),$$

with $\hat{H}_\tau(dr, s)$ a measure of mass not greater than 1. After enlarging the space $\mathcal{H}$ by adjoining a point $\hat{w}$, one can introduce Markoff transition measures $\hat{P}_\tau(dr, s)$ that yield the $\hat{H}_\tau(dr, s)$ when both arguments are restricted to $\mathcal{H}$. These transition measures satisfy hypotheses (A) and (C); the corresponding processes will be denoted by $\hat{X}$.

The relation

$$\int h(\gamma, r, s)\xi(ds) \equiv H_\gamma(r, \mathcal{H}) \leqq 1, \qquad \gamma \in \Gamma_1,$$

shows that ξ dominates the measure $\hat{H}_\gamma\,\xi$ for every γ in Γ_1. Thus, taking f to be a positive function in $\mathcal{C}(\mathcal{H})$ and letting γ run through a fundamental sequence, one obtains the inequalities

$$\int f(r)\hat{H}_\tau\,\xi.(dr) \leqq \liminf \int f\hat{H}_\tau\hat{H}_\gamma.(s)\xi(ds) \leqq \int f(s)\xi(ds),$$

by Fatou's lemma. The measure ξ is therefore excessive relative to the kernels $\hat{H}_\tau(dr, s)$.

It is now evident that the semigroups H_τ and $\hat{H}_\tau$ have equal roles in the situation being discussed; they are said to be dual relative to the basic measure ξ. Propositions and conventions are usually stated for the semigroup H_τ and the associated processes X; they are to be carried over to the dual semigroup and processes in the obvious manner. The element of measure $\xi(dr)$ is written dr from now on, and phrases like *integrable* or *almost everywhere* are understood to be defined in terms of the basic measure.

Hypothesis (F) amounts to saying, $h(\gamma, r, s)$ is the density of $H(\gamma, r, ds)$ when considered as a function of s, the density of $\hat{H}(\gamma, dr, s)$ when considered as a function of r. In the simplest circumstances, discussed in the next few paragraphs, these relations correspond to a certain identity of the dual processes.

For the moment suppose the $H_\tau(r, ds)$ to be true Markoff transition measures and ξ to be invariant, that is to say,

$$\xi(B) = \int \xi(dr)H_\tau(r, B), \qquad B \subset \mathcal{H}.$$

Let $\mathcal{Y}$ be the set of functions from the reals to $\mathcal{H}$ which are continuous on the right and have limits from the left; take $\mathcal{B}(\mathcal{Y})$ to be the least Borel field which includes all sets defined by a single relation $y(\sigma) \in B$, with B a Borel set in $\mathcal{H}$. There is a measure m on $\mathcal{B}(\mathcal{Y})$ determined by the condition that

$$m(A) = \int_{B_0} dr \int_{B_1} H_{\Delta_1}(r, ds_1) \int_{B_2} \cdots \int_{B_n} H_{\Delta_n}(s_{n-1}, ds_n),$$

if A is a subset of $\mathcal{Y}$ defined by the relations $y(\sigma_i) \in B_i$, the σ_i forming a finite increasing sequence of real numbers and Δ_i standing for $\sigma_i - \sigma_{i-1}$. In particular, the set defined by a single relation $y(\sigma) \in B$ has measure $\xi(B)$. The shift transformations $y(\sigma) \rightarrow y(\sigma + \tau)$ obviously leave the measure invariant.

Suppose further that ξ is bounded. One may as well take $\xi(\mathcal{H})$, hence also the measure of $\mathcal{Y}$, to be 1. The triple $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}), m)$ is then the standard realization of a strictly stationary Markoff process having the $H_\tau(r, ds)$ for transition measures. Now, a Markoff process remains one when the direction of time is reversed. Under (F) and the present hypotheses the transition measures of the reversed Markoff process are precisely the measures $\hat{H}_\tau(dr, s)$; unfortunately, the sample paths are continuous on the left, not on the right, so that one must alter the new paths slightly as well as change the direction of time.

Consider the set in $\mathcal{Y}$ defined by the two relations $y(\sigma) \in B$ and $y(\sigma + \tau) \in C$ with τ a positive number; its measure, which does not depend on σ, will be denoted by $\varphi(B, C, \tau)$. The set and its measure can be interpreted two ways, either the original process or the reversed one being taken as primitive; the sets B and C exchange roles in the two interpretations, and one may suppose the reversed process to be continuous on the right because there are no fixed discontinuities. Now,

$$\int_B \int_C h(\gamma, r, s)\, dr\, ds = \int \varphi(B, C, \tau)\gamma(\tau)\, d\tau,$$

an equation that defines $h(\gamma, r, s)$ for almost all r and s. If γ belongs to Γ_1, the equation leads obviously to dual interpretations of $h(\gamma, r, s)$ as a conditional probability density.

All that has been said remains true even if $\xi(\mathcal{H})$ is infinite, provided one permits probability spaces of infinite measure to be used in defining a Markoff process. The theories of the Newtonian potential, the potentials of Marcel Riesz, and the heat potential are included in the extension, with ξ the relevant Lebesgue measure.

We shall not attempt to relate the dual processes to one another in the general situation, for the description requires notions that are not used in this paper.

Dual systems of terminal times are studied in the remainder of the section. A positive function and an analytic set determine one system for processes X, another for processes $\hat{X}$; let $K_\tau(r, ds)$ and $\hat{K}_\tau(dr, s)$ be the associated relative transition measures. Functions $k(\gamma, r, s)$ and $\hat{k}(\gamma, r, s)$ exist, well defined but usually not continuous, which satisfy the relations

$$\int \gamma(\tau)K_\tau(r, B)\, d\tau = \int_B k(\gamma, r, s)\, ds,$$

$$\int \gamma(\tau)\hat{K}_\tau(C, s)\, d\tau = \int_C \hat{k}(\gamma, r, s)\, dr.$$

The two functions turn out to be the same. This fundamental equality can be established for almost all r and s in a number of ways. In the situation discussed a moment ago, for example, the equality in this sense hardly needs proving; but even there the absence of exceptional points is not immediate. We shall follow this course: The definition of $h(\gamma, r, s)$ is first extended to lower semicontinuous γ. The functions $k(\gamma, r, s)$ are then defined and some of their elementary properties discussed. It is shown that $k(\gamma, r, s)$ can be approximated naturally by continuous functions, provided γ is suitably restricted; the approximation is the means of establishing a relation without exceptions once it is known almost everywhere. The equality of $k(\gamma, r, s)$ and $\hat{k}(\gamma, r, s)$ almost everywhere, for certain systems of terminal times, is next proved by studying the semigroups induced by $K_\tau(r, ds)$ and $\hat{K}_\tau(dr, s)$ on the space of square integrable functions. The strict equality is finally proved by several passages to the limit.

The composition $f * g$ of two positive functions on $\mathcal{H} \times \mathcal{H}$ is defined to be

$$f * g.(r, s) \equiv \iint f(r, t)g(t, s)\, dt, \qquad r, s \in \mathcal{H}.$$

The operation is associative but not commutative.

We take $h(\alpha, r, s)$, for α positive lower semicontinuous, to be the supremum of $h(\gamma, r, s)$ as γ ranges over the functions in Γ nowhere exceeding α; it will often be denoted by $h(\alpha)$ or h_α when considered as a function on $\mathcal{H} \times \mathcal{H}$. Clearly, $h(\alpha, r, s)$ is lower semicontinuous in r and in s, and linear in α in the sense that $h_{\alpha+\beta}$ coincides with $h_\alpha + h_\beta$ and $h_{\lambda\alpha}$ with λh_α. Also, h_α increases to h_β as α increases to β, and $h_{\alpha*\beta}$ coincides with $h_\alpha * h_\beta$.

Let a be a positive function on $\mathcal{H}$, A an analytic subset of $\mathcal{H}$, and $\mathfrak{R}$ a system of terminal times for the processes X determined by a and $A \cup \{w\}$; we shall say, simplifying the language, that $\mathfrak{R}$ is determined by a and A. The transition measures relative to $\mathfrak{R}$ are denoted by $K_\tau(r, ds)$, as in §10, and K_τ, K_α, $K(\alpha, r, ds)$ are defined in the same manner as H_τ, H_α, $H(\alpha, r, ds)$. The notation of §10 will be used, with a few changes in meaning of no consequence. I is the interval $0 \leqq \sigma < \infty$, and I_τ the interval $0 \leqq \sigma < \tau$. Now $H_\mathfrak{R}(r, d\sigma, ds)$ is regarded, for each r in $\mathcal{H}$, as the measure on $I \times \mathcal{H}$ defined as

$$(17.5) \qquad H_\mathfrak{R}(r, C, D) = \mathcal{P}\{R \in C, X(R) \in D\}, \qquad C \subset I, \quad D \subset \mathcal{H},$$

with X a process starting at r and R the associated terminal time; this definition agrees with (10.5) because only subsets of $\mathcal{H}$ are considered. Equation (10.6) reads

$$H_\tau(r, B) = K_\tau(r, B) + \int_{I_\tau \times \mathcal{H}} H_\mathfrak{R}(r, d\sigma, dt)H_{\tau-\sigma}(t, B) + \mathcal{P}\{R = \tau, X(\tau) \in B\}.$$

Since the last term on the right vanishes for all but countably many values of τ, one obtains

$$H(\alpha, r, B) = K(\alpha, r, B) + \int_{I\times\mathcal{K}} H_{\mathfrak{R}}(r, d\sigma, dt)H(\alpha^{\sigma}, t, B)$$

after integrating with respect to the positive measure $\alpha(\tau)d\tau$; here α^{σ} is the translated function defined as $\alpha(\tau + \sigma)$ for τ in J. Since $K(\alpha, r, ds)$ is dominated by $H(\alpha, r, ds)$, it is absolutely continuous relative to ξ if α belongs to Γ.

For given r in $\mathcal{K}$ and α in Γ, any choice of the density function satisfies the equation

$$h(\alpha, r, s) = k(\alpha, r, s) + \int_{I\times\mathcal{K}} H_{\mathfrak{R}}(r, d\sigma, dt)h(\alpha^{\sigma}, t, s) \tag{17.6}$$

for almost all s; we define $k(\alpha, r, s)$ unambiguously as the one solution of the equation. The integral is positive, lower semicontinuous in s, and dominated by $h(\alpha, r, s)$ for all s because it is so for almost all s. Thus, $k(\alpha, r, s)$ is positive, upper semicontinuous in s, and not greater than $h(\alpha, r, s)$. It is linear and increasing in α. It vanishes identically in α and s if r is regular for $\mathfrak{R}$, because the integral then reduces to $h(\alpha, r, s)$. It increases to $k(\beta, r, s)$ as α increases to a function β that also belongs to Γ, for h_α and the integral have as limits h_β and the integral written for β instead of α.

We define k_α, for α positive lower semicontinuous, to be the supremum of k_γ as γ ranges over the functions in Γ nowhere exceeding α; here k_α is understood to be $k(\alpha, r, s)$ regarded as a function on $\mathcal{K} \times \mathcal{K}$. The properties mentioned in the last four sentences of the preceding paragraph carry over at once, except for the upper semicontinuity in s. In addition, $k(\alpha, r, s)$ is always the density function of $K(\alpha, r, ds)$, in the sense that

$$K(\alpha, r, B) = \int_B k(\alpha, r, s)\, ds, \qquad B \subset \mathcal{K},$$

and it satisfies equation (17.6) if the integral on the right is finite—in particular, if $h(\alpha, r, s)$ is finite.

The function $h_\alpha * k_\beta$ is easily seen to be continuous in its first space variable if α and β belong to Γ: The lower semicontinuity of $h_\alpha * k_\beta$ and $h_\alpha * (h_\beta - k_\beta)$ in that variable follows from Fatou's lemma; the sum of these functions is $h(\alpha * \beta)$, continuous in each space variable; so the two functions must be continuous in the first variable. Similarly, $k_\beta * h_\gamma$ is continuous in the second space variable and $h_\alpha * k_\beta * h_\gamma$ is continuous in each, provided γ also belongs to Γ.

The functions α, β, γ are assumed to be in Γ during this paragraph, α and γ having unit integrals; α', β', γ' are restricted similarly and are supposed to satisfy the relations $\alpha \prec \alpha'$ and $\gamma \prec \gamma'$. We shall prove that $k_{\beta'} * h_{\gamma'}$ dominates $k_\beta * h_\gamma$ if $\beta' * \gamma'$ coincides with $\beta * \gamma$. It is first of all clear that the kernel $K_\sigma H_\tau$ decreases with the number τ, the sum $\sigma + \tau$ being held fast; the kernel $K_\beta H_\gamma$ is the integral of $K_\sigma H_\tau$ with respect to the measure $\beta(\sigma)\gamma(\tau)\, d\sigma\, d\tau$;

the domination of $K_\beta H_\gamma$ by $K_{\beta'} H_{\gamma'}$ follows from the relative position of the supports of $\beta(\sigma)\gamma(\tau)$ and $\beta'(\sigma)\gamma'(\tau)$; the assertion to be proved is then immediate, since $k_\beta * h_\gamma$ is the density corresponding to the kernel $K_\beta H_\gamma$ and is continuous in the second space variable. Similarly, the inequality

$$h_\alpha * k_\beta * h_\gamma \leqq h_{\alpha'} * k_{\beta'} * h_{\gamma'}$$

holds if $\alpha * \beta * \gamma$ coincides with $\alpha' * \beta' * \gamma'$; here α or γ may be allowed to coincide with α' or γ'. The argument also establishes the inequality

$$h_\alpha * k_{\beta}.(r, s) \leqq h_{\alpha'} * k_{\beta'}.(r, s),$$

for r fixed and almost all s, provided $\alpha * \beta$ coincides with $\alpha' * \beta'$. The relation holds in fact without exception, as we shall see at the end of the section.

A function γ in Γ is said to be special if it can be written as $\alpha_m * \beta_{mn} * \alpha_n$, for every pair of positive integers m and n, with β_{mn} belonging to Γ and the α_m forming a fundamental sequence in Γ_1. The function $\beta_{mn} * \alpha_n$, which is independent of n, will be denoted by β_m; evidently γ can be written as $\alpha_m * \beta_m$ also. Every function in Γ, hence every positive lower semicontinuous function on J, is the limit of an increasing sequence of special functions; results proved for special functions are therefore easily translated to more general ones.

Proposition 17.1. *If γ is special, then*

$$k(\beta_n) * h(\alpha_n), \qquad h(\alpha_m) * k(\beta_{mn}) * h(\alpha_n)$$

both decrease to $k(\gamma)$ as m and n increase to infinity.

The functions have already been shown to decrease as m and n increase.

On writing equation (17.6) with β_n in place of α and then composing every term with $h(\alpha_n)$, one obtains

$$h(\gamma, r, s) = k_\beta * h_{\alpha}.(r, s) + \int H_{\mathfrak{R}}(r, d\sigma, dt) h(\beta^\sigma * \alpha, t, s),$$

where α and β are understood to have the subscript n. Now, $\beta_n^\sigma * \alpha_n$ increases to γ^σ, so that $k(\beta_n) * h(\alpha_n)$ indeed decreases to $k(\gamma)$.

The next step of the proof uses a remark or two that should have been made at the moment terminal times were defined. Let X be a process, Z a positive random variable independent of X and having $e^{-\sigma}$ as density function for positive σ, and S the infimum of the strictly positive τ satisfying the inequality

$$\int_0^\tau a(X(\sigma))\, d\sigma \geqq Z.$$

Then, for every strictly positive ε, the strict inequality

$$\int_0^{S+\varepsilon} a(X(\sigma))\, d\sigma > Z$$

holds almost everywhere on the set where S is finite. Otherwise, as one easily sees, the equation

$$\int_0^\tau a(X(\sigma))\,d\sigma = Z$$

would hold with strictly positive probability for some τ; now, this is absurd, for the two members are independent and Z has a continuous distribution. Consequently, the terminal time R, assigned to X by the pair (a, A) and the auxiliary variable Z, coincides with the infimum of the strictly positive τ for which one of the assertions

$$X(\tau) \in A, \qquad \int_0^\tau a(X(\sigma))\,d\sigma > Z,$$

is true, provided one neglects a set of probability null.

Consider a sequence of stopping times T_n for X that decrease to 0; suppose X, Z, T_n to be independent; and take R_n to be the terminal time assigned to the shifted process $X(\tau + T_n)$ by the pair (a, A), with Z as auxiliary variable. The sum $T_n + R_n$ obviously decreases as n increases; according to the paragraph above, the limit is R with probability 1. Another choice of auxiliary variables yields a little more. Let Z' be a positive random variable distributed like Z and independent of the random quantities mentioned so far. Take as auxiliary variable in defining R_n the one that coincides with

$$Z - \int_0^{T_n} a(X(\sigma))\,d\sigma$$

on the set where R exceeds T_n and with Z' on the complement. Again $T_n + R_n$ decreases to R with probability 1; this time it coincides with R on the set where R exceeds T_n, a set which increases with n and which ultimately exhausts Ω up to a null set if X starts from a point not regular for the system $\mathfrak{R}$ determined by a and A.

We shall next prove a weak statement concerning the behavior of $h(\alpha_m) * k(\beta_m)$. Let X be a process starting at the point r, with associated terminal time R. In order to simplify the interpretation, the function γ—hence every β_m also—will be assumed to have unit integral. Let S be a positive random variable having γ for density function, and write it as the sum of independent random variables T_m and U_m having α_m and β_m for density functions. We assume these random variables to take on values in the supports of the corresponding density functions and to be independent of X and the auxiliary variable Z used in defining R. Clearly, T_n decreases to 0 as n increases, and U_n remains bounded away from 0. The extended Markoff property implies the relation

$$\int_B ds \int_{\mathfrak{X}} h(\alpha_m, r, t)k(\beta_m, t, s)\,dt = \mathcal{P}\{X(S) \in B, R_m > U_m\},$$

where R_m has the same meaning as before and B is a set in $\mathcal{H}$. On the other hand,

$$\int_B k(\gamma, r, s)\, ds = \mathcal{P}\{X(S) \in B, R > S\}.$$

Now, by the foregoing paragraph, the first probability decreases to the second. Consequently, the integral $\int h(\alpha_m, r, t)k(\beta_m, t, s)\, dt$ decreases to $k(\gamma, r, s)$ for r fixed and almost all s. In this statement one may obviously replace β_m by β_{mn} and γ by β_n; we shall use this fact immediately.

The equations

$$\begin{aligned} \lim_{m,n} h(\alpha_m) * k(\beta_{mn}) * h(\alpha_n) &= \lim_n \lim_m h(\alpha_m) * k(\beta_{mn}) * h(\alpha_n) \\ &= \lim_n k(\beta_n) * h(\alpha_n) \\ &= k(\gamma) \end{aligned}$$

complete the proof of the proposition. The transformations are justified by monotonicity, the paragraph above, and the first step of the proof.

It will be seen at the end of the section that also $h(\alpha_m) * k(\beta_m)$ decreases to $k(\gamma)$.

We can now prove that $k_\alpha * k_\beta$ coincides with $k_{\alpha*\beta}$ whenever α and β are positive lower semicontinuous. It is enough to prove the assertion for a function δ in Γ and a special function γ, since the general assertion follows on considering increasing sequences. First note that $\delta * \gamma$ is special, the factorizations being $\alpha_m * (\delta * \beta_{mn}) * \alpha_n$ in the notation used above. The equation

$$k(\delta) * k(\beta_n) * h(\alpha_n) = k(\delta * \beta_n) * h(\alpha_n)$$

holds without exceptions, since it holds almost everywhere in the second space variable once the first is fixed and since both members are continuous in the second space variable. According to the proposition, the left member decreases to $k_\delta * k_\gamma$ and the right one to $k_{\delta*\gamma}$.

We shall consider next the semigroups induced by the relative transition measures on certain Banach spaces. Let $\mathcal{K}$ be the set of points in $\mathcal{H}$ not regular for $\mathfrak{N}$, and let ξ' be the restriction to $\mathcal{K}$ of the basic measure ξ. The Banach space $\mathcal{L}^p(\mathcal{K})$ is defined in terms of ξ', with p restricted by the inequalities $1 \leqq p < \infty$, and $\mathfrak{M}(\mathcal{K})$ is the Banach space of bounded functions on $\mathcal{K}$, two functions being identified if they coincide almost everywhere on $\mathcal{K}$. These spaces may be considered subspaces of the corresponding spaces $\mathcal{L}^p(\mathcal{H})$ and $\mathfrak{M}(\mathcal{H})$, defined in terms of the basic measure; they are indeed the subspaces comprising the functions vanishing outside $\mathcal{K}$.

It has been noted that $k(\alpha, r, s)$ vanishes identically in α and s if r lies outside $\mathcal{K}$. The analogous statement for s outside $\mathcal{K}$ is usually false. For given α and r, however, $k(\alpha, r, s)$ vanishes almost everywhere on the complement of $\mathcal{K}$. In verifying this statement we shall assume α to have unit integral, for simplicity. Let X be a process starting at r, let R be the corresponding terminal time, and let U be a positive random variable having α for density function and independent of X and R. The mass attributed to $\mathfrak{K} - \mathcal{K}$ by the measure $k(\alpha, r, s)\, ds$ is just the probability of the joint event that $X(U)$ belongs to $\mathfrak{K} - \mathcal{K}$ and R exceeds U; now, this probability obviously vanishes, by the definition of $\mathcal{K}$, so that the statement is true. It is therefore unnecessary to specify the domain of integration to be $\mathcal{K}$, rather than $\mathfrak{K}$, when one uses the measure $k(\alpha, r, s)\, ds$.

If γ belongs to Γ and f to $\mathfrak{L}^p(\mathcal{K})$ or $\mathfrak{M}(\mathcal{K})$, the integral

$$K_\gamma f.(r) \equiv \int k(\gamma, r, s) f(s)\, ds \tag{17.7}$$

is finite for every r, because $k(\gamma, r, s)$ belongs both to $\mathfrak{L}^1(\mathcal{K})$ and to $\mathfrak{M}(\mathcal{K})$; and the integral vanishes for r outside $\mathcal{K}$. Since $k(\gamma, r, s)$, considered as a function of either space variable, has an integral bounded by that of γ, one may take the equation to define a linear transformation $f \to K_\gamma f$ of either $\mathfrak{L}^1(\mathcal{K})$ or $\mathfrak{M}(\mathcal{K})$ into itself, the bound of the transformation not exceeding the integral of γ. This statement remains true with $\mathfrak{L}^1(\mathcal{K})$ replaced by $\mathfrak{L}^p(\mathcal{K})$, for every value of p, according to the convexity theorem of Marcel Riesz. These transformations will be denoted indifferently by K_γ.

Let g be the restriction to $\mathcal{K}$ of a positive continuous function with compact support in $\mathfrak{K}$. The function $K_\tau g$ tends to g at every point of $\mathcal{K}$, as $\tau \to 0$; hence $K_\gamma g$ does so as γ runs through a fundamental sequence. The integral of $K_\gamma g$ is moreover bounded by the integral of g, since γ has unit integral. As a consequence, $K_\gamma g$ approaches g in the norm of each space $\mathfrak{L}^p(\mathcal{K})$ as γ runs through a fundamental sequence. Now, linear combinations of functions like g are dense in every $\mathfrak{L}^p(\mathcal{K})$. Therefore K_γ, considered as a transformation on a particular space $\mathfrak{L}^p(\mathcal{K})$, tends strongly to the identity transformation as γ runs through a fundamental sequence. One can now apply the argument yielding the existence of the semigroup $\hat{H}_\tau$. On each space $\mathfrak{L}^p(\mathcal{K})$ there is a semigroup of transformations K_τ such that the bound of K_τ does not exceed 1, such that K_τ tends strongly to the identity transformation as $\tau \to 0$, and such that

$$K_\gamma f = \int \gamma(\tau) K_\tau f\, d\tau, \qquad \gamma \in \Gamma, \quad f \in \mathfrak{L}^p(\mathcal{K}).$$

Ordinarily, a semigroup with these propertes does not exist on $\mathfrak{M}(\mathcal{K})$.

On starting with the functions $h(\gamma, r, s)$ one obtains a similar semigroup of transformations H_τ on the space $\mathfrak{L}^p(\mathfrak{K})$. If $\mathcal{K}$ coincides with $\mathfrak{K}$—that is to say, if no point is regular for $\mathfrak{R}$—two semigroups acting on the same space have been defined. We proceed to relate their infinitesimal generators, assuming a further restriction on the system of terminal times.

Let us suppose $\mathfrak{R}$ to be determined by a bounded function a, the set A being empty; clearly, no point is regular for this system. Denote by I the infinitesimal generator of the semigroup H_τ acting on $\mathfrak{L}^p(\mathfrak{K})$. The infinitesimal generator of the semigroup K_τ acting on the same space will be shown to have the same domain as I and to be precisely $I - a$, where a stands for multiplication by the function a.

Given a function f in $\mathfrak{L}^p(\mathfrak{K})$, define a family of functions g_τ by setting

$$g_\tau \equiv \frac{1}{\tau}(H_\tau f - f) - \frac{1}{\tau}(K_\tau f - f), \qquad \tau > 0.$$

The assertion concerning the infinitesimal generators will clearly be proved once g_τ is shown to approach af in the norm of $\mathfrak{L}^p(\mathfrak{K})$ as $\tau \to 0$. Now, g_τ may also be expressed as

$$g_\tau(r) = \frac{1}{\tau}\int_\Omega \left[1 - \exp\left\{-\int_0^\tau a(X(\rho))\,d\rho\right\}\right] f(X(\tau))\,d\omega,$$

where X is a process starting at r. Since a is bounded, one has the estimate

$$1 - \exp\left\{-\int_0^\tau a(X(\rho))\,d\rho\right\} = \int_0^\tau a(X(\rho))\,d\rho + O(\tau^2),$$

with $O(\tau^2)$ uniform in ω and r. Consequently, $g_\tau(r)$ may be written

$$\frac{1}{\tau}\int_0^\tau d\rho \int_\Omega f(X(\tau))a(X(\rho))\,d\omega + O(\tau)\int_\Omega f(X(\tau))\,d\omega.$$

The second term approaches zero in norm, since the integral is $H_\tau f$. The first term approaches af in norm, as one sees on writing it in the form

$$\frac{1}{\tau}\int_0^\tau H_\rho(aH_{\tau-\rho}f)\,d\rho \equiv \frac{1}{\tau}\int_0^\tau H_\rho(af)\,d\rho + \frac{1}{\tau}\int_0^\tau H_\rho(aH_{\tau-\rho}f - af)\,d\rho$$

and noting that multiplication by a is a bounded operation.

The identity of the functions k_γ and $\hat{k}_\gamma$ is easily proved, for systems $\mathfrak{R}$ and $\hat{\mathfrak{R}}$ determined by a bounded function a, by applying the foregoing result to the semigroups K_τ and $\hat{K}_\tau$ acting on $\mathfrak{L}^2(\mathfrak{K})$. We denote by $\langle f, g\rangle$ the inner product of two elements of $\mathfrak{L}^2(\mathfrak{K})$. Both $\langle f, H_\gamma g\rangle$ and $\langle f\hat{H}_\gamma, g\rangle$ have the expression

$$\iint f(r)h(\gamma, r, s)g(s)\,dr\,ds,$$

so that the transformations H_γ and $\hat{H}_\gamma$ are adjoint. Then H_τ and $\hat{H}_\tau$ are adjoint for all τ, so also the infinitesimal generators I and $\hat{I}$. It follows that $I - a$ and $\hat{I} - a$ are adjoint, for multiplication by a is a bounded self-adjoint operator. The semigroups K_τ and $\hat{K}_\tau$ are therefore adjoint. The resulting equations

$$\begin{aligned}\iint f(r)k(\gamma, r, s)g(s)\,dr\,ds &= \langle f, K_\gamma g\rangle \\ &= \langle f\hat{K}_\gamma, g\rangle = \iint f(r)\hat{k}(\gamma, r, s)g(s)\,dr\,ds\end{aligned}$$

show that k_γ and $\hat{k}_\gamma$ coincide almost everywhere on the product space $\mathfrak{K} \times \mathfrak{K}$, if that space is given the product measure $\xi \times \xi$.

We shall remove the restriction on the systems of terminal times before eliminating the possibility of exceptional points. First, let the bounded function a increase to the arbitrary positive function a'. The terminal time R assigned to a process X by a decreases with probability 1 to the time R' assigned by a', provided the same auxiliary variable is used throughout. Let us take γ to have unit integral, and choose a positive random variable U having γ for density function and independent of the auxiliary variable as well as of the process X, which we suppose to start at the point r. Then

$$\int_B k(\gamma, r, s)\, ds \equiv \mathcal{P}\{X(U) \in B, R > U\} \tag{17.8}$$

decreases to

$$\int_B k'(\gamma, r, s)\, ds \equiv \mathcal{P}\{X(U) \in B, R' > U\} \tag{17.9}$$

for every set B in $\mathfrak{K}$, because R or R' can coincide with U only with probability null. Consequently, for r fixed and for almost all s, $k(\gamma, r, s)$ decreases to $k'(\gamma, r, s)$. Since $\hat{k}(\gamma, r, s)$ converges correspondingly to $\hat{k}'(\gamma, r, s)$, defined by the system determined by a' for the processes $\hat{X}$, the functions k'_γ and $\hat{k}'_\gamma$ must agree almost everywhere on $\mathfrak{K} \times \mathfrak{K}$ according to the preceding paragraph.

The result shows that k_γ and $\hat{k}_\gamma$ agree almost everywhere whenever the dual systems of terminal times are determined by a positive function a and an open set G, for one may dispense with G in the definition by taking a to be infinite at every point of G. Let us denote by $\mathfrak{R}$ the system determined for the processes X by such a pair (a, G), and by $\mathfrak{R}'$ the system determined by a and a compact set F, the same auxiliary variables being used in defining corresponding terminal times. We now let G run through a decreasing sequence of neighborhoods of F that have compact closures shrinking to F. Almost all points of F are regular for F, hence regular for $\mathfrak{R}$ and $\mathfrak{R}'$; if r is such a point, then $k(\gamma, r, s)$ and $k'(\gamma, r, s)$ both vanish identically in s. Consider a point r outside F. The terminal time assigned by $\mathfrak{R}$ to a process starting at r increases with probability 1 to the time assigned by $\mathfrak{R}'$, so that again the integral in (17.8) approaches the one in (17.9), but through an increasing sequence in the present circumstances. Thus k_γ increases to k'_γ almost everywhere on $\mathfrak{K} \times \mathfrak{K}$. It is now clear that k'_γ and $\hat{k}'_\gamma$, determined by the system dual to $\mathfrak{R}'$, agree almost everywhere.

The next passage to the limit requires a strong version of Proposition 2.1 which will be proved using hypothesis (F), although weaker assumptions would suffice. Let X be a process having initial distribution $g(r)\, dr$, with g a strictly positive function. Given an analytic set A, choose an increasing sequence of compact subsets F_n of A so that the time X hits F_n decreases with probability 1 to the time X hits A; this can be done, according to Proposition 2.1. Let X be an arbitrary process. We shall prove that the time T_n at which X hits F_n increases with probability 1 to the time T at which X hits A. The statement is obvious for a process whose initial distribution is absolutely continuous relative to the basic measure. To treat an arbitrary process, choose a fundamental sequence of functions α_m in Γ_1 and corresponding random variables U_m that have the α_m for density functions and are independent of X. Let T^m or T_n^m be the time the process $X(\tau + U_m)$ hits A or F_n; this process has an absolutely continuous initial distribution, except for a small mass at w, so that T_n^m decreases to T^m with probability 1 as n increases. Now, we have seen earlier that $U_m + T^m$ decreases to T with probability 1 as m increases. Therefore, T_n must decrease to T with probability 1, since it is dominated by $U_m + T_n^m$ for every m.

Consider now the dual systems $\mathfrak{R}$ and $\hat{\mathfrak{R}}$ determined by a positive function a and an analytic set A. A repetition of the arguments above, using the systems determined by a and the compact sets F_n of the preceding paragraph, establishes the agreement of the functions k_γ and $\hat{k}_\gamma$ almost everywhere on $\mathfrak{X} \times \mathfrak{X}$. We shall prove there are no exceptional points.

Suppose first γ to be special, with factorizations $\alpha_m * \beta_{mn} * \alpha_n$. The equation

$$h(\alpha_n) * k(\beta_{nn}) * h(\alpha_n) = h(\alpha_n) * \hat{k}(\beta_{nn}) * h(\alpha_n)$$

holds almost everywhere on $\mathfrak{X} \times \mathfrak{X}$, because the functions $k(\beta_{nn})$ and $\hat{k}(\beta_{nn})$ coincide almost everywhere. The two members are continuous in each space variable; so the equation must hold without exceptions. On passing to the limit with the help of Proposition 17.1, we see that k_γ and $\hat{k}_\gamma$ must indeed be the same function. This result extends at once to γ positive lower semicontinuous, for such a function is the limit of an increasing sequence of special functions.

Some of the results obtained in the course of the proof can now be completed. Proposition 17.1, for example, should also include the statement that $h(\alpha_m) * k(\beta_m)$ decreases to k_γ. Such consequences of the identity of k_γ and $\hat{k}_\gamma$ will be used frequently without special mention. The equation

$$(17.10) \qquad \int_{I\times\mathfrak{X}} H_{\mathfrak{R}}(r, d\sigma, dt)h(\alpha^\sigma, t, s) = \int_{I\times\mathfrak{X}} h(\alpha^\sigma, r, t)\hat{H}_{\hat{\mathfrak{R}}}(d\sigma, dt, s),$$

valid for α positive lower semicontinuous, follows at once from the identity of k_α and $\hat{k}_\alpha$ and the definitions of these functions, provided α belongs to Γ. It is extended to the more general functions by a passage to the limit.

18. Potentials and excessive functions

Let φ be a function on $\mathfrak{X}$ excessive relative to the semigroup H_τ and let γ be an element of Γ_1. The equations

$$H_\gamma \varphi = \int \gamma(\tau) H_\tau \varphi \, d\tau = \int h(\gamma, r, s)\varphi(s) \, ds$$

show first that $H_\gamma \varphi$ increases to φ as γ runs through a fundamental sequence, and next that φ is lower semicontinuous, for the second integral is lower semicontinuous in r by Fatou's lemma. Hypothesis (F) thus binds excessive functions more closely to the topology of $\mathcal{K}$ than do the earlier hypotheses, which imply a little less than Borel measurability.

Suppose φ to be finite on a set dense in $\mathcal{K}$; it must then be integrable on some neighborhood of any given point s. Indeed, choose γ in Γ_1 so that $h(\gamma, r, s)$ does not vanish identically in r, then fix r so that $\varphi(r)$ is finite and $h(\gamma, r, s)$ strictly positive; the inequality

$$\varphi(r) \geqq \int h(\gamma, r, t)\varphi(t)\, dt$$

implies the integrability of φ on some neighborhood of s, because $h(\gamma, r, t)$ is continuous in t and strictly positive for t near s. On coupling this result with the fact that a set is approximately null if and only if it is a null set for the basic measure ξ, we see that saying φ is finite on a dense set, or finite almost everywhere, or finite except on a set negligible for the processes X, or integrable over compact subsets of $\mathcal{K}$ are four equivalent ways of stating the natural requirement of finiteness.

Let us denote by ζ the measure $\varphi(s)\, ds$, with φ excessive for the H_τ and integrable over compact sets. The measure $\hat{H}_\gamma\, \zeta$,

$$\hat{H}_\gamma\, \zeta.(B) \equiv \int \hat{H}_\gamma(B, s)\zeta(ds), \qquad B \subset \mathcal{K}, \quad \gamma \in \Gamma_1,$$

is excessive relative to the semigroup $\hat{H}_\tau$, because the density of $\hat{H}_\tau(\hat{H}_\gamma\, \zeta)$ is the function $H_\gamma(H_\tau\, \varphi)$, and it increases to ζ as γ runs through a fundamental sequence. The measure ζ itself is therefore excessive for the semigroup $\hat{H}_\tau$, since it is finite on compact sets.

Conversely, let ζ be a measure excessive relative to the semigroup $\hat{H}_\tau$, and let γ be an element of Γ_1. The measure $\hat{H}_\gamma\, \zeta$ is the indefinite integral of the function φ_γ,

$$\varphi_\gamma(r) \equiv \int h(\gamma, r, s)\zeta(ds).$$

This function is excessive for the H_τ, because

$$H_\tau \varphi_\gamma.(r) \equiv \int h(\gamma, r, s)\hat{H}_\tau\, \zeta.(ds)$$

increases to $\varphi_\gamma(r)$ as τ decreases to 0, by the definition of excessive measure. Suppose the relation $\gamma \prec \gamma'$ to hold, so that $\hat{H}_\gamma\, \zeta$ dominates $\hat{H}_{\gamma'}\, \zeta$; then φ_γ dominates $\varphi_{\gamma'}$ everywhere, because it does so almost everywhere and the two functions are excessive. Thus φ_γ increases to an excessive function φ as γ runs through a fundamental sequence, and the limit does not depend upon the particular fundamental sequence chosen. The equation

$$\zeta(B) = \int_B \varphi(s)\, ds, \qquad B \subset \mathcal{K}, \tag{18.1}$$

holds identically, by monotone convergence. Since $\hat{\xi}$ is finite on compact sets, φ is finite on a dense set as well as excessive relative to the semigroup H_τ.

By the last two paragraphs, equation (18.1) and its dual establish a one-to-one correspondence between measures that are excessive for one of the dual semigroups and functions that are excessive for the other semigroup and finite almost everywhere. The correspondence will be used to eliminate excessive measures, statements being phrased in terms of the corresponding excessive functions. This amounts to little more than a straightforward translation, with a loss of intuitive content on occasion. A few statements become more precise, however, partly because a natural density function is finer than the corresponding measure, and partly because the situation being treated is more regular than the one discussed in the second installment.

From now on, an excessive function is understood to be finite on a set dense in $\mathcal{K}$. A function or measure excessive relative to the semigroup H_τ is said to be right excessive; a point is said to be right regular for an analytic set if it is regular for the system of terminal times defined by the set for the processes X. Left excessive and left regular are defined similarly, in terms of the semigroup $\hat{H}_\tau$ and the processes $\hat{X}$. Propositions are usually stated for right excessive functions.

In keeping with the replacement of excessive measures by functions, the kernel $U(r, s)$ for potentials is taken to be $h(\alpha, r, s)$, with α the constant 1 on J. It is positive lower semicontinuous in each variable, even right excessive in r and left excessive in s if the requirements of finiteness are met.

Let $\mathfrak{R}$ and $\hat{\mathfrak{R}}$ be the dual systems of terminal times determined by a positive function and an analytic set. The kernel $\hat{H}_{\hat{\mathfrak{R}}}(dr, s)$ is defined by the formula.

$$\hat{H}_{\hat{\mathfrak{R}}}(B, s) \equiv \mathcal{P}\{\hat{X}(\hat{R}) \in B, \hat{R} < \infty\}, \qquad B \subset \mathcal{K}, \tag{18.2}$$

where $\hat{X}$ is a process starting at the point s and $\hat{R}$ is the time assigned by the system $\hat{\mathfrak{R}}$. The kernel $H_{\mathfrak{R}}(r, ds)$ is defined similarly. We write $H_E(r, ds)$ and $\hat{H}_E(dr, s)$ if the systems are defined by the analytic set E and the null function. Equation (17.10) implies the important relation

$$\int H_{\mathfrak{R}}(r, dt)U(t, s) = \int U(r, t)\hat{H}_{\hat{\mathfrak{R}}}(dt, s), \tag{18.3}$$

which is often abbreviated to $H_{\mathfrak{R}} U = U\hat{H}_{\hat{\mathfrak{R}}}$. It is also clear that $H_\tau U$ coincides with $U\hat{H}_\tau$. In the preceding definitions there are tacit conventions regarding the points w and $\hat{w}$ introduced in the last section. A right excessive function, whether finite almost everywhere or not, is defined on the augmented space $\mathcal{K} \cup \{w\}$ and vanishes at w. Similarly, a left excessive function is defined at $\hat{w}$ and vanishes there. In the expression $U(r, s)$, consequently, r may be the point w or s the point $\hat{w}$, the kernel being null in either event. One should permit s to be $\hat{w}$ and B to include w in the definition (18.2), with the understanding that $\hat{H}_{\hat{\mathfrak{R}}}(B, s)$ vanishes if B reduces to w or s coincides with $\hat{w}$. The definition of $H_{\mathfrak{R}}(r, ds)$ is completed in like manner. Such conventions, which are needed only in carrying over the results of earlier sections, will seldom be mentioned in the future. Note also that $U(r, s)$ is not a kernel in the sense of §3, but rather the corresponding density function; more exactly, there are two kernels in the old sense, $U(r, s)\, ds$ and $U(r, s)\, dr$, corresponding to the two semigroups H_τ and $\hat{H}_\tau$, and $U(r, s)$ is the family of density functions derived from either kernel. We shall retain the old notation for the composition of kernels, as in writing $H_{\mathfrak{R}}\, U$ above, when there is little chance of ambiguity.

The results of §§10–14 obviously hold for right excessive functions; the space is $\mathcal{K} \cup \{w\}$, the processes are the processes X terminated on reaching w, and the special sets are the open sets with compact closure in $\mathcal{K}$. One may also apply the simple theory of §§4–9, with the terminal time identically infinite, for a right excessive function is one which is excessive relative to the measures $P_\tau(r, ds)$ introduced in the last section and which vanishes at the point w.

The right potential $U\nu$ of a measure ν on $\mathcal{K}$ is the function

$$U\nu.(r) \equiv \int U(r, s)\nu(ds),$$

which is positive lower semicontinuous, and right excessive if finite almost everywhere. It is of course the density function of the measure that was formerly named the potential of ν relative to the semigroup $\hat{H}_\tau$. If ν has the form $f(s)\, ds$, then $U\nu$ is also the potential, in the old sense, of the function f relative to the semigroup H_τ; the notion of right potential may therefore be considered an extension of the old notion of potential of a function. The second point of view accords with our treatment, and we shall suppose right potentials to be defined at w and to vanish there.

The left potential νU is the function

$$\nu U.(s) \equiv \int \nu(dr) U(r, s),$$

which also is positive lower semicontinuous and has two interpretations. It is taken to vanish at $\hat{w}$.

The measure ν must be bounded if it has compact support and if $U\nu$ is finite on a dense set; this statement follows from Proposition 12.1 and the integrability of the potential over compact subsets of $\mathcal{K}$.

If ν is concentrated on the set of points left regular for a certain analytic set E, then $\hat{H}_E\, \nu$ is evidently the same measure as ν; according to relation (18.3), the functions $H_E\, U\nu$ and $U\nu$ must also coincide.

Conversely, suppose $U\nu$ to be finite on a dense set and E to be an analytic set such that $H_E\, U\nu$ coincides with $U\nu$. Then ν must be concentrated on the set of points left regular for E. In the proof we assume ν to have unit mass, without losing generality. Take $\hat{X}$ to be a process having ν as initial distribution, and let $\hat{T}$ be the time $\hat{X}$ hits E. Suppose that ν is not concentrated on the set of points left regular for E or, what is the same, that $\hat{T}$ does not vanish with probability 1. There exists then a compact subset F of $\mathfrak{K}$, with characteristic function χ, such that the integral

$$\int_\Omega d\omega \int_0^{\hat{T}} \chi(\hat{X}(\tau))\, d\tau$$

has a strictly positive value ρ. The integral of $U\nu$ over F is finite, because $U\nu$ is finite on a dense set, and it exceeds the integral of $U\hat{H}_E\,\nu$ over F by exactly the amount ρ. Thus $U\nu$ exceeds $H_E\,U\nu$ at some point of F, by (18.3), and the assertion is proved.

According to the last two paragraphs, if $U\nu$ is finite on a dense set, then the support of ν is the least closed set F in $\mathfrak{K}$ such that $H_G\,U\nu$ coincides with $U\nu$ whenever G is a neighborhood of F.

We shall ordinarily discuss potentials assuming a version of hypothesis (E) of §12 to hold. Suppose (E) to hold for the semigroup H_τ, so that an integral $\int U(r, s)f(s)\, ds$ is bounded in r if f is a bounded function vanishing outside a compact subset of $\mathfrak{K}$. The function $\int g(r)U(r, s)\, dr$ of s must then be integrable over every compact subset of $\mathfrak{K}$ whenever g is integrable over $\mathfrak{K}$, and it is not much of an additional restriction to require the semigroup $\hat{H}_\tau$ to satisfy (E). In order to preserve symmetry, we often assume the following hypothesis (G) rather than impose (E) on one of the dual semigroups.

(G) *If F is a compact subset of $\mathfrak{K}$, then the integrals*

$$\int_F U(r, s)\, ds, \qquad \int_F U(r, s)\, dr,$$

are bounded in r and in s.

Suppose (G) to be true. Then the semigroups H_τ and $\hat{H}_\tau$ satisfy (D), by Proposition 12.5. The right and the left potentials of a bounded measure are finite on a dense set, and either determines the measure, according to Proposition 14.1. A right excessive function is the limit of an increasing sequence of potentials $U\nu_n$, where ν_n may be taken bounded and absolutely continuous relative to the basic measure ξ; this result is contained in Theorem 12.2.

Just as in preceding installments, it is sometimes advantageous to replace the semigroups H_τ and $\hat{H}_\tau$ by the semigroups

$$H_\tau^\lambda \equiv e^{-\lambda\tau} H_\tau\,, \qquad \hat{H}_\tau^\lambda \equiv e^{-\lambda\tau}\hat{H}_\tau\,,$$

with λ a positive parameter. Hypothesis (F) carries over with the same basic measure ξ; indeed, the only change is the replacement of $h(\gamma, r, s)$ by $h(\gamma', r, s)$, where γ' is the function $e^{-\lambda\tau}\gamma(\tau)$. The new semigroups obviously satisfy (G) if λ is strictly positive. Quantities defined in terms of these semigroups will be denoted by $U^\lambda(r, s)$, $H_E^\lambda(r, ds)$, and so on.

We shall now complete the correspondence (18.1) between right excessive functions and left excessive measures by showing that if φ corresponds to $\hat{\xi}$ then $H_E \varphi$ corresponds to $\hat{M}_E \hat{\xi}$; here E is an analytic set and $\hat{M}_E$ is the operation defined as in §14 but in terms of the semigroup $\hat{H}_\tau$. The statement follows at once from (18.3) and the definition of $\hat{M}_E$ if φ is a right potential. The extension to arbitrary right excessive functions, under hypothesis (G), then follows from the definition of $\hat{M}_E$. Since $\hat{M}_E \hat{\xi}$ is defined generally by introducing the parameter λ and taking a limit as λ decreases, the statement obviously holds even when (G) is not satisfied.

In order to develop the consequences of (G) we shall prove, using only (F), that $H_\alpha f$ is continuous on $\mathcal{H}$ as well as bounded if f is a bounded function on $\mathcal{H}$ and α a positive integrable function on J. The function f is of course assumed measurable over the field $\mathcal{A}(\mathcal{H})$, so that $H_\alpha f$ is defined unambiguously by the formula

$$H_\alpha f.(r) \equiv \int H(\alpha, r, ds)f(s) \equiv \int \alpha(\tau)H_\tau f.(r)\, d\tau.$$

Indeed, it is enough to suppose f measurable with respect to the basic measure, for one can easily prove $H(\alpha, r, ds)$ to be absolutely continuous. Since the mass of $H_\tau(r, ds)$ does not exceed 1, we have the estimate

$$\sup | H_\alpha f | \leqq \sup | f | \int \alpha(\tau)\, d\tau, \tag{18.4}$$

which shows that $H_\alpha f$ is bounded if f is bounded and that $H_\beta f$ approaches $H_\alpha f$ uniformly if β varies so that

$$\int | \alpha(\tau) - \beta(\tau) | \, d\tau \to 0. \tag{18.5}$$

The function $H_\alpha f$ belongs to $\mathcal{C}(\mathcal{H})$ if f does so, because $H_\tau f$ is a bounded continuous function of τ in the norm of $\mathcal{C}(\mathcal{H})$. Consequently, $H_\alpha g$ belongs to $\mathcal{C}(\mathcal{H})$ if α is lower semicontinuous and if g is positive and dominated by a function f in $\mathcal{C}(\mathcal{H})$. For, both $H_\alpha g$ and $H_\alpha(f - g)$ are lower semicontinuous, and their sum belongs to $\mathcal{C}(\mathcal{H})$; the lower semicontinuity of $H_\alpha g$, for example, follows from the representation

$$H_\alpha g.(r) = \int h(\alpha, r, s)g(s)\, ds$$

and the lower semicontinuity of $h(\alpha, r, s)$ in the variable r. The lower semicontinuity of α may be dropped from the hypotheses, since a positive integrable function can be approximated in the sense of (18.5) by positive lower semicontinuous functions. In particular, $H_\alpha g$ belongs to $\mathcal{C}(\mathcal{H})$ if α is integrable and if g is bounded and vanishes outside a compact subset of $\mathcal{H}$.

Let F be a compact set, G a neighborhood of F with compact closure in $\mathcal{H}$, and ε a strictly positive number. A simple argument, using only the behavior of the semigroup H_τ on $\mathcal{C}(\mathcal{H})$, yields the existence of a strictly positive δ with the property that

(18.6) $$H_\tau(r, \mathcal{H} - G) < \varepsilon \quad \text{for} \quad \tau < \delta, r \in F.$$

Suppose f to be bounded by 1 and α to have the form $\beta * \gamma$, where α, β, γ are positive functions with unit integrals and $\beta(\tau)$ vanishes for τ greater than δ. Write

$$H_\alpha f = H_\beta\{\chi H_\gamma f\} + H_\beta\{(1 - \chi)H_\gamma f\},$$

with χ the characteristic function of G. The first term on the right is continuous, by the preceding paragraph, and the second term is bounded by ε on F, by the inequality (18.6). The restriction of $H_\alpha f$ to F therefore differs by less than ε from a continuous function.

Let α be any positive function on J with unit integral, and δ any strictly positive number. Clearly, α can be approximated in the sense of (18.5) by functions of the form $\beta * \gamma$, where β and γ are positive functions with unit integrals and $\beta(\tau)$ vanishes for τ greater than δ. This remark, relation (18.4), and the preceding paragraph imply that $H_\alpha f$ is continuous at all points of $\mathcal{H}$ whenever α is integrable and f bounded.

Proposition 18.1. *Let (G) hold, and let f be a bounded function vanishing outside a compact subset of* $\mathcal{H}$. *Then the integrals*

$$\int U(r, s)f(s)\, ds, \qquad \int f(r)U(r, s)\, dr,$$

are continuous in r and in s at all points of $\mathcal{H}$, *as well as bounded.*

It is enough to consider the first integral. Boundedness follows at once from (G). By the definition of the kernel for potentials, the integral is precisely the function $H_\alpha f$, with α identically 1 on J. The function $H_\gamma H_\alpha f$ is continuous on $\mathcal{H}$ if γ belongs to Γ_1, according to what has just been proved, and it converges uniformly to $H_\alpha f$ as γ runs through a fundamental sequence, because f is bounded. So $H_\alpha f$ also is continuous on $\mathcal{H}$.

The first integral is defined naturally at the point w and vanishes there, but usually it is not continuous at w in the topology of the extended space $\mathcal{H} \cup \{w\}$. Similarly, the second integral need not be continuous at $\hat{w}$.

Proposition 18.2. *Let (G) hold, let F be a compact subset of* $\mathcal{H}$, *and let the* ν_n *be a sequence of measures concentrated on F. If* $U\nu_n$ *increases and* $\nu_n(F)$ *remains bounded, as n increases, then* ν_n *converges weakly to a bounded measure* ν *on F, and* $U\nu_n$ *increases everywhere to* $U\nu$.

A subsequence of the ν_n converges weakly to some bounded measure, say ν, with support in F. If f is a bounded function vanishing outside a compact set, then

$$\int \nu_n(ds) \int f(r)U(r, s)\, dr \to \int \nu(ds) \int f(r)U(r, s)\, dr,$$

as ν_n runs through the subsequence, because the inner integral is continuous in s. Thus $U\nu_n$ increases to $U\nu$ almost everywhere; the convergence must take place everywhere, in fact, because both $U\nu$ and the limit function are right excessive. Finally, the measure ν is determined by its potential; so the full sequence ν_n must converge weakly to ν.

PROPOSITION 18.3. *Let (G) hold, let F be a compact subset of* $\mathcal{H}$, *and let the* ν_n *be a sequence of bounded measures concentrated on F. If* $U\nu_n$ *decreases as n increases, then* ν_n *tends weakly to a bounded measure* ν *on F, and* $U\nu_n$ *decreases to* $U\nu$ *almost everywhere.*

By Proposition 12.1, applied to the semigroup $\hat{H}_\tau$, the mass of ν_n is a bounded function of n; indeed, it decreases as n increases. The rest of the proof goes like the preceding one, since a right excessive function is completely determined once it is known almost everywhere.

The compact set F may be replaced by $\mathcal{H}$ in the last two propositions, if in the hypotheses (G) is replaced by the requirement that (E^*) hold for the semigroup $\hat{H}_\tau$. In particular, the measures may be distributed on $\mathcal{H}$ if the parameter λ is taken strictly positive.

PROPOSITION 18.4. *Let (G) hold, let* φ *be a right excessive function, and let E be an analytic set with compact closure in* $\mathcal{H}$. *Then* $H_E\,\varphi$ *is the right potential of a bounded measure* ν *concentrated on the union of E with the set of points left regular for E.*

Since (G) holds, φ is the limit of an increasing sequence of potentials $U\mu_n$. According to (18.3),

$$H_E\,U\mu_n = U\hat{H}_E\,\mu_n\,,$$

so that $H_E\,\varphi$ is the limit of the increasing sequence of potentials $U\nu_n$, where ν_n is the measure $\hat{H}_E\,\mu_n$. The support of ν_n is certainly included in the closure of E. Since the right excessive function φ is integrable over compact sets, Proposition 12.1 implies that the mass of ν_n is bounded uniformly in n. By Proposition 18.2, therefore, $H_E\,\varphi$ is the right potential of a measure ν with support in the closure of E. To see that ν is borne in fact by the union of E with the set of points left regular for E, consider a compact neighborhood G of E. The kernel $H_E\,H_G$ evidently coincides with H_E and, according to what has already been proved, $H_G\,\varphi$ is the right potential of some measure μ. Thus

$$H_E\,\varphi = H_E\,H_G\,\varphi = H_E\,U\mu = U\hat{H}_E\,\mu,$$

so that ν is precisely the measure $\hat{H}_E\,\mu$, which is concentrated on the union of E with the set of points left regular for E since the measure $\hat{H}_E(dr, s)$ is so concentrated for every choice of the point s.

Under (G), a right excessive function φ having a compact subset F of $\mathcal{H}$ for a determining set must be the right potential of a measure with support in F. In proof, consider a compact neighborhood G of F. According to the last proposition, $H_G\,\varphi$ is the right potential of a bounded measure μ concentrated on G, and it coincides with φ by assumption. As G shrinks to F

through a decreasing sequence, $U\mu$ decreases almost everywhere to a potential $U\nu$, according to Proposition 18.3, and the support of ν is included in F because it is included in every neighborhood of F. Since $U\mu$ always coincides with φ, the two functions $U\nu$ and φ must agree almost everywhere; they therefore coincide, being excessive functions. Incidentally, each measure μ reduces to ν.

As we have noted before, the support of a measure ν is a determining set for $U\nu$ even when (G) does not hold, and it is the least determining set if $U\nu$ is finite on a dense set.

Let us take φ to be the constant 1 in the last proposition, E still being included in a compact subset of $\mathfrak{K}$. Then $H_E\,\varphi.(r)$ is the probability that a process X starting at r sometime hits E. The probability will be denoted by $\Phi_E(r)$ as in §5. According to the proposition, Φ_E is the right potential of a bounded measure π_E concentrated on the union of E with the set of points left regular for E. We shall often speak of Φ_E and π_E as the natural right capacitary potential and measure of E. Clearly, π_E vanishes if and only if Φ_E vanishes, that is to say, if and only if E is negligible for the processes X. The natural left capacitary potential and measure, $\hat{\Phi}_E$ and $\hat{\pi}_E$ are defined similarly in terms of the semigroup $\hat{H}_\tau$ and the processes $\hat{X}$. The appropriateness of the names will become clear in the next section.

Proposition 18.5. *Let E and F be analytic sets such that E includes F and such that every point of F is left regular for E, and let T be the time a process X hits E. Then the point $X(T(\omega), \omega)$ is right regular for F for almost all ω such that $T(\omega)$ is finite and $X(T(\omega), \omega)$ belongs to F.*

In the proof we shall suppose F to be compact; no generality is lost thereby, since the proposition holds for an analytic set F if it holds for every compact subset. We shall also suppose (G) to hold and Φ_F to be strictly less than 1 at every point that is not right regular for F; it has already been mentioned in §6, during the discussion of hypothesis (B), that these additional hypotheses can always be achieved by going over to the semigroups H_τ^λ and $\hat{H}_\tau^\lambda$, with λ strictly positive. According to (18.3), one has

$$H_E\,\Phi_F = H_E\,U\pi_F = U\hat{H}_E\,\pi_F = U\pi_F = \Phi_F\,,$$

because π_F is concentrated on F and every point of F is left regular for E. The proposition is proved by the equality of the extreme members and a repetition of the argument following hypothesis (B) in §6.

The conclusion of the proposition amounts to saying, $H_E^\lambda\,H_F^\lambda$ coincides with H_F^λ for every value of λ. In the proposition we have tacitly assumed E and F to be in $\mathfrak{K}$; the conclusion remains valid, however, if the point w is adjoined to E or to both E and F, since w is right regular for the set $\{w\}$. Thus (F) implies hypothesis (B) for the semigroups P_τ and $\hat{P}_\tau$ introduced in the preceding section, because every point of an open subset of $\mathfrak{K}$ is both right and left regular for the set.

Let us suppose (G) to hold and E to be an analytic set having compact closure in $\mathcal{K}$. The measures π_E and $\hat{\pi}_E$ are then defined, the first being concentrated on the union of E with the set of points left regular for E, by what has just been proved, the second being concentrated on the union of E with the set of points right regular for E. If F is a subset of E and if every point of F is both right and left regular for E, then π_F coincides with $\hat{H}_F \pi_E$ and $\hat{\pi}_F$ with $\hat{\pi}_E H_F$; the proof is contained in the restatement of Proposition 18.5. These relations imply that π_E and $\hat{\pi}_E$ have the same mass. To see this, fix G as any neighborhood of E having compact closure in $\mathcal{K}$. Then $U\pi_G$ is identically 1 on the set where $\hat{\pi}_E$ is concentrated, so that

$$\hat{\pi}_E(\mathcal{K}) = \int \hat{\pi}_E(dr) U\pi_G.(r) = \iint \hat{\pi}_G(dr) H_E U.(r, s) \pi_G(ds).$$

One evidently obtains the mass of π_E on replacing $H_E U$ by $U\hat{H}_E$, a change which has no effect on the value of the last integral. We shall speak of the common value of the masses as the natural capacity of E. It is now clear that E is negligible for the processes X if and only if it is negligible for the processes $\hat{X}$; obviously, the condition that E have compact closure is unnecessary in this statement.

Under (G), the condition that a right excessive function φ be finite on a set dense in $\mathcal{K}$ can be expressed in terms of the measures $\hat{\pi}_F$, with F ranging over the compact subsets of $\mathcal{K}$. First suppose φ to be finite on a dense set, and let G be a compact neighborhood of the compact set F. Then $H_G \varphi$ is the right potential of some bounded measure ν and coincides with φ on F. Therefore,

$$\int \hat{\pi}_F(dr)\varphi(r) = \int \hat{\pi}_F(dr) U\nu.(r) = \int \hat{\pi}_F U.(s)\nu(ds) \leqq \nu(\mathcal{K}),$$

because $\hat{\pi}_F$ is concentrated on F and $\hat{\pi}_F U$ nowhere exceeds 1. Let us drop the hypothesis on the finiteness of φ, but keep the hypotheses concerning F and G. The potential μU, with μ the restriction of the basic measure ξ to F, is dominated by $\alpha \hat{\pi}_G U$, provided the constant α is large enough, because the latter function exceeds μU on a neighborhood of F. The inequality

$$\int \mu(dr)\varphi(r) \leqq \alpha \int \hat{\pi}_G(dr)\varphi(r)$$

holds in these circumstances, as noted in the proof of Proposition 14.4. So φ is integrable over F if the right member of the inequality is finite. Consequently, the condition that a right excessive function be finite on a dense set amounts to saying that it is integrable with respect to the natural left capacitary measure of every compact set.

We shall denote by $\mathfrak{M}$ the class of measures ν on $\mathcal{K}$ such that $\hat{H}_F \nu$ is a bounded measure whenever F is a compact subset of $\mathcal{K}$, and by $\hat{\mathfrak{M}}$ the class defined similarly in terms of the kernels H_F. Note that $\hat{\mathfrak{M}}$ corresponds to the class $\mathfrak{M}$ of §7 and the cass $\mathfrak{N}$ of §14. The identity

$$\hat{H}_F \nu.(\mathcal{K}) = \int \hat{\Phi}_F(s)\nu(ds)$$

obviously holds, because $\hat{\Phi}_F(s)$ is just another way of writing $\hat{H}_F(\mathcal{K}, s)$. So the measure ν belongs to $\mathfrak{M}$ if and only if the integral on the right is finite for every compact subset F of $\mathcal{K}$. When (G) holds, the condition can be expressed in terms of the right potential of ν. Since $\hat{\Phi}_F$ is then the left potential of $\hat{\pi}_F$, we have

$$\int \hat{\Phi}_F(s)\nu(ds) = \int \hat{\pi}_F(dr)U\nu.(r),$$

so that ν belongs to $\mathfrak{M}$ if and only if $U\nu$ is integrable with respect to the natural left capacitary measure of every compact subset of $\mathcal{K}$. The last condition is equivalent, under (G), to the finiteness of $U\nu$ on a set dense in $\mathcal{K}$, according to the preceding paragraph.

In the remainder of the section we review the principal results of the second installment, rephrasing some to fit the present circumstances. It should be kept in mind that (F) implies (A), (C), and a strong version of (B) for each of the dual semigroups, and that open sets with compact closure in $\mathcal{K}$ play the part of special sets. We shall speak only of analytic sets, omitting the extensions to nearly analytic sets, because excessive functions are now Borel measurable.

The approximation of excessive functions by potentials given in Theorem 12.4 has two supplements. First, Proposition 18.4 and the subsequent remarks show that, under (G), a right excessive function is the right potential of a measure if it is finite on a dense set and has a compact subset of $\mathcal{K}$ for a determining set. Next, a right potential $U\nu$ is the limit of an increasing sequence of potentials of functions, even when (G) does not hold; one has only to note that $U\hat{H}_\gamma \nu$ increases to $U\nu$ as γ runs through a fundamental sequence in Γ_1 and that $\hat{H}_\gamma \nu$ is the indefinite integral of the function f_γ,

$$f_\gamma(r) \equiv \int h(\gamma, r, s)\nu(ds).$$

Using the second observation, one can establish for right potentials that are finite almost everywhere several statements that we shall prove assuming (G) to hold.

Theorem 11.5 needs no change.

The times R and T mentioned in Theorem 11.3 reduce to the time a process reaches the point w. Under the present conventions, a right excessive function φ vanishes at w and is finite except on a negligible set. Since a particle remains at w once it arrives there, the random variables of the lower semimartingale discussed after the theorem reduce to $\varphi(X(\tau))$, the factor $\mathcal{P}\{R > \tau \mid \mathfrak{F}_\tau\}$ being superfluous. The theorem and the subsequent discussion therefore yield the following statement.

THEOREM 18.6. *Let φ be a right excessive function and X a process. Then, with probability* 1, *the function $\varphi(X(\tau))$ of τ is continuous on the right, has finite limits from the left, has a finite limit as τ becomes infinite, and is finite for all strictly positive τ. The random variables $\varphi(X(\tau))$ form a lower semimartingale if the expectation of $\varphi(X(0))$ is finite.*

Let φ be a right potential, not necessarily finite almost everywhere, and let X be a process starting at a point r where φ is finite. The assertions of the theorem hold for $\varphi(X(\tau))$ and, in addition, $\varphi(X(\tau))$ vanishes in the limit with probability 1, as τ becomes infinite. The second assertion follows from a standard martingale theorem, since

$$\mathcal{E}\{\varphi(X(\tau))\} \equiv H_\tau \varphi.(r) \to 0, \qquad \tau \to \infty,$$

if φ is a right potential and finite at r.

The next theorem translates Theorems 14.6 and 14.9, for left excessive measures, into the language of this section.

THEOREM 18.7. *Let* (G) *hold. Then every right excessive function φ can be written in just one way as $U\nu + \psi$, with ν a measure in $\mathfrak{M}$ and ψ a right excessive function having the property that $H_E \psi$ coincides with ψ whenever E is the complement of a compact subset of $\mathcal{K}$. A right excessive function is a right potential if it is dominated by the right potential of a measure in $\mathfrak{M}$.*

Fix a sequence of open subsets G_n of $\mathcal{K}$ so that the closure of G_n is a compact subset of G_{n+1} and the union of the G_n is $\mathcal{K}$, and denote the complement of G_n by F_n. The proof of Theorem 14.6 implies that $H_F \varphi$ decreases almost everywhere to ψ as F runs through the sequence F_n. We proceed to strengthen this result.

THEOREM 18.8. *Let* (G) *hold, and let ν belong to $\mathfrak{M}$. Then $H_F U\nu$ decreases to* 0 *as F runs through the sequence F_n, except perhaps at the points where $U\nu$ is infinite.*

In the proof we shall use Theorem 13.5, the set E of the theorem being the compact set $\{w\}$. Let us adjoin w to each set F_n. The F_n then form a decreasing sequence of compact neighborhoods of w in the space $\mathcal{K} \cup \{w\}$, and their intersection reduces to $\{w\}$; these sets obviously serve as well as the decreasing sequence of open neighborhoods used in proving Theorem 13.5 for a compact set E. The limit of $H_F U\nu$ as F runs through the F_n therefore coincides, except at the points where $U\nu$ is infinite, with a right excessive function which has $\{w\}$ for a determining set. The excessive function must be the right potential of a measure μ on $\mathcal{K}$, because it is dominated by $U\nu$. Now, the least determining set for $U\mu$ is the support of μ in $\mathcal{K} \cup \{w\}$, so that μ must be the null measure. The proof is now complete.

The theorem can be stated in another manner. Let X be a process starting at a point r where $U\nu$ is finite, T_n the time it hits $F_n \cup \{w\}$, and T the supremum of the τ for which $X(\tau)$ belongs to $\mathcal{H}$; if $T(\omega)$ is finite it is the time the path $X(\tau, \omega)$ reaches w. Define $Y_n(\omega)$ to be the value of $U\nu$ at $X(T_n(\omega), \omega)$ if $T_n(\omega)$ is finite, to be 0 otherwise. The random variables Y_n form a lower semimartingale, and

$$\mathcal{E}\{Y_n\} = H_{F_n} U\nu.(r) \to 0, \qquad n \to \infty.$$

Consequently, Y_n vanishes in the limit with probability 1. Let Ω' be the set of ω such that $X(\tau, \omega)$ approaches w as τ increases to $T(\omega)$; the set comprises precisely those ω for which every $T_n(\omega)$ is less than $T(\omega)$. For almost every ω not in Ω', the time $T(\omega)$ is finite and coincides with some $T_n(\omega)$, because hypothesis (D) is a consequence of the present assumptions (F) and (G). According to Theorem 18.6 the value of $U\nu$ at the point $X(\tau, \omega)$ has a finite limit as τ increases to $T(\omega)$, for almost all ω; by the remarks just made, the limit vanishes for almost all ω in Ω'. It is perhaps more striking, and unfortunately less clear, to say that the value of $U\nu$ at the wandering point $X(\tau, \omega)$ almost certainly approaches 0 as the point approaches the boundary w of $\mathcal{H}$ continuously. The results of this paragraph, unlike Theorem 18.8, extend at once to processes starting at arbitrary points.

The last theorem enables one to complete Theorem 18.7. The function $H_F \varphi$ decreases as F runs through the sequence F_n, and the limit coincides with ψ except perhaps at the points where $U\nu$ is infinite. In particular, φ reduces to $U\nu$ if the limit vanishes almost everywhere.

A simple example shows that the exceptional points in Theorem 18.8 cannot be eliminated without further hypotheses. Take $\mathcal{H}$ to be real number space of dimension four, ξ to be Lebesgue measure. Let ν be a measure of unit mass concentrated on the rational points of $\mathcal{H}$, symmetric in the sense that $\nu(-r)$ coincides with $\nu(r)$, and attributing strictly positive mass to each rational point. The measures ν_k are defined by recursion, ν_0 being the unit mass placed at the origin of $\mathcal{H}$, and ν_{k+1} the convolution of ν_k with ν. The measures μ_τ, defined as

$$\mu_\tau \equiv e^{-\tau} \sum_{0 \leq k < \infty} \frac{\tau^k}{k!} \nu_k, \qquad 0 \leqq \tau < \infty,$$

then form a semigroup of probability measures under convolution. Now set

$$p_\tau(r, s) \equiv \int g_\tau(r - t, s)\mu_\tau(dt), \qquad \tau > 0,$$

where $g_\tau(r, s)$ stands for the transition probability density of Brownian motion in four-space, that is to say,

$$g_\tau(r, s) \equiv \frac{1}{(2\pi\tau)^2} \exp\left\{ - \frac{|r - s|^2}{2\tau} \right\}.$$

Clearly, $p_\tau(r, s)$ is symmetric in r, and s, continuous in the pair (r, s), small for $|r - s|$ large, and less than τ^{-2}. Thus hypotheses (A), (F), (G) are more than satisfied by the stationary Markoff transition measures $p_\tau(r, s)\,ds$. In addition, the dual semigroups reduce to one because of the symmetry of the transition probability densities in r and s. If $r - s$ is a rational point, then $p_\tau(r, s)$ exceeds α/τ for small values of τ, with α some strictly positive number, so that $U(r, s)$ is infinite. Consider the potential $U\nu$, which is finite almost everywhere. If G is open and not empty, then $H_G\, U\nu$ is infinite at every rational point since it exceeds the right potential of the restriction of ν to G. In particular, $H_F\, U\nu$ stays infinite at all rational points as F runs through the sequence F_n.

In order to eliminate such examples one must restrict the behavior of the kernel for potentials off the diagonal of $\mathcal{H} \times \mathcal{H}$. We shall not do so, because kernels like the one just mentioned possess all the important properties of the classical kernels for potentials, as we shall see in §20.

The following remark supplements Theorem 13.2. Under (F) the set of points right regular for an analytic set E is a countable intersection of open sets. In proving this statement we suppose Φ_E to attain its maximum value 1 only at the points right regular for E, going over to the semigroup H_τ^λ if necessary; the set where Φ_E exceeds $1 - 1/n$, which is open because the function is lower semicontinuous, then decreases to the set of right regular points as n becomes infinite. The exceptional set mentioned in the theorem is therefore analytic, and even a countable union of closed sets if E is such a set.

In the present circumstances Theorems 13.5 and 14.11 are easily seen to be slightly different versions of one result. We shall state only the version that uses hypothesis (G) to replace excessive functions with compact determining set by potentials of measures with compact support.

THEOREM 18.9. *Let (G) hold, let E be an analytic set, and let φ be a right excessive function. The infimum of the right excessive functions that dominate φ on a neighborhood of E coincides, except perhaps at the points outside E where φ is infinite and at the points of E that are not right regular for E, with the supremum of the potentials $U\mu$, where μ is concentrated on a compact subset of E and $U\mu$ is dominated everywhere by φ. There is a sequence of such measures μ_n such that $U\mu_n$ increases everywhere to the supremum. The supremum itself is the right potential of a measure concentrated on the union of E with the set of points left regular for E, if φ is the right potential of a measure in $\mathfrak{M}$, or if E has compact closure in $\mathcal{H}$.*

The second sentence is an immediate consequence of Theorem 13.5 and the remark following Proposition 18.4. The example discussed a moment ago shows that the points outside E where φ is infinite may indeed be exceptional; one need only consider the set E reduced to a point which has irrational coordinates.

If φ is the right potental of a measure ν in $\mathfrak{M}$, the proof of the fourth sentence is a repetition of the proof of Proposition 14.10; the supremum is the right potential of the measure $\nu_1 + \hat{H}_E\, \nu_2$, where ν_1 is the restriction of ν

to E and ν_2 is the remaining part of ν. If E has compact closure in $\mathcal{K}$, one considers $H_G \varphi$ for some compact neighborhood G of E; this function is the right potential of a bounded measure, according to Proposition 18.4, so that the proof of Proposition 14.10 is applicable again.

The following proposition is used in proving the third sentence of the theorem.

Proposition 18.10. *Let (G) hold, and let μ_1 and μ_2 be two measures in $\mathfrak{M}$. Then the least right excessive function dominating both $U\mu_1$ and $U\mu_2$ is a potential $U\nu$, with ν a measure concentrated on the union of the supports of μ_1 and μ_2.*

Let μ be the sum of μ_1 and μ_2. Since $U\mu$ dominates the potentials of μ_1 and μ_2 and since the minimum of $U\mu$ and a right excessive function is a potential, we need only consider potentials dominated by $U\mu$. Choose a strictly positive function g so that the product $gU\mu$ is integrable over $\mathcal{K}$, and set

$$\alpha = \inf_{\nu} \int g(r) U\nu.(r)\, dr,$$

where ν ranges over the measures whose potentials dominate the potentials of μ_1 and μ_2. Choose a sequence of such measures ν_n so that the potential $U\nu_n$ decreases as n increases and the integral of $gU\nu_n$ decreases to α. The limit of the potentials $U\nu_n$ coincides almost everywhere with a right excessive function ψ, by Proposition 13.4, because it is Borel measurable and satisfies the defining inequality. The function ψ clearly dominates the potentials of μ_1 and μ_2, since it does so almost everywhere; it is the least such right excessive function, because the integral of $g\psi$ has the least possible value α and g is strictly positive. Let G be any open set including the supports of μ_1 and μ_2. Then $H_G \psi$ dominates the potentials of μ_1 and μ_2, since it does so on a neighborhood of the supports of the measures; so $H_G \psi$ coincides with ψ. Now, ψ is the potential of some measure ν, because it is dominated by $U\mu$, and the support of ν must be included in the union of the supports of μ_1 and μ_2 according to what has just been noted.

It is now easy to prove the third sentence of the theorem. Let h be a strictly positive function for which the integral of $h\varphi$ is finite. Choose a sequence of measures μ_n' such that the support of μ_n' is a compact subset of E and the potential of μ_n' is dominated by φ, and such that the integral of $hU\mu_n'$ tends to the supremum of the possible values under these restrictions. According to the proposition, there are measures μ_n, satisfying the same restrictions, whose right potentials form an increasing sequence. These potentials clearly increase to the supremum mentioned in the theorem.

The next two propositions, the analogues of the last two in §9, are needed later on.

Proposition 18.11. *Let (G) hold, and let ν and the μ_n be measures in $\mathfrak{M}$. If*

$$U\mu_n \leqq U\mu_{n+1} \leqq U\nu$$

for all n, then μ_n converges weakly to some measure μ in $\mathfrak{M}$, and $U\mu_n$ increases to $U\mu$. Also, $\int \hat{\varphi}\mu_n(ds)$ increases to $\int \hat{\varphi}\mu(ds)$, for every left excessive function $\hat{\varphi}$.

The limit of the $U\mu_n$ is the right potential of some measure in $\mathfrak{M}$, because it is right excessive and dominated by the potential of ν. By Proposition 18.2, the measure $\hat{H}_D\,\mu_n$ tends weakly to $\hat{H}_D\,\mu$ if D is a set with compact closure, because $H_D\,U\mu_n$ increases to $H_D\,U\mu$. The weak convergence of μ_n to μ is clearly implied by this fact and the following statement: For every compact set F and every strictly positive number ε, there is a compact neighborhood D of F such that

$$\hat{H}_D\,\mu_n.(F) - \mu_n(F) < \varepsilon, \qquad n = 1, 2, \cdots \tag{18.7}$$

To prove the statement, first choose a compact set E so that

$$\int_E U(r, s)\,dr \geqq \rho > 0, \qquad s \in F.$$

Then choose an open set G so that the complement of G is a compact set including F and so that the integral of $U\hat{H}_G\,\mu$ over E is less than $\rho\varepsilon$; Proposition 18.8 ensures the existence of G. If D is any compact neighborhood of the complement of G, then the integral of $U\hat{H}_D\,\hat{H}_G\,\mu_n$ over E is less than $\rho\varepsilon$. The mass assigned to F by $\hat{H}_D\,\hat{H}_G\,\mu_n$ is therefore less than ε, and (18.7) follows at once.

Given a left excessive function $\hat{\varphi}$, choose a sequence of measures ν_n whose left potentials increase to $\hat{\varphi}$. The last sentence of the proposition is proved by the chain of equations

$$\begin{aligned}
\lim_n \int \hat{\varphi}\mu_n(ds) &= \lim_n \lim_m \iint \nu_m(dr)U(r, s)\mu_n(ds) \\
&= \lim_m \lim_n \iint \nu_m(dr)U(r, s)\mu_n(ds) \\
&= \lim_m \iint \nu_m(dr)U(r, s)\mu(ds) \\
&= \int \hat{\varphi}\mu(ds),
\end{aligned}$$

which are justified by monotone convergence.

PROPOSITION 18.12. *Let (G) hold, and let the μ_n be measures in $\mathfrak{M}$ whose right potentials form a decreasing sequence. Then μ_n converges weakly to a measure μ in $\mathfrak{M}$, and $U\mu_n$ decreases almost everywhere to $U\mu$.*

The proof is omitted because it is very like the preceding one.

Hypothesis (F), as we have seen, enables one to strengthen several results of the second installment. We mention without proof a few more instances, chosen more or less at random and not used in the paper.

A positive function φ is excessive relative to the semigroup H_r if and only if it is lower semicontinuous and satisfies the inequality $H_\tau \varphi \leqq \varphi$ for all τ. The condition that $H_\tau \varphi$ approach φ as τ decreases may therefore be replaced by the lower semicontinuity of φ, in the definition of excessive function. Consequently, a good part of the early discussion of excessive functions becomes simpler or even unnecessary; the minimum of two excessive functions, for example, is immediately seen to be excessive.

In Proposition 12.7 the intersection of $\mathcal{K}_\beta$ with a compact set need not be assumed empty for β sufficiently small; this consequence was noted in the last paragraph of §12.

In several passages to the limit we have considered only sequences of excessive functions in order to ensure the measurability of the limit function. Under (F) one can establish, for increasing or decreasing families of excessive functions, results like those obtained by H. Cartan for positive superharmonic functions (Bull. Soc. Math. France, t. 73). The analogue is clear for an increasing family. The precise analogue for a decreasing family requires hypothesis (H) of §20, but a weaker version, with sets of measure null replacing sets of capacity null, holds under (F) alone.

Finally, the lower semicontinuity of excessive functions is reflected in the behavior of $H_E(r, ds)$, the distribution of hits of a set E by a process starting at the point r. It is easily proved, by the argument of §4.4 of [10], that as r approaches a point right regular for E, the distribution of hits tends weakly to the unit mass placed at the limit point. This fact is useful in verifying that hypothesis (F) holds for relative transition measures if the systems of terminal times are suitably restricted.

19. Capacity

In this section we study a capacity of analytic sets which is defined in terms of a given right excessive function α and a given left excessive function $\hat{\alpha}$. One obtains the natural capacity of the preceding section on taking both functions to have the constant value 1 on $\mathcal{H}$; this choice gives the classical notions of capacity, and it leads to the simplest probabilistic interpretation. The functions α and $\hat{\alpha}$ will be restricted in such a way that the capacity behaves essentially like the Newtonian capacity.

Hypotheses (F) and (G) are assumed to hold, except during the discussion of the general situation at the end of the section.

Matters are particularly simple when α and $\hat{\alpha}$ are potentials. In order to take advantage of this fact, we shall at first limit our attention to subsets of a fixed open set D. A right excessive function α is said to be admissible for D if it is a right potential, if it is bounded on D, and if it satisfies the following condition: *For every compact subset F of D, the function $H_G\, \alpha$ decreases almost everywhere to $H_F\, \alpha$ as G runs through a decreasing sequence of compact neighborhoods of F having F as intersection.* Admissible left excessive functions are defined similarly.

The statement in italics is certainly true if α is continuous on D, for the convergence then takes place except at the points belonging to F but not right regular for F. Doob [6] has shown the condition to be no restriction at all for bounded superharmonic functions and Brownian motion, and his result will be extended in §20.

If the sum of two right excessive functions is admissible for D, then both functions are also admissible. This remark will be used in the next paragraph.

Let β be a measure in $\mathfrak{M}$ whose right potential α is bounded on D. We shall prove that α is admissible for D if and only if, for every compact subset F of D, the measure β assigns no mass to the set B comprising the points of F that are not left regular for F. First suppose $\beta(B)$ to vanish. Then $\hat{H}_G\,\beta$ tends weakly to $\hat{H}_F\,\beta$ as G runs through a decreasing sequence of compact neighborhoods of F that shrink to F. Consequently,

$$\int \hat{H}_G\beta.(ds) \int g(r)U(r,\,s)\;dr \to \int \hat{H}_F\beta.(ds) \int g(r)U(r,\,s)\;dr,$$

whenever g is bounded and vanishes outside a compact set, for the inner integral is then continuous in s by Proposition 18.1. Moreover, $H_G\,\alpha$ coincides with the right potential of $\hat{H}_G\,\beta$, decreases with G, and dominates $H_F\,\alpha$, which is the right potential of $\hat{H}_F\,\beta$. So $H_G\,\alpha$ decreases to $H_F\,\alpha$ almost everywhere, and α is admissible. Next, suppose $\beta(B)$ to be strictly positive for some choice of F. In proving α not to be admissible we may assume β to be concentrated on B, by the remark above, and even to have unit mass. Then $H_G\,\alpha$ coincides with α whenever G is a neighborhood of F, for $\hat{H}_G\,\beta$ and β are clearly the same measure. On the other hand, $H_F\,\alpha$ is less than α on a set of strictly positive measure. To see this, let $\hat{X}$ be a process having β for initial distribution, and let $\hat{T}$ be the time $\hat{X}$ hits F. Since $\hat{T}$ is strictly positive with probability 1, the integral

$$\int_\Omega d\omega \int_0^{\hat{T}} \chi(\hat{X}(\tau))\;d\tau$$

has a strictly positive value ρ, with χ the characteristic function of some compact set. Now, the integral of α over that set is finite and exceeds the integral of $H_F\,\alpha$ over the same set by exactly the amount ρ. So α is not admissible.

In particular, α is admissible for D if the restriction of β to D is absolutely continuous relative to the basic measure ξ. Also, if α is admissible for D, then $\beta(B)$ vanishes whenever B is the set of points that belong to an analytic subset E of D but are not left regular for E; one has only to consider the compact subsets of B.

Let α be admissible for D, X a process, T the time it hits the compact subset F of D, and Ω' the set of ω satisfying the conditions

$$0 < T(\omega) < \infty, \qquad \lim_{\tau\to T(\omega)} X(\tau,\,\omega) = X(T(\omega),\,\omega).$$

We shall prove that $\alpha(X(\tau, \omega))$ is continuous at $T(\omega)$, for almost all ω in Ω'. Since $\alpha(X(\tau))$ is continuous on the right and has limits from the left, with probability 1, we need only investigate what happens as T is approached from below through some sequence. Choose a decreasing sequence of compact neighborhoods G_n of F in D, with F as intersection, and denote by T_n the time X hits G_n, by Ω_n the set where T_n is finite, and by Ω'' the set of ω for which $T(\omega)$ is finite and coincides with some $T_n(\omega)$. If X has an absolutely continuous initial distribution, say $g(r)\,dr$, then

$$\int_{\Omega_n} \alpha(X(T_n))\,d\omega \equiv \int g(r)H_{G_n}\alpha.(r)\,dr$$
$$\to \int g(r)H_F\alpha.(r)\,dr \equiv \int_{\Omega'\cup\Omega''} \alpha(X(T))\,d\omega,$$

because α is admissible and because the union of Ω' and Ω'' is precisely the set where T is finite. This relation and the lower semicontinuity of α imply that the value of α at $X(T_n(\omega), \omega)$ approaches the value at $X(T(\omega), \omega)$, for almost all ω in Ω', and the proof is complete for a process with absolutely continuous initial distribution. The restriction is easily removed. For a process starting at a point right regular for F, there is nothing to prove. If X is a process starting at a point not right regular for F, consider the process $X(\tau + Z)$, with Z a random variable having an element γ of Γ for density function. The initial distribution of this process is absolutely continuous when restricted to $\mathcal{H}$. The time the process hits F tends to T as γ runs through a fundamental sequence, and even coincides with $T - Z$ at some finite stage, with probability 1. So the assertion concerning X is easily derived from those concerning the processes $X(\tau + Z)$.

Let E be an analytic set whose closure is included in D. The time T at which a process X hits E is dominated with probability 1 by the time X hits a given compact subset F of the union of E with the set of points right regular for E, and the probability that T differs from the time of hitting F can be made arbitrarily small by a suitable choice of F. So the value of an admissible function α at $X(\tau, \omega)$ approaches the value at $X(T(\omega), \omega)$ as τ tends to $T(\omega)$, for almost all ω such that $T(\omega)$ is finite and such that $X(\tau, \omega)$ is continuous in τ at $T(\omega)$. Let us now suppose that the closure of E is compact and that the initial distribution of X attributes no mass to the set of points belonging to E but not right regular for E. Choose a decreasing sequence of neighborhoods G_n of E so that the closure of G_n is a compact subset of D and so that T_n, the time X hits G_n, increases to T with probability 1, and let Ω' or Ω_n be the set where T or T_n is finite. The probability of Ω_n decreases to that of Ω', because hypothesis (D) holds, and $\alpha(X(T_n))$ approaches $\alpha(X(T))$ almost everywhere on Ω', either because T_n ultimately coincides with T or because α has the continuity proved above. Matters being so, the relation

$$\int_{\Omega_n} \alpha(X(T_n))\,d\omega \to \int_{\Omega'} \alpha(X(T))\,d\omega, \qquad n \to \infty,$$

follows from dominated convergence. Thus, for every measure μ that attributes no mass to the set of points belonging to E but not right regular for E, the function $H_G\,\alpha$ decreases to $H_E\,\alpha$, except on a set that is null for μ, as G runs through some decreasing sequence of neighborhoods of E; in particular, μ may be taken as the basic measure ξ. These results extend at once, by means of the inequality (11.10), to every analytic set with closure included in E. The proof, one should observe, makes no use of the fact that α is a potential.

Given two admissible functions for D,

$$\alpha \equiv U\beta, \qquad \hat{\alpha} \equiv \hat{\beta}U,$$

we define the capacity of an analytic E with closure included in D by the formula

$$C(E) \equiv \int \hat{\beta}(dr)H_E\,\alpha.(r) \equiv \int \hat{\alpha}(s)\hat{H}_E\,\beta.(ds). \tag{19.1}$$

The two integrals have the same value, according to (18.3). The dual formula,

$$C(E) \equiv \int \hat{\alpha}\hat{H}_E.(s)\beta(ds) \equiv \int \hat{\beta}H_E.(dr)\alpha(r), \tag{19.2}$$

is also valid. Clearly, $C(E)$ is finite if either $\hat{H}_E\,\beta$ or $\hat{\beta}H_E$ is a bounded measure; this is so, in particular, if the closure of E is compact. The capacity of a set increases with the set, and it is alternating of order infinity in the sense of Choquet, by Theorem 11.5. Let E and E' be two analytic sets with closures included in D, the capacity of E' being finite. If

$$E = \bigcup E_n\,, \qquad E' = \bigcup E'_n\,, \qquad E_n \supset E'_n\,,$$

then the inequality

$$C(E) - C(E') \leqq \sum\,[C(E_n) - C(E'_n)] \tag{19.3}$$

holds, according to (11.10). In particular, $C(E)$ does not exceed $\sum C(E_n)$.

What has been said so far holds for all analytic sets in $\mathcal{K}$ and for all choices of α and $\hat{\alpha}$ as potentials, perhaps not admissible for D. The restrictions imposed above are needed in proving that the capacity of a set is the extreme value of certain classes of integrals.

Let E be an analytic set with closure included in D. We shall prove that

$$C(E) = \inf_{\varphi} \int \hat{\beta}(dr)\varphi(r), \tag{19.4}$$

where φ ranges over the right excessive functions that dominate α on a neighborhood of E. The infimum is at least $C(E)$, for each function φ dominates $H_E\,\alpha$ everywhere. In proving the converse we assume E to have finite capacity, since otherwise there is nothing to prove. First suppose the closure of E to be compact, and let G be a neighborhood of E that also has compact closure in D. The function $H_G\,\alpha$ is integrable with respect to $\hat{\beta}$, it coincides with α on the interior of G, and as G runs through a certain decreasing sequence $H_G\,\alpha$ decreases to $H_E\,\alpha$ except on a set that is null for $\hat{\beta}$, because $\hat{\beta}$ attributes no mass to the set of points belonging to E but not right regular for E. So $C(G)$ decreases to $C(E)$. Now let E be an analytic set whose closure is not compact, though it is of course included in D, and write E as the union of sets E_n with compact closures. According to (19.3), one obtains a neighborhood of E with capacity arbitrarily close to that of E by taking the union of neighborhoods of the sets E_n having capacities sufficiently close to those of the E_n. So $C(G)$ decreases to $C(E)$ as G runs through some decreasing sequence of neighborhoods of E in D. Relation (19.4) follows at once, for $H_G\,\alpha$ coincides with α on the interior of G.

The range of φ in taking the infimum in (19.4) may be enlarged to the right excessive functions that dominate α on E, for such functions dominate $H_E\,\alpha$ everywhere. Only right potentials need be considered, because α itself is a potential. Therefore (19.4) may be expressed equivalently as

$$C(E) = \inf_{\mu} \int \hat{\alpha}(s)\mu(ds), \tag{19.5}$$

where μ ranges either over the measures whose right potentials dominate α on a neighborhood of E, or over the measures whose right potentials dominate α on E.

One may also write

$$C(E) = \sup_{\varphi} \int \hat{\beta}(dr)\varphi(r) = \sup_{\mu} \int \hat{\alpha}(s)\mu(ds), \tag{19.6}$$

φ ranging over the right excessive functions that nowhere exceed α and have compact subsets of E for determining sets, μ ranging over the measures that have right potentials bounded by α and compact subsets of E for support. The two suprema are equal because, under (F) and (G), an excessive function has a compact set for determining set if and only if it is the right potential of a measure on that set. The equality of $C(E)$ and the first supremum follows from Theorem 18.9 and what has been proved earlier. Indeed, $H_E\,\alpha$ coincides almost everywhere with the infimum of the right excessive functions that dominate α on a neighborhood of E; it is therefore the supremum of the right excessive functions considered in taking the supremum, and a sequence of such functions increases to $H_E\,\alpha$.

According to the strong version of Proposition 2.1 proved at the end of §17, there is an increasing sequence of compact subsets F_n of E having the property that $H_F\,\psi$ increases to $H_E\,\psi$ as F runs through the sequence, no matter how the right excessive function ψ be chosen. The capacity of F_n therefore increases to the capacity of E. This statement is somewhat stronger than the corresponding one derived from Theorem 18.9 and relation (19.6), since the F_n do not depend upon the functions α and $\hat{\alpha}$ entering in the definition of capacity.

One obtains other expressions for the capacity on interchanging the roles of α and $\hat{\alpha}$. The statements concerning capacity, it is to be observed, are all derived from somewhat stronger ones concerning the functions $H_E\,\alpha$ and $\hat{\alpha}\hat{H}_E$.

We shall now remove the condition that α and $\hat{\alpha}$ be potentials. The right excessive function α is assumed to be bounded on compact sets and to have the following property: *If F is compact, then $H_G\,\alpha$ decreases almost everywhere to $H_F\,\alpha$ as G runs through a decreasing sequence of compact neighborhoods of F that shrink to F.* The left excessive function $\hat{\alpha}$ is restricted similarly.

The capacity of an analytic set E with compact closure is defined in the following way. Choose an open set D that includes the closure of E and has compact closure itself. The functions $H_D\,\alpha$ and $\hat{\alpha}\hat{H}_D$ are potentials, say of the measures β and $\hat{\beta}$, and coincide with α and $\hat{\alpha}$ on D; they are therefore admissible for D. The capacity of E is now defined in terms of D, $H_D\,\alpha$ and $\hat{\alpha}\hat{H}_D$. Because $H_E\,H_D\,\alpha$ coincides with $H_E\,\alpha$, for example, one has the same formulas (19.1) and (19.2) as before, β and $\hat{\beta}$ being given their present meanings. In verifying that the definition does not depend on the choice of neighborhood, it is enough to consider a second neighborhood D' that includes D; the equivalence of the two definitions then follows at once from the fact that the kernel $H_D\,H_{D'}$ coincides with H_D. The capacity thus defined for sets with compact closure evidently has all the properties of the capacity discussed above. Moreover, the expression of $C(E)$ as an infimum in (19.5) and the second expression as a supremum in (19.6) do not require a neighborhood of E to be chosen.

The capacity of an arbitrary analytic set is defined to be the supremum of the capacities of its compact subsets. This definition agrees with the preceding one if the set has compact closure. Also, if D is any open set for which $H_D\,\alpha$ is the bounded right potential of some measure β, and if E is an analytic set whose closure is included in D, then $C(E)$ has for expression the second integral in (19.1), the first integral in (19.2), the infimum in (19.5), and the second supremum in (19.6). If in addition $\hat{\alpha}\hat{H}_D$ is the bounded left potential of a measure $\hat{\beta}$, then $C(E)$ is just the capacity of E as defined before in terms of the open set D and the admissible functions $H_D\,\alpha$ and $\hat{\alpha}\hat{H}_D$. These statements follow quickly from what has been proved above.

Consider an increasing sequence of open sets G_n whose closures are compact and whose union is $\mathcal{H}$. The capacity of $E \cap G_n$ increases to that of E, by the definition of capacity, for every analytic set E. Using this approximation, one proves that capacity is an alternating function, that the inequality (19.3) holds, and that the second supremum in (19.6) always gives the capacity of E. It follows immediately from (19.3) and the properties of capacity for sets with compact closure that the capacity of an analytic set is the infimum of the capacities of its open neighborhoods. Relation (19.6), however, does not hold generally.

In the following remarks, which concern the natural capacity, α and $\hat{\alpha}$ are to be taken as the constant function 1 on $\mathfrak{H}$. The function $H_E\,\alpha$ then reduces to Φ_E, the probability of hitting E. This function is the supremum of the right potentials $U\mu$, as μ ranges over the measures that have compact supports in E and right potentials bounded by 1; it is the limit of an increasing sequence of such potentials; and the capacity of E is the supremum of the masses of the measures μ. On the other hand, Φ_E coincides, except at the points of E not right regular for E, with the infimum of the right excessive functions that dominate 1 on E, or on a neighborhood of E; and given any measure that attributes no mass to the set of points belonging to E but not right regular for E, the function Φ_G decreases to Φ_E, except on a set that is null for the measure, as G runs through a certain decreasing sequence of neighborhoods of E. If Φ_E is a potential, say $U\pi_E$, we speak of π_E as the natural right capacitary measure of E; its total mass is $C(E)$, by Proposition 18.11, because Φ_E is the limit of an increasing sequence of right potentials of measures whose total masses increase to the capacity of E. The natural left capacitary measure $\hat{\pi}_E$ is defined under similar circumstances; both capacitary measures exist if the set has compact closure.

If F is a compact subset of the compact set E, then π_F dominates the restriction of π_E to F. In proving this statement, take β to be the natural right capacitary measure of a neighborhood of E, and denote the restriction of β to F by β_1, the restriction to $E - F$ by β_2, and the remaining part of β by β_3. Since β attributes no mass to the set of points belonging to F but not left regular for F, or to the corresponding part of E, one may write

$$\pi_F = \beta_1 + \hat{H}_F(\beta_2 + \beta_3), \qquad \pi_E' = \beta_1 + (\hat{H}_E\,\beta_3)',$$

the prime signifying that a measure is restricted to F. The assertion to be proved follows at once from these representations.

Now let E be an analytic set for which π_E exists, F the part of E lying in an open set G. Then π_F also exists, for Φ_F is dominated by Φ_E. We shall prove that π_F dominates the restriction of π_E to G. The assertion for E compact and G replaced by a closed set C is merely a restatement of what was proved in the preceding paragraph. In order to prove the assertion for E compact and G open, let C run through an increasing sequence of compact subsets of G whose interiors exhaust G; the probability of hitting $E \cap C$ then increases to Φ_F, so that the right capacitary measure of $E \cap C$ tends weakly to π_F, by Proposition 18.11; consequently, π_F dominates the restriction of π_E to each set C, and therefore the restriction to G. The extension to analytic sets is proved by a similar argument in which the set E is approximated by compact subsets.

The measure π_E, provided it exists, is concentrated on the union of E with the set of points left regular for E. This is easy to see if E has compact closure, for then π_E has the form $\hat{H}_E\,\pi_D$, with D some neighborhood of E; the result for more general sets follows from the preceding paragraph.

The statement that π_E exists if E has finite natural capacity is not always true; one has only to consider uniform motion on a line, taking ξ to be Lebesgue measure and E to be the whole space. The statement is true, however, if a left potential of the form $\int f(r)U(r, s)\, dr$ vanishes at infinity provided f is bounded and vanishes outside a compact set, that is to say, if the semigroup $\hat{H}_\tau$ satisfies condition (E^*) of §14. Indeed, the existence of π_E, for sets of finite natural capacity, is then assured by Proposition 14.8 and the fact that Φ_E is the limit of an increasing sequence of right potentials of measures having masses bounded by $C(E)$. The theories of the Newtonian, Riesz, and heat potentials, with ξ taken as Lebesgue measure, are instances of this situation.

Suppose for the moment that every analytic set of finite natural capacity has a natural right capacitary measure. Such a set E has a neighborhood D of finite natural capacity, and Φ_D is a right potential under the hypothesis of the moment. Hence $C(E)$ can be expressed as the infimum in (19.5), by an earlier remark. On the other hand, the infimum is certainly infinite whenever $C(E)$ is infinite.

Let F be a compact set. Of the measures on F with right potentials bounded by 1, the measure π_F has the greatest potential. There may be other measures on F, however, having mass $C(E)$ and right potential bounded by 1. By way of illustration, let $\mathcal{H}$ comprise only two points a and b, and let $H_\tau(r, ds)$ be defined as

$$H_\tau(a, a) = e^{-2\tau}, \qquad H_\tau(b, a) = 0,$$
$$H_\tau(a, b) = e^{-\tau} - e^{-2\tau}, \qquad H_\tau(b, b) = e^{-\tau}.$$

Any measure of the form

$$\xi(a) = \alpha, \qquad \xi(b) = \alpha + \beta, \qquad \alpha > 0, \quad \beta > 0$$

may be taken as the basic measure. The natural capacitary measures of $\mathcal{H}$ itself are then

$$\pi(a) = \alpha, \quad \pi(b) = \alpha + \beta, \qquad \hat{\pi}(a) = 2\alpha, \quad \hat{\pi}(b) = \beta,$$

and the natural capacity of $\mathcal{H}$ is $2\alpha + \beta$. Clearly, $\hat{\pi}$ is the only measure with mass $C(\mathcal{H})$ and left potential not exceeding 1. The corresponding statement is true of π, provided β is strictly positive; if β vanishes, however, every measure ν satisfying

$$\nu(a) + \nu(b) = 2\alpha, \qquad \nu(b) \leqq \alpha,$$

has mass $C(\mathcal{H})$ and right potential not exceeding 1. The example gives point to the next paragraph.

Let D be an open set with compact closure, F a closed subset of D, and μ and ν two bounded measures satisfying the relations

$$\mu(F) = \nu(F), \qquad U\mu \leqq U\nu.$$

The extreme members in the chain

$$\mu(F) = \int \hat{\Phi}_D \mu(ds) = \int \hat{\pi}_D(dr) U\mu \leqq \int \hat{\pi}_D(dr) U\nu = \int \hat{\Phi}_D \nu(ds) = \nu(F)$$

coincide, so that the right potentials of μ and ν agree on D except for a null set relative to $\hat{\pi}_D$. Suppose in addition that the restriction of ξ to D is absolutely continuous with respect to $\hat{\pi}_D$. Then $U\mu$ and $U\nu$ coincide on D, since they do so up to a set of basic measure null; hence μ and ν are the same measure, for D is a neighborhood of their supports. In particular, we may take ν to be π_F and μ to be any measure on F whose right potential is bounded by 1. In many examples of interest—the theory of Riesz potentials, say, with ξ Lebesgue measure—the restriction of ξ to any open set D with compact closure is absolutely continuous relative to $\hat{\pi}_D$. In Brownian motion, with ξ taken as Lebesgue measure, the capacitary measure of every set is concentrated on the boundary; for every compact set F, however, the measure π_F is the only measure on F having mass $C(F)$ and right potential bounded by 1.

It is sometimes useful to know how the capacitary measures vary with the parameter λ. In this paragraph and the next two, we assume the semigroup H_τ and the basic measure ξ to satisfy (F) but perhaps not (G). The dual semigroups

$$H_\tau^\lambda \equiv e^{-\lambda\tau} H_\tau, \qquad \hat{H}_\tau^\lambda \equiv e^{-\lambda\tau} \hat{H}_\tau,$$

with the same basic measure ξ, satisfy both (F) and (G) for λ strictly positive; in fact, they both satisfy hypothesis (E^*) of §14. Let ρ be a strictly positive value of the parameter, and λ a smaller value; λ is permitted to vanish if hypothesis (G) holds then. Let E be an analytic set for which π_E^λ exists. We shall prove that in these circumstances π_E^ρ exists and is given by the formula

$$\pi_E^\rho = \pi_E^\lambda + \hat{H}_E^\rho \nu_E, \tag{19.7}$$

where ν_E is the absolutely continuous measure having $(\rho - \lambda)\Phi_E^\lambda$ for density function. Since U^λ coincides with $U^\rho + (\rho - \lambda)U^\lambda U^\rho$, we may write

$$\Phi_E^\lambda = U^\rho \pi_E^\lambda + U^\rho \nu_E. \tag{19.8}$$

If E is open, then $H_E^\rho \Phi_E^\lambda$ coincides with Φ_E^ρ, so that (19.7) follows from (18.3) and the fact that π_E^λ is concentrated on the union of E with the set of points left regular for E. Now suppose E to be compact, and let D be an open neighborhood of E with compact closure. As D decreases so that its closure shrinks to E, the functions Φ_D^λ and Φ_D^ρ decrease almost everywhere to Φ_E^λ and Φ_E^ρ; hence π_D^λ and π_D^ρ tend weakly to π_E^λ and π_E^ρ, by Proposition 18.12. The measure $\hat{H}_D^\rho(dr, s)$ tends weakly to $\hat{H}_E^\rho(dr, s)$ unless s is a point of E not left regular for E; the measure ν_D decreases to ν_E, and neither attributes any mass to the set of points belonging to E but not left regular for E. The weak convergence of $\hat{H}_D^\rho \nu_D$ to $\hat{H}_E^\rho \nu_E$ follows from these facts if λ is strictly positive, for then the measures ν_D are bounded, but an additional argument is needed if λ vanishes. Let C be a compact set that includes all the sets D considered in the passage to the limit. Given any continuous function f with compact support, one can clearly find another such function g with the property that $g\hat{H}_C^\rho$ dominates every function $f\hat{H}_D^\rho$. Now, the integral $\int (g\hat{H}_C^\rho)\nu_C(ds)$ is finite, and it provides the domination needed in passing to the limit. So the proof of (19.7) for a compact set is complete. The proof is similar if E is an analytic set with a natural right capacitary measure. One approximates E by an increasing sequence of closed subsets, using Proposition 18.11. The passage to the limit gives (19.7) if E has compact closure, but otherwise it shows only that the left member dominates the right. The proof is completed in this way: Let f be a positive continuous function with compact support, F the part of E lying in an open set G whose closure is compact and includes the support of f. Evidently $f\hat{H}_F^\rho$ decreases as G increases. The measures π_F^λ and π_F^ρ tend weakly to π_E^λ and π_E^ρ as G increases to $\mathcal{H}$, and the inequality

$$\pi_F^\rho = \pi_F^\lambda + \hat{H}_F^\rho \nu_F \leqq \pi_F^\lambda + \hat{H}_F^\rho \nu_E$$

holds by what has already been proved. On integrating f with respect to these measures and passing to the limit, we find that the right member of (19.7) dominates the left. The proof of (19.7) is now complete.

It is clear that π_E^ρ increases with ρ. If E has compact closure, the only restriction on λ is that it be strictly positive and less than ρ; it follows that π_E^ρ dominates the restriction to E of the measure $\rho\xi$, because ν_E dominates the restriction of $(\rho - \lambda)\xi$ to E. The statement holds, in fact, whenever π_E^ρ exists; for π_E^ρ is the weak limit of π_F^ρ as F runs through some increasing sequence of compact sets that exhaust E up to a null set. As an earlier remark shows, for every compact set F the measure π_F^ρ is the only one on F having mass $C(F)$ and right potential bounded by 1.

We shall now prove that the mass assigned by π_E^λ to a set with compact closure varies continuously with λ. It is enough to prove that π_E^λ varies continuously with λ in the weak topology of measures, for the measure increases with λ. Let σ be greater than ρ. Then Φ_E^λ and Φ_E^ρ are the right σ-potentials of the measures

$$\pi_E^\lambda(ds) + (\sigma - \lambda)\Phi_E^\lambda\, ds, \qquad \pi_E^\rho(ds) + (\sigma - \rho)\Phi_E^\rho\, ds.$$

By Proposition 18.12, the first measure converges weakly to the second as λ increases to ρ, because Φ_E^λ decreases to Φ_E^ρ; since $(\sigma - \lambda)\Phi_E^\lambda\, ds$ evidently decreases to $(\sigma - \rho)\Phi_E^\rho\, ds$, the measure π_E^λ must tend weakly to π_E^ρ. One establishes weak convergence for ρ decreasing to λ by a similar argument, using Proposition 18.11.

Let us now assume that (G) holds for the semigroup H_τ^0 and that E is an analytic set for which π_E^0 exists. Then π_E^λ decreases to π_E^0 as λ decreases to 0.

This relation is sometimes useful in establishing properties of capacitary measures, as we shall see in a moment.

Suppose $H_\tau(r, \mathcal{K})$ to be identically 1. The right λ-potential of the measure $\lambda\xi$ then has the constant value 1 on $\mathcal{K}$, for λ strictly positive, so that π_E^λ exists for every analytic set and coincides with $\hat{H}_E^\lambda(\lambda\xi)$. We may clearly neglect the restriction of ξ to any fixed compact set in passing to the limit to find π_E^0 ; thus, if C is any compact set and if B is any Borel set having the property that the measure $\hat{H}_E^0(dr, s)$ is concentrated on B for every point s outside C, then the measure π_E^0 itself is concentrated on B. In the theory of Newtonian potentials, the capacitary measure of a compact set E is accordingly borne by the accessible points of the boundary of the component of $\mathcal{K} - E$ that extends to infinity, for the sample paths of Brownian motion are continuous. A little more can be said about capacitary measures in the theory of the heat potentials, since the time coordinate of a Brownian motion in space-time varies monotonically. Let I be a compact time interval $[\rho, \sigma]$, and J a compact space interval; the natural right capacitary measure π of $I \times J$ is concentrated on the boundary, with no mass on the face $\rho \times J$. Let E be the boundary of $I \times J$ with the face $\sigma \times J$ deleted; the measure $\hat{H}_E\,\pi$ is the natural right capacitary measure of E and has the same mass as π, provided ξ is Lebesgue measure and the parameter λ vanishes. Observe that π and $\hat{H}_E\,\pi$ are distinct measures on the compact set $I \times J$ and that both have mass $C(I \times J)$ and right potential bounded by 1.

The mass $\hat{H}_\tau(\mathcal{K}, s)$ is identically 1 if and only if ξH_τ coincides with ξ for all τ. Results dual to the ones above are valid when this condition is satisfied.

The notion of capacity can be introduced in a setting more general than the one treated here. Consider the situation discussed in §§4–9, for simplicity, assuming the parameter to be strictly positive and hypotheses (A), (B), (C) to hold, and suppose given an excessive function φ and an excessive measure ζ. If G is an open set with compact closure, then ζH_G is the potential of a measure μ, and the capacity of G is defined naturally to be

$$C(G) \equiv \int \varphi(r)\mu(dr). \tag{19.9}$$

There is no dual definition unless $H_G\,\varphi$ happens to be the potential of a function. The theorems of §§4–9 imply almost at once, however, that $C(G)$ is the infimum of $\int \varphi\nu(dr)$ as ν ranges over the measures whose potentials dominate ζ on G, or the supremum as ν ranges over the measures on G whose potentials are dominated by ζ. Dually, $C(G)$ is the infimum of $\int \zeta(ds)f(s)$ as f ranges over the positive functions whose potentials dominate φ on G (provided at least one such function exists), or the supremum as f ranges over the positive functions that vanish outside G and have potentials dominated by φ. There are several other expressions of $C(G)$ as an extremum. The various expressions for capacity usually become inequivalent, however, on leaving the class of open sets; there are several definitions of capacity for Borel sets which agree on the open sets with compact closures and are alternating of order infinity in the sense of Choquet. There is of course only one way of extending the definition (19.9) so that the capacity is continuous on the right; one takes the capacity of E to be the supremum of $\int \varphi(r)\nu(dr)$ as ν ranges over the measures with compact supports in E and potentials dominated by ζ. This extension, unfortunately, is not always the one of interest. For example, take φ to be the constant 1 and ζ to be the potential of the unit mass placed at a given point r; most probabilists would extend the definition (19.9) to a Borel set E by writing down the similar formula or, what is the same, by taking $C(E)$ to be $\Phi_E(r)$, the probability of hitting E after starting at r, which is continuous on the left in E rather than on the right.

20. Regular points

None of the hypotheses discussed so far implies the following statement, as one sees by considering uniform motion on a line.

(H) *If F is compact and not negligible, then some point is regular for F.*

We shall develop certain consequences of (H) that de la Vallée Poussin and Henri Cartan have shown to be fundamental in the theory of the Newtonian potential; from the point we leave off, it is clear sailing over a well-travelled course. The verification of (H) is treated in the latter half of the section. We obtain three effective criteria; they lack generality, but suffice in discussing most examples. More satisfactory conditions can be found in terms of the infinitesimal generator of the semigroup of transformations defined by the transition measures.

THEOREM 20.1. *Under (H), the points of an analytic set E that are not regular for E form a negligible set.*

Let F be a compact subset of E that includes no point regular for E. Then no point is regular for F, so that F must be negligible. The set mentioned in the theorem must therefore be negligible also.

For every analytic set E, the measure $H_E^\lambda(r, ds)$ is concentrated on the union of E with the set of points regular for E. Either r is regular for E—and then the measure is the unit mass placed at r—or else the measure vanishes on negligible sets. Under (H), consequently, the kernels $H_E^\lambda H_F^\lambda$ and $H_F^\lambda H_E^\lambda$ both coincide with H_F^λ if F is an analytic subset of E, as one sees on reviewing the discussion of hypothesis (B) in §6. In particular, the transformations defined by H_E^λ are idempotent. Conversely, these relations for λ fixed and strictly positive imply statement (H).

What has been said so far is valid under hypothesis (A) alone. In the remainder of the section we assume (F) and use the notation of §§17–19, interpreting (H) as saying that some point is right regular for a compact subset of $\mathcal{K}$ if the set is not negligible. The point w is, of course, right regular for any set to which it belongs. The word *regular* in Theorem 20.1 similarly stands for *right regular*.

Hypothesis (F) is assumed to hold for the semigroup H_τ and the basic measure ξ. The semigroups

$$H_\tau^\lambda \equiv e^{-\lambda\tau} H_\tau\,, \qquad \hat{H}_\tau^\lambda \equiv e^{-\lambda\tau}\hat{H}_\tau\,, \qquad \lambda \geqq 0,$$

with ξ still the basic measure, then satisfy both (F) and (G) for λ strictly positive, and even hypothesis (E^*) of §14. For λ strictly positive, moreover, Φ_E^λ attains the value 1 only at the points right regular for E.

THEOREM 20.2. *Under (F) and (H), the points of an analytic set E that are not left regular for E form a negligible set.*

According to the discussion above and relation (18.3), the kernels $\hat{H}_E^\lambda\,\hat{H}_F^\lambda$ and $\hat{H}_F^\lambda\,\hat{H}_E^\lambda$ coincide with $\hat{H}_F^\lambda$ if F is an analytic subset of the analytic set E and if λ is strictly positive; a passage to the limit shows that the identity persists for vanishing λ. Therefore (H) implies its dual, so also the present theorem.

Let E be an analytic set, ν a measure on E that vanishes on the set of points belonging to E but not left regular for E. Then $H_E\,U\nu$ coincides with $U\nu$, by relation (18.3), and $H_F\,U\nu$ increases to $H_E\,U\nu$ as F runs through a certain sequence of compact subsets of E. Consequently, the right potential of ν is determined by its restriction to E. One proves in the same manner that a right excessive function dominates $U\nu$ everywhere if it does so on E. These remarks and Theorem 20.2 give the following proposition.

PROPOSITION 20.3. *Let (F) and (H) hold, and let ν be a measure that is concentrated on the analytic set E and vanishes on every negligible set. Then $U\nu$ and $H_E\,U\nu$ coincide, $U\nu$ is determined by its restriction to E, and a right excessive function dominates $U\nu$ everywhere if it does so on E.*

Observe that a measure vanishes on negligible sets if its right or left potential is bounded.

Some other consequences of (F) and (H) are most easily derived in the same manner from stronger statements that assume only (F) but restrict certain functions or measures. We shall therefore disregard (H) momentarily.

Let (G) hold, and let ν be a measure which has compact support F and which vanishes on the set of points belonging to F but not left regular for F. We shall prove that the right potential φ of ν is continuous if its restriction to F is bounded and continuous. Since φ coincides with $H_F\,\varphi$, as we have seen above, it is bounded by its maximum on F. Thus $H_\gamma\,\varphi$ is continuous if γ belongs to Γ_1, according to a result preceding Proposition 18.1; so we need only prove that $H_\gamma\,\varphi$ approaches φ uniformly as γ runs through a fundamental sequence. The potential $U^\lambda\nu$ has a continuous restriction to F, since φ is the sum of $U^\lambda\nu$ and another lower semicontinuous function; it nowhere exceeds the maximum it attains on F, by the argument above; and it decreases everywhere to the null function as λ increases to infinity. By Dini's theorem the convergence is uniform on F, therefore on $\mathcal{K}$. We shall bound the difference $\varphi - H_\gamma \varphi$ in terms of these potentials. Denote by α the constant function 1 on the interval $0 < \tau < \infty$, by β the function $\alpha - \gamma * \alpha$, and by σ the supremum of the τ for which $\gamma(\tau)$ is strictly positive. Since $\beta(\tau)$ is bounded by 1 and vanishes for τ greater than σ, it is dominated by $e^{\lambda\sigma}e^{-\lambda\tau}$ for all λ. Consequently $h(\beta, r, s)$ is dominated by $e^{\lambda\sigma}U^\lambda(r, s)$, because $U^\lambda(r, s)$ is just $h(\delta, r, s)$ with δ the function $e^{-\lambda\tau}$. On specializing λ to $1/\sigma$, we obtain the relation

$$\varphi - H_\gamma\varphi \equiv \int h(\beta, r, s)\nu(ds) \leqq eU^{1/\sigma}\nu.$$

Now, σ decreases to 0 as γ runs through a fundamental sequence. Hence $H_\gamma \varphi$ indeed approaches φ uniformly.

Let (G) hold and let ν be a measure in $\mathfrak{M}$ that vanishes on negligible sets and has the following property:

(α) *Let μ be the restriction of ν to a compact set. Then $U\mu$ is continuous if it is bounded and if its restriction to the support of μ is continuous.*

We shall prove the existence of an open set G, of arbitrarily small natural capacity, such that the restriction of $U\nu$ to $\mathcal{K} - G$ is continuous. In the proof and later on, we shall need to bound the natural capacity of the open set B on which a potential $U\zeta$ exceeds a given positive number ρ. If F is a compact subset of B and $\hat{\pi}$ its natural left capacitary measure, then

$$\rho C(F) \leqq \int \hat{\pi}(dr)U\zeta.(r) \leqq \int \hat{\Phi}_F(s)\zeta(ds) \leqq \zeta(\mathcal{K}).$$

So the natural capacity of B does not exceed $\zeta(\mathcal{K})/\rho$.

First suppose ν to have compact support. Given a strictly positive number δ, choose an open set A such that $\nu(A)$ is less than δ and such that the restriction of $U\nu$ to $\mathcal{K} - A$ is bounded and continuous; the choice is possible because ν is a bounded measure vanishing on negligible sets and $U\nu$ is infinite on a negligible set at most. Let ν_1 be the restriction of ν to $\mathcal{K} - A$, and ν_2 the restriction to A. The support of ν_1 is a compact subset of $\mathcal{K} - A$, and the restriction of $U\nu_1$ to $\mathcal{K} - A$ is continuous, for the right potentials of ν_1 and ν_2 are lower semicontinuous and have $U\nu$ as their sum; hence $U\nu_1$ is continuous on $\mathcal{K}$. The set $G(\delta, \rho)$ on which $U\nu_2$ exceeds ρ is an open set of natural capacity not greater than δ/ρ. Clearly, the required set G may be taken to be the union of the sets $G(4^{-n}\varepsilon, 2^{-n})$, with ε sufficiently small.

Suppose now that the support of ν is not compact, and let D be any open set with compact closure. We shall demonstrate the existence of an open set A, of arbitrarily small natural capacity, such that the restriction of $U\nu$ to $D - A$ is continuous; the theorem evidently follows from this weaker version and the properties of capacity. Let ν_1 be the restriction of ν to a large compact set B, and ν_2 the restriction to the complement of B. The sum of $\hat{H}_D \nu_1$ and $\hat{H}_D \nu_2$ is the bounded measure $\hat{H}_D \nu$, and $\hat{H}_D \nu_1$ increases to $\hat{H}_D \nu$ as B increases to $\mathcal{K}$; the mass of $\hat{H}_D \nu_2$ can therefore be made arbitrarily small. Given a strictly positive number δ, choose B so that $\hat{H}_D \nu_2$ has mass less than δ^2; the set $A_2(\delta)$ on which the right potential of this measure exceeds δ then has natural capacity less than δ. Since ν_1 has compact support and satisfies (α), there is an open set $A_1(\delta)$, of natural capacity less than δ, such that the restriction of $U\nu_1$ to the complement of $A_1(\delta)$ is continuous. Let $A(\delta)$ be the union of $A_1(\delta)$ and $A_2(\delta)$. It has natural capacity less than 2δ, and $U\nu$ differs from $U\nu_1$ on $D - A(\delta)$ by not more than δ, because the right potentials of ν_1 and $\hat{H}_D \nu_1$ agree on D. Consequently, the required set A may be taken to be the union of the sets $A(2^{-n}\varepsilon)$, with ε sufficiently small. This completes the proof of the theorem.

Note that a measure in $\mathfrak{M}$ vanishes on negligible sets and satisfies (α) if it vanishes on every set comprising the points of an analytic set not left regular for the set. The right potential of such a measure therefore becomes continuous on being restricted to the complement of a certain open set of arbitrarily small natural capacity.

We are now in position to consider the relation of (H) to the three following statements:

(I) *Let μ be a bounded measure with compact support F. Then $U\mu$ is continuous if it is bounded and if its restriction to F is continuous.*

(J) *If φ is right excessive, there is an open set G, of arbitrarily small natural capacity, such that the restriction of φ to $\mathcal{K} - G$ is continuous.*

(K) *Let φ be a right excessive function, X a process, R the supremum of the τ for which $X(\tau)$ belongs to $\mathcal{K}$. Consider the function $\varphi(X(\tau, \omega))$ on the interval $0 < \tau < R(\omega)$. For almost all ω, this function is continuous wherever $X(\tau, \omega)$ is continuous.*

Hypothesis (F) must hold for (I) to make sense, but the truth of the statement does not depend upon the choice of the basic measure. Both (F) and (G) are needed in discussing (J), since it implicitly assumes the natural capacity to be defined; one can easily dispense with (G), however, by supposing the natural capacity to be defined in terms of the semigroup H_τ^λ, with λ strictly positive. Finally, (H) and (K) make sense under (A) alone.

THEOREM 20.4. *Under (F) and (G), the four statements (H) through (K) are equivalent.*

First, (H) implies (I). Let μ have compact support F, and let its right potential be bounded and have a continuous restriction to F. Then μ vanishes on negligible sets, because its potential is bounded; under (H), therefore, μ vanishes on the set of points belonging to F but not left regular for F. So (I) follows from an earlier result.

Next, (I) implies (J). If φ is a bounded potential, say $U\nu$, then ν vanishes on negligible sets; under (I), moreover, ν has the property (α) stated above. The conclusion of (J) has already been proved for the potentials of such measures. Suppose φ to be the right potential of a bounded measure μ. The minimum of φ and a positive constant ρ is a potential for which the conclusion of (J) holds, by what has just been proved, and φ differs from the minimum on an open set of natural capacity not greater than $\mu(\mathcal{H})/\rho$. So potentials of bounded measures satisfy the conclusion of (J). Finally, consider a right excessive function φ and an open set D with compact closure. The function $H_D\varphi$ is the right potential of a bounded measure and coincides with φ on D. There is an open set A, of arbitrarily small natural capacity, such that $H_D\varphi$ is bounded and continuous on the complement of A; hence φ is bounded and continuous on $D - A$. Clearly, (J) follows at once from this result and the properties of capacity.

That (J) implies (K) has been established by Doob for superharmonic functions and Brownian motion. The proof carries over unchanged; it is repeated here only for completeness. Let (J) hold, let φ, X, R have the meanings given in (K), and let (β) be the statement that $\varphi(X(\tau, \omega))$ is continuous at all points of the interval $0 < \tau < R(\omega)$ where $X(\tau, \omega)$ is continuous. Suppose X to have initial distribution $g(r)\,ds$, with g a bounded function that vanishes outside a compact set. The left potential of $g(r)\,dr$ is bounded, say by α. If E is any set having a natural right capacitary measure, then

$$\int g(r)\Phi_E^\lambda(r)\,dr = \int \pi_E(ds) \int g(r)U(r, s)\,ds \leqq \alpha C(E).$$

The inequality relating the extreme members holds for all analytic sets, by a passage to the limit. It states that the probability of $X(\tau)$ belonging to E for some strictly positive τ is bounded by $\alpha C(E)$. Let us take E to be the set G mentioned in (J). Since (β) holds if $X(\tau, \omega)$ lies outside G for all strictly positive τ and since the natural capacity of G can be made arbitrarily small, statement (β) must hold with probability 1. This conclusion extends at once to a process with an absolutely continuous distribution. In dealing with an arbitrary process X, let Z be a positive random variable which is independent of X and has an element γ of Γ_1 for density function. The process $X(\tau + Z)$ has an absolutely continuous initial distribution, if one considers only the ω for which $Z(\omega)$ is less than $R(\omega)$. On letting γ run through a fundamental sequence, we see that (β) holds again with probability 1. Statement (K) follows from this result and Theorem 11.3.

Finally, (K) implies (H). We shall prove a somewhat stronger assertion, valid in the simple theory of §§4–9. Assume (A) and (B) to hold, λ to be strictly positive, F to be a compact set for which no point is regular, and X to be a process which hits F with strictly positive probability. Then Φ_F^λ nowhere attains the value 1, and the set F is approximately null. We may assume the initial distribution of X to vanish on F, going over to the process $X(\tau + Z)$ if necessary, with Z a small positive random variable that is independent of X and has a continuous density function. Let T be the time X hits F, T' the time it hits the compact neighborhood G of F, and Ω^* or Ω' the set where T or T' is less than the simple terminal time S^λ. Since no point is regular for F and since (B) holds, T exceeds T' almost everywhere on Ω^*. The time T' increases to T as G shrinks to F; for almost all ω in Ω^*, the point $X(T'(\omega), \omega)$ approaches $X(T(\omega), \omega)$, and the value of Φ_F^λ at $X(T'(\omega), \omega)$ approaches 1, by Theorem 6.2. On the other hand, the value of Φ_F^λ at $X(T(\omega), \omega)$ is less than 1, so that $\Phi_F^\lambda(X(\tau))$ is not with probability 1 continuous wherever $X(\tau)$ is continuous. Turning matters around we see that (H) holds, under (A) and (B), if $\Phi_F^\lambda(X(\tau))$ is with probability 1 continuous wherever $X(\tau)$ is continuous, for some strictly positive λ and for all compact sets F and all processes X.

In the next step of the argument we assume (A) and (E) to hold and ψ to be excessive relative to the semigroup H_τ^λ. The function $\psi + \lambda U^0\psi$ is then excessive relative to the semigroup H_τ^0, as one sees by writing the equation

$$U^0 f = U^\lambda f + \lambda U^0 U^\lambda f, \qquad f \geqq 0,$$

and then letting f vary so that $U^\lambda f$ increases to ψ. Under (E), the right member of the equation is bounded if f is bounded and vanishes outside a compact set; consequently $\psi + \lambda U^0\psi$ is bounded if ψ is dominated by $U^\lambda f$, with f a bounded function that vanishes outside a compact set. We see in particular that Φ_F^λ, for F compact, is the difference of two bounded functions, both excessive relative to the semigroup H_τ^0. According to this remark and the foregoing paragraph, (H) follows from (A, B, E, K). It therefore follows from the stronger set of hypotheses (F, G, K).

There are a number of remarks complementing the theorem just proved. First, some hypothesis like (G) is needed in passing from (I), (J), or (K) to (H), in order to eliminate the possibility that all excessive functions are constant. On the other hand, (G) is superfluous if one starts from (F) and (H). Statement (K) then follows from the remarks that a function is excessive for the semigroup H_τ^λ if it is excessive for H_τ^0 and that (G) holds for the semigroup H_τ^λ and the basic measure ξ, provided λ is strictly positive. The discussion of (I) is a little more complicated, but the same in principle. Suppose the support F of μ to be compact, $U^0\mu$ to be bounded, and the restriction of $U^0\mu$ to F to be continuous. The equation

$$U^0\mu = U^\lambda\mu + \lambda U^0 U^\lambda\mu,$$

shows that $U^\lambda\mu$ is bounded and that its restriction to F is continuous, for both functions on the right are lower semicontinuous. If λ is strictly positive, the potential $U^\lambda\mu$ is therefore continuous, since (G) holds for the pair H^λ_τ and ξ. The continuity of $\lambda U^0 U^\lambda \mu$ also is easily established, so that $U^0\mu$ itself must be continuous; alternatively, $U^\lambda\mu$ tends uniformly to $U^0\mu$ because $\lambda U^0 U^\lambda\mu$ obviously decreases uniformly to the null function as λ decreases to 0. As for (J), only a sensible interpretation of capacity is needed. Let (J^λ) be statement (J) with the natural capacity of G defined in terms of the semigroup H^λ_τ and the basic measure ξ, and with φ understood to be right excessive for the semigroup H^0_τ; here λ is required to be strictly positive, unless (G) holds for vanishing λ. The statement (J^λ) apparently grows stronger as λ increases, because the natural capacity of a set increases with λ; but the statements are easily proved to be equivalent for all strictly positive λ, or for all λ if (G) holds when λ vanishes. The statements are also independent of the choice of basic measure. Now, by the theorem just proved, (F) and (H) imply (J^λ) for every strictly positive λ.

One can derive (K) from (A, C, H) in the setting of §§4–9, by a proof quite different from the one we have gone through. Statements (I) and (J) as they stand make sense only under (F). One can frame similar assertions in the setting of §§4–9, however, and prove an analogue of Theorem 20.4.

Let (F) and (G) hold, and let ν be a measure in $\mathfrak{M}$ that vanishes on every set comprising the points of an analytic set not left regular for the set. The left potential φ of ν satisfies the conclusion of (J), as we have seen; so it also satisfies the conclusion of (K), by the argument used in proving Theorem 20.4.

Let (F) and (G) hold, and let φ, X, R have the meanings given in (K). If φ is a right potential, then $\varphi(X(\tau, \omega))$ approaches 0 as τ increases to $R(\omega)$, for almost all ω such that $X(\tau, \omega)$ approaches the point w. Therefore, one may replace R by ∞ in stating (K) for potentials, under (F) and (G).

Statement (H) holds for Brownian motion in n-space, but not for Brownian motion in space-time. The theory of superharmonic functions therefore differs profoundly from the theory of superparabolic functions. So far as it concerns this paper, however, the difference is rather one of terminology and elegance: Under (H) the exceptional sets mentioned in several theorems are negligible, and need not be described in more detail; but without (H) they may not be negligible, and must be described explicitly—for example, as the set of points belonging to a certain analytic set but not right regular for the set.

Let us discuss conditions ensuring the truth of (H).

In this paragraph (F) and (G) are assumed to hold, and F is the set reduced to a single point t. The representation

$$\Phi_F(r) \equiv C(F)U(r, t),$$

with $C(F)$ the natural capacity of F, shows that F is not negligible if and only if $U(r, t)$ is bounded in r. We shall suppose $U(r, t)$ to be bounded in r for the rest of the paragraph. If t is right regular for F, then Φ_F attains its maximum at t, and $U(t, t)$ therefore bounds $U(r, t)$. The converse is also true. Suppose, for the purpose of argument, that $\Phi_F(t)$ is 1 but that t is not right regular for F. A process X starting at t then hits F at a finite and strictly positive time, with probability 1; this fact and the extended Markoff property imply that with probability 1 the point $X(\tau, \omega)$ coincides with t for arbitrarily great values of τ. Hence the integral of $U(t, r)\, dr$ over any neighborhood of t is infinite, by an argument like the one beginning the proof of Proposition 12.6. This result contradicts (G). So t must in fact be right regular for F.

Thus (H) holds, under (F) and (G), if $U(s, s)$ is finite for all s and bounds $U(r, s)$ for all r and s. Incidentally, $U(r, s)$ is then continuous in r and in s, because (I) must also be true by Theorem 20.4. Instances of this situation are the stable processes on the line with exponent greater than 1; one must take the parameter λ strictly positive to ensure (G).

Matters are less satisfactory if the kernel is unbounded. Let (F) and (G) hold, and let $U(r, s)$ have the following properties:

(i) *$U(s, s)$ bounds $U(r, s)$ for all r and s.*

(ii) *If A and B are disjoint compact sets, then $U(r, s)$ is bounded in the pair (r, s) and continuous in r for s fixed, as r and s range over A and B.*

(iii) *$\mathcal{H}$ can be covered by distinguished open sets.*

A set G is said to be distinguished if a constant α exists for which this statement is true: *For every compact subset F of G and for every point r of G, there is a point r' of F such that the inequality*

$$U(r, s) \leqq \alpha U(r', s)$$

holds for all s in F. The set G is distinguished, for example, if $\alpha U(t, s)$ bounds $U(r, s)$ whenever r, s, t are three points of G satisfying the inequality

$$\rho(r, s) \geqq \rho(r, t),$$

with ρ a metric inducing the topology of G; for then r' may be taken as the point of F nearest r.

Clearly, (ii) implies that $U\nu$ is continuous on $\mathcal{H} - F$ if ν is a bounded measure on the compact set F. Let G be a distinguished open set, α the associated constant, F a compact subset of G, and ν a measure on F. Then $U\nu$ is bounded on G by α times the supremum of $U\nu$ on F; the same bound serves on $\mathcal{H}$, because G is a neighborhood of the support of ν.

Statement (I) is easily proved under these hypotheses, and (H) then follows from Theorem 20.4. First of all, $U(r, s)$ is continuous in r if $U(s, s)$ is finite; for it is continuous away from s by (ii), upper semicontinuous at s by (i), and always lower semicontinuous. Thus one may assume the measure mentioned in (I) to attribute no mass to a single point. Next, the compact set F mentioned in (I) may be assumed to be a subset of some distinguished open set, because the sum of two lower semicontinuous functions is continuous only if both functions are continuous. With these simplifications the proof of (I) is a mere repetition of the familiar one for Newtonian potentials.

According to the preceding criterion, (H) holds for Brownian motion and for the processes associated with the potentials of Marcel Riesz. The next criterion establishes this result with less computation.

Statement (H) holds if (F) holds and if the function $h(\gamma, r, s)$ is symmetric,

$$h(\gamma, r, s) \equiv h(\gamma, s, r), \qquad r, s \in \mathcal{K}, \quad \gamma \in \Gamma.$$

We shall first sketch a simple proof, assuming each measure $H_\tau(r, ds)$ to have total mass 1. The measure ξ then coincides with ξH_τ for all τ, and one may consider the standard realization $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}), m)$ of a stationary process with transition measures $H_\tau(r, ds)$, defined more precisely in §17. The semigroups H_τ^λ and $\hat{H}_\tau^\lambda$ coincide, because of the symmetry assumed. Let φ be excessive for some semigroup H_τ^λ. We know from Theorem 11.3 that $\varphi(y(\tau))$ is continuous on the right in τ for almost all elements of $\mathcal{Y}$. The transformation sending the path $y(\tau)$ into the path

$$\hat{y}(\tau) = \lim_{\sigma \searrow \tau} y(-\sigma),$$

preserves the measure m, because of the symmetry assumed. So Theorem 11.3 implies that $\varphi(\hat{y}(\tau))$ is continuous on the right for almost all elements of $\mathcal{Y}$. Consequently, $\varphi(y(\tau))$ is continuous wherever $y(\tau)$ is continuous, a set of paths of measure null being neglected. It quickly follows that, for every process X and with probability 1, the function $\varphi(X(\tau))$ is continuous wherever $X(\tau)$ is continuous. Thus (K) holds for each semigroup H_τ^λ, so that (H) must indeed be true. In this proof, hypothesis (F) is not really needed. It is enough that (A) and (B) hold. that the Markoff transition measures be absolutely continuous relative to the invariant measure ξ, and that the stationary Markoff process constructed from ξ and the transition measures have the same transition measures after the direction of time is reversed.

We shall establish (H), under (F) and the symmetry of $h(\gamma, r, s)$, by means of Theorem 20.4 and an argument taken from a paper by Henri Cartan on the Newtonian potential, Bull. Soc. Math. France, t. 73, pp. 74–106. Hypothesis (G) will be assumed during the proof; no generality is lost thereby, for the semigroup H_τ may be replaced by the semigroup H_τ^λ if necessary, the basic measure being held fast. The adjectives *left* and *right* are unnecessary now, because of the symmetry.

Let β be the function $(\pi\tau)^{-1/2}$ on the interval $0 < \tau < \infty$. The convolution of this function with itself has the constant value 1, so that U may be written as $h_\beta * h_\beta$ in the notation of §17; this relation may also be expressed as

$$U(r, s) \equiv \int h(\beta, r, t) h(\beta, s, t)\, dt, \tag{20.1}$$

because $h(\beta, r, s)$ is symmetric in r and s. If μ is a positive measure on $\mathcal{H}$, let $W\mu$ be the function

$$W\mu.(r) \equiv \int h(\beta, r, s)\mu\,(ds), \qquad r \in \mathcal{H}. \tag{20.2}$$

We take $\mathcal{E}^+$ to be the class of positive measures μ for which the functions $W\mu$ are square integrable, and define the inner product of two elements of $\mathcal{E}^+$ to be

$$\begin{aligned}\langle \mu, \nu \rangle &\equiv \int (W\mu)(W\nu)\,dr \\ &\equiv \iint \mu(dr)U(r, s)\nu(ds).\end{aligned} \tag{20.3}$$

By (20.1) the two integrals are in fact equal for all positive measures μ and ν. The class $\mathcal{E}^+$ is closed under addition and multiplication by positive constants. The inner product is finite, symmetric, and bilinear, and it obviously satisfies the inequality

$$\langle \mu, \nu \rangle \leqq |\mu|\,|\nu|, \tag{20.4}$$

where $|\mu|$ stands for the positive square root of $\langle \mu, \mu \rangle$.

A bounded measure having a bounded potential clearly belongs to $\mathcal{E}^+$, by the second expression of the inner product. Let E be an analytic set, of finite natural capacity, whose natural capacitary measure exists. The measure π_E belongs to $\mathcal{E}^+$ and is concentrated on the union of E with the set of points regular for E; so the natural capacity of E is precisely $|\pi_E|^2$ by the second definition of the inner product. We shall now prove that a measure μ in $\mathcal{E}^+$ belongs to $\mathfrak{M}$ and vanishes on negligible sets. If F is a compact set on which $U\mu$ exceeds the positive constant ρ, then

$$\rho C(F) \leqq \langle \pi_F\,, \mu \rangle \leqq |\pi_F|\,|\mu| = C(F)^{1/2}|\mu|,$$

so that $C(F)$ does not exceed $|\mu|^2/\rho^2$. The same number evidently bounds the natural capacity of the open set on which $U\mu$ exceeds ρ. In particular, $U\mu$ is infinite on a negligible set at most, so that μ belongs to $\mathfrak{M}$. Let E be a negligible set with compact closure. If G is a compact neighborhood of E, then

$$\mu(E) \leqq \mu(G) \leqq \langle \pi_G\,, \mu \rangle \leqq |\pi_G|\,|\mu| \leqq C(G)^{1/2}|\mu|,$$

because the potential of π_G has the value 1 at all points of G. Since the natural capacity of G can be made arbitrarily small, $\mu(E)$ must vanish; therefore μ vanishes on every negligible set.

The difference of two measures in $\mathcal{E}^+$ is a set function, well defined and countably additive on sets with compact closures and vanishing on negligible sets. Let $\mathcal{E}$ be the vector space formed by these set functions. The potential of an element ν of $\mathcal{E}$ is defined up to a negligible set, the set where the potentials of the positive and negative parts of ν are both infinite. We again take (20.3) to define the inner product of two elements of $\mathcal{E}$. The two integrals make sense and have the same value, by what has just been proved; the first one shows that (20.4) still holds, $|\mu|$ being again the positive square root of the positive number $\langle\mu, \mu\rangle$. It will be shown in a moment that $|\mu|$ vanishes only if μ vanishes, but this fact is not really needed.

Let μ and ν belong to $\mathcal{E}$, and let B be the set where $|U\nu - U\mu|$ exceeds the positive number ρ or one of the potentials is undefined. We shall prove the inequality

$$C(B) \leqq 2|\nu - \mu|^2/\rho^2, \tag{20.5}$$

where $C(B)$ stands for the natural capacity of B. Let F be a compact set on which both potentials are defined and $U\nu$ exceeds $\rho + U\mu$. By the inequality

$$\rho C(F) \leqq \int \pi_F(dr)[U\nu - U\mu] = \langle\pi_F, \nu - \mu\rangle \leqq |\pi_F|\,|\nu - \mu|,$$

and the fact that $C(F)$ bounds $|\pi_F|^2$, the natural capacity of F does not exceed $|\nu - \mu|^2/\rho^2$. The same number evidently bounds the natural capacity of the full set on which both potentials are defined and $U\nu$ exceeds $\rho + U\mu$. A similar inequality holds with μ and ν interchanged, and the potentials of μ and ν are defined except on a negligible set. So (20.5) is established. As a first application, let us take ν to be the null measure and $|\mu|$ to vanish. The right member of (20.5) then vanishes for all strictly positive ρ, so that $U\mu$ vanishes except on a negligible set. The potentials of the positive and negative parts of u consequently agree except on a negligible set; these two measures therefore coincide, and μ must be the null measure.

It is now an easy matter to prove statement (J). Let ν be a positive bounded measure with a bounded potential, and let γ be an element of Γ_1. Then $H_\gamma U\nu$ is continuous, by Proposition 18.1, and $|\nu|$ bounds $|\hat{H}_\gamma \nu|$ because U dominates $H_\gamma U\hat{H}_\gamma$. Therefore,

$$|\nu - \hat{H}_\gamma \nu|^2 \leqq 2|\nu|^2 - 2\langle\nu, \hat{H}_\gamma \nu\rangle = 2\int [U\nu - H_\gamma U\nu]\nu(dr).$$

The last member decreases to 0 as γ runs through a fundamental sequence, because $H_\gamma U\nu$ increases to $U\nu$. Consequently, by (20.5), the continuous function $H_\gamma U\nu$ converges uniformly to $U\nu$ on the complement of an open set whose natural capacity can be made arbitrarily small. Thus there is an open set G, of arbitrarily small natural capacity, such that the restriction of $U\nu$ to $\mathcal{K} - G$ is continuous. One extends this result to the potential of a bounded measure by considering the minimum of the potential and a constant, and to an excessive function φ by considering the potential $H_D\varphi$, with D a compact set. So (J) holds.

Statement (H) now follows from Theorem 20.4.

We shall continue the discussion a little further. Given a compact set F, denote by $\mathcal{E}_F^+$ the subclass of $\mathcal{E}^+$ comprising the measures concentrated on F, and by $\mathcal{E}_F$ the subspace of $\mathcal{E}$ spanned by $\mathcal{E}_F^+$. If ν is any measure in $\mathcal{E}^+$, then $\hat{H}_F\,\nu$ belongs to $\mathcal{E}_F^+$, and its potential agrees with the potential of ν everywhere on F, with the exception of a negligible set. Thus $\langle \mu, \nu - \hat{H}_F\,\nu \rangle$ vanishes for every μ in $\mathcal{E}_F$, so that $\hat{H}_F\,\nu$ is precisely the projection of ν on $\mathcal{E}_F$, in the sense of the inner product defined on $\mathcal{E}$. The part of the modern theory of potentials that exploits the inner product of $\mathcal{E}$ can therefore be carried over to the present situation; for example, Cartan's discussion of the various modes of convergence of measures remains valid.

21. The relative theory

For a moment let A be a closed set in Euclidean space, $\mathcal{H}$ the complement of A, and ξ the restriction of Lebesgue measure to $\mathcal{H}$. The space $\mathcal{H}$, the measure ξ, and the transition measures of Brownian motion relative to the time of hitting A satisfy hypothesis (F) if and only if every point of A is regular for A. Thus the theory of the classical potentials on an arbitrary domain is not included in the treatment of §§17–20, so that it is necessary to discuss the relative theory in the spirit of §§10–14, but within the framework of (F).

In this section the space $\mathcal{H}$, the semigroup H_τ, and the basic measure ξ satisfy (F), while $\mathfrak{R}$ and $\hat{\mathfrak{R}}$ are the dual systems of terminal times determined by a positive function a and an analytic set A as in §17. We shall use the notation introduced in §17 for the relative transition measures and density functions. The kernels K_τ^λ, defined as $e^{-\lambda\tau}K_\tau$ for λ positive, can be interpreted as the transition measures relative to the system determined by $a + \lambda$ and A for the processes corresponding to the semigroup H_τ, or as the transition measures relative to the system determined by a and A for the processes corresponding to the semigroup H_τ^λ. The dual kernels $\hat{K}_\tau^\lambda$ are defined similarly. Observe that the density functions $k^\lambda(\alpha, r, s)$ are symmetric in r and s if the functions $h(\alpha, r, s)$ are symmetric. The parameter λ will be displayed only in a few definitions and proofs to avoid ambiguity.

The set $\mathcal{K}$ comprises all points of $\mathcal{H}$ not regular for $\mathfrak{R}$, while $\hat{\mathcal{K}}$ comprises all points of $\mathcal{H}$ not regular for $\hat{\mathfrak{R}}$. The two sets obviously coincide if the functions $h(\alpha, r, s)$ are symmetric, and it is important that they never differ by much.

Proposition 21.1. *The sets $\mathcal{K}$ and $\hat{\mathcal{K}}$ differ at most by a null set, or by a negligible set if (H) holds.*

In the proof we take λ strictly positive, so that the measure $H_{\mathfrak{R}}^\lambda(r, ds)$, defined as in (18.2), has mass less than 1 whenever r belongs to $\mathcal{K}$. Let F be a compact subset of $\mathcal{K} - \hat{\mathcal{K}}$. The relation

$$\Phi_F^\lambda = U^\lambda \pi_F^\lambda = U^\lambda \hat{H}_{\hat{\mathfrak{R}}}^\lambda \pi_F^\lambda = H_{\mathfrak{R}}^\lambda \Phi_F^\lambda$$

holds because π_F^λ is concentrated on F, hence on the set of points regular for $\hat{\mathfrak{R}}$. The right member is less than 1 on F, by the first sentence of the proof, so that no point of F is regular for F. Thus F is null, or negligible under (H); consequently $\mathcal{K} - \hat{\mathfrak{K}}$ is null or negligible, so also $\hat{\mathfrak{K}} - \mathcal{K}$ by duality.

The discussion of the sets $\mathcal{K}_\beta$ at the end of §10 and of special sets at the beginning of §11 requires a few supplements. Take $\Phi^1_{\mathfrak{R}}$ to be the function

$$\Phi^1_{\mathfrak{R}}(r) \equiv \mathcal{P}\{R < T, R < S\},$$

where R is the time assigned by $\mathfrak{R}$ to a process starting at r, T is the time the process arrives at w, and S is a positive random variable independent of R and T and having $e^{-\sigma}$ as density function for positive σ. This function is lower semicontinuous, since it is excessive for the semigroup H^1_τ. The set $\mathcal{K}$ is the part of $\mathcal{H}$ where $\Phi^1_{\mathfrak{R}}$ is less than 1, while the set $\mathcal{K}_\beta$, defined for β less than 1, is the part of $\mathcal{H}$ where $\Phi^1_{\mathfrak{R}}$ is less than β. If γ exceeds β, then $\mathcal{K}_\gamma$ includes the closure of $\mathcal{K}_\beta$ because $\Phi^1_{\mathfrak{R}}$ is lower semicontinuous. A right special set is an analytic set which is nearly open for the processes X, which is included in some $\mathcal{K}_\beta$, and which has compact closure in $\mathcal{H}$. The closure of a right special, set is evidently included in some right special set; in the present circumstances, indeed, one could replace right special sets by compact subsets of the $\mathcal{K}_\beta$. The sets $\hat{\mathfrak{K}}_\beta$ and left special sets are defined similarly.

The kernel for potentials $V^\lambda(r, s)$ is taken to be $k^\lambda(\alpha, r, s)$, with α identically 1. It vanishes unless r belongs to $\mathcal{K}$ and s to $\hat{\mathfrak{K}}$, and it is related to $U^\lambda(r, s)$ by the equations

$$U^\lambda = V^\lambda + H^\lambda_{\mathfrak{R}} U^\lambda = V^\lambda + U^\lambda \hat{H}^\lambda_{\hat{\mathfrak{R}}},$$

where $H^\lambda_{\mathfrak{R}}$ and $\hat{H}^\lambda_{\hat{\mathfrak{R}}}$ are defined as in (18.2). These facts were proved in §§17–18.

Let $\mathfrak{T}$ and $\hat{\mathfrak{T}}$ be dual systems of terminal times, relatively independent of $\mathfrak{R}$ and $\hat{\mathfrak{R}}$. The kernel $K^\lambda_{\mathfrak{T}}$ is defined as in (10.16), and $\hat{K}^\lambda_{\hat{\mathfrak{T}}}$ is defined dually. The identity

$$K^\lambda_{\mathfrak{T}} V^\lambda.(r, s) = V^\lambda \hat{K}^\lambda_{\hat{\mathfrak{T}}}.(r, s)$$

is easily proved. The two members are excessive for K^λ_τ in the variable r, excessive for $\hat{K}^\lambda_\tau$ in the variable s; so it is enough to establish the relation almost everywhere on the product space $\mathcal{H} \times \mathcal{H}$. To do this, consider the dual systems $\mathfrak{S}$ and $\hat{\mathfrak{S}}$, which are respectively the minimum of $\mathfrak{R}$ and $\mathfrak{T}$ or the minimum of $\hat{\mathfrak{R}}$ and $\hat{\mathfrak{T}}$. For λ strictly positive, the equations

$$U^\lambda = W_1 + H^\lambda_{\mathfrak{S}} U^\lambda, \qquad V^\lambda = W_2 + K^\lambda_{\mathfrak{T}} V^\lambda,$$

$$U^\lambda = W_3 + U^\lambda \hat{H}^\lambda_{\hat{\mathfrak{S}}}, \qquad V^\lambda = W_4 + V^\lambda \hat{K}^\lambda_{\hat{\mathfrak{T}}},$$

determine the four functions W_i almost everywhere on $\mathcal{H} \times \mathcal{H}$. By (18.3), the functions W_1 and W_3 agree almost everywhere. By the probabilistic interpretation, W_1 and W_2 agree almost everywhere in the second variable once the first is fixed, while W_3 and W_4 agree almost everywhere in the first variable once the second is fixed. Thus W_2 and W_4 coincide almost everywhere on the product space, so that the relation to be established holds for

λ strictly positive. It follows for vanishing λ by a passage to the limit. In a thorough presentation, of course, the general version of (17.10) would be established by the methods of §17, but we shall not need the result.

We shall write K_E^λ if $\mathfrak{T}$ is determined by the analytic set E. A measure ν belongs to the class $\mathfrak{N}^\lambda$ if it is concentrated on $\hat{\mathfrak{K}}$ and if $\hat{K}_{\hat{D}}^\lambda \nu$ is a bounded measure whenever $\hat{D}$ is a left special set. The class $\hat{\mathfrak{N}}^\lambda$ is defined dually. The place of hypothesis (G) is taken by the following statement:

(G^*) *If D is a right special set and $\hat{D}$ a left special set, the integrals*

$$\int_D V(r, s)\, ds, \qquad \int_{\hat{D}} V(r, s)\, dr,$$

are bounded in s and in r.

Under (G^*), a potential $V\nu$ is integrable over left special sets—hence finite almost everywhere—if ν belongs to $\mathfrak{N}$, and the potential determines ν. For λ strictly positive, not only does (G^*) hold, but the integrals also vanish at infinity, because U^λ dominates V^λ.

A set E is easily seen to be approximately null relative to $\mathfrak{R}$ if and only if $E \cap \mathcal{K}$ is a null set for the basic measure ξ. By Proposition 21.1, therefore, a set is approximately null relative to $\mathfrak{R}$ if and only if it is approximately null relative to $\hat{\mathfrak{R}}$.

Let φ be excessive for K_τ and finite almost everywhere. It is then finite except on a set that is negligible relative to $\mathfrak{R}$, and we shall prove that it is integrable over every left special set. First, choose a strictly positive function f so that the product $f\varphi$ is integrable over $\mathcal{K}$. Next, observe that the inequality

$$\lambda \int V^\lambda(r, s)\varphi(s)\, ds \leqq \varphi(r), \qquad \lambda > 0,$$

follows at once from the definition of excessive function. Given a left special set $\hat{D}$, fix λ strictly positive and choose a positive function g, bounded and vanishing outside some left special set, so that the integral $\int g(r)V^\lambda(r, s)\, dr$ exceeds 2 whenever s belongs to $\hat{D}$; the existence of g is assured by the dual of Proposition 12.1. Denote by ν or ν_n the measure $g(r)\, dr$ or $g_n(r)\, dr$, with g_n the minimum of g and nf. The potential $\nu_n U^\lambda$ increases to νU^λ as n increases, and both potentials are bounded continuous functions by Proposition 18.1. Using Dini's theorem, let us fix n so that νU^λ exceeds $\nu_n U^\lambda$ by less than 1 at all points of $\hat{D}$. Then $\nu_n V^\lambda$ differs from νV^λ by less than 1 on $\hat{D}$, and $\nu_n V^\lambda$ consequently exceeds 1 at all points of $\hat{D}$. Matters being so, we have

$$\begin{aligned}\int_{\hat{D}} \varphi(s)\, ds &\leqq \iint g_n(r)V^\lambda(r, s)\varphi(s)\, dr\, ds \\ &\leqq \frac{1}{\lambda}\int g_n(r)\varphi(r)\, dr \leqq \frac{n}{\lambda}\int f(r)\varphi(r)\, dr,\end{aligned}$$

and the last member is finite by the choice of f.

The only requirement of finiteness imposed on a measure excessive for $\hat{K}_\tau$ is that it attribute finite mass to every left special set. Thus, using the preceding result and arguing as in §18, we find that there is a one-to-one correspondence $\varphi \leftrightarrow \varphi\, ds$ between such measures and functions that are excessive for K_τ and finite almost everywhere. One verifies as before that $K_E \varphi$ corresponds to $\hat{M}_E(\varphi\, ds)$, the transformation $\hat{M}_E$ being defined as in §14. From now on we suppose excessive functions to be finite almost everywhere.

The integral $\int k(\alpha, r, s)f(s)\, ds$ is upper semicontinuous in r if f is a bounded positive function and α an element of Γ, because the integrand is upper semicontinuous in r and the integral with $k(\alpha, r, s)$ replaced by $h(\alpha, r, s)$ was proved in §18 to be continuous. The assertion remains true if α is only positive and integrable, by the argument immediately preceding Proposition 18.1.

Proposition 21.2. *Let (G^*) hold, and let f be a positive bounded function that vanishes outside some right special set. The integral $\int V(r, s)f(s)\, ds$ is then upper semicontinuous in r as well as bounded.*

The proof is like that of Proposition 18.1. The proposition is of course not so useful as the former one.

Proposition 21.3. *Let (G^*) hold, let F be a compact set of some $\hat{\mathfrak{K}}_\beta$, and let the ν_n be measures concentrated on F. If $V\nu_n$ increases and $\nu_n(F)$ remains bounded, as n increases, then ν_n converges weakly to a measure ν on F, and $V\nu_n$ increases to $V\nu$.*

We shall first prove the proposition assuming that ν_n converges weakly to a measure ν. By the dual of Proposition 21.2,

$$\lim \int f(r)\, dr \int V(r, s)\nu_n(ds) \leqq \int f(r)\, dr \int V(r, s)\nu(ds)$$

for every positive bounded function f that vanishes outside some left special set. Therefore $V\nu$ dominates the limit of $V\nu_n$ almost everywhere, hence everywhere, the two functions being excessive.

In the next step of the proof we fix λ strictly positive and choose a left special set $\hat{D}$ that includes F. Let μ_n and μ be the measures

$$\mu_n(ds) = (\lambda V\nu_n)\, ds, \qquad \mu(ds) = \lambda\varphi\, ds,$$

where φ is the limit of $V\nu_n$. The measure $\hat{K}^\lambda_{\hat{D}}\, \nu_n$ coincides with ν_n, the measure $\hat{K}^\lambda_{\hat{D}}\, \mu_n$ increases to the bounded measure $\hat{K}^\lambda_{\hat{D}}\, \mu$, and $\nu_n + \hat{K}^\lambda_{\hat{D}}\, \mu_n$ tends weakly to $\nu + \hat{K}^\lambda_{\hat{D}}\, \mu$. Also, the potentials

$$V^\lambda(\nu_n + \hat{K}^\lambda_{\hat{D}}\, \mu_n) = K^\lambda_{\hat{D}} V^\lambda(\nu_n + \mu_n) = K^\lambda_{\hat{D}} V\nu_n$$

increase with n. Consequently, by Proposition 14.4,

$$V^\lambda(\nu_n + \hat{K}^\lambda_{\hat{D}}\, \mu_n) \nearrow V^\lambda(\nu + \hat{K}^\lambda_{\hat{D}}\, \mu), \qquad n \to \infty,$$

and the inequality

$$\varphi = \lim V\nu_n \geqq \lim V^\lambda(\nu_n + \hat{K}^\lambda_{\hat{D}}\, \mu_n) \geqq V^\lambda\nu$$

follows immediately. On letting λ approach 0 we find that φ dominates $V\nu$. So the two functions must coincide, and the proof is complete under the supplementary hypothesis that the ν_n converge weakly. The full proposition follows from this partial result, because the limit measure is determined by its potential.

PROPOSITION 21.4. *Let (G^*) hold, let F be a compact subset of some $\hat{\mathfrak{K}}_\beta$, and let the ν_n be bounded measures on F. If $V\nu_n$ decreases as n increases, then ν_n converges weakly to a measure ν on F, and $V\nu_n$ decreases almost everywhere to $V\nu$.*

The proof is like the preceding one and will be omitted. According to Proposition 14.4 and the subsequent remark, the set F may be replaced by $\hat{\mathfrak{K}}$ in the last two propositions if the parameter λ is strictly positive.

According to Proposition 18.5 and the subsequent discussion, the kernels $K_E K_F$ and K_F coincide if E includes F and every point of F is left regular for E. The next result is proved in the same manner as Proposition 18.4.

PROPOSITION 21.5. *Let (G^*) hold, let φ be excessive for K_τ, and let E be an analytic subset of some left special set. Then $K_E\varphi$ is a potential $V\nu$, with ν a bounded measure concentrated on the union of E with the set of points left regular for E.*

An analytic set E determines two functions which are excessive for K_τ,

$$\Psi_E(r) \equiv \mathcal{P}\{T < R\}, \qquad \Theta_E(r) \equiv \mathcal{P}\{R > 0,\, T \leqq R,\, T < \infty\},$$

where R is the terminal time assigned to a process X starting at r and T is the time X hits E. These functions generalize in different ways the natural capacitary potential discussed in §19. We shall treat first Ψ_E, the right projective potential of E, and reserve until the end of the section a few remarks concerning Θ_E, the right capacitary potential of E.

PROPOSITION 21.6. *The set $\mathcal{K} - \hat{\mathfrak{K}}$ is negligible relative to $\mathfrak{R}$.*

Let F be a compact subset of $\mathcal{K} - \hat{\mathfrak{K}}$, take λ strictly positive, and consider a bounded potential $U^\lambda\nu$, with ν a measure on F. The identity

$$U^\lambda\nu - H^\lambda_{\mathfrak{R}} U^\lambda\nu \equiv V^\lambda\nu \equiv 0$$

holds, because ν is concentrated on the set of points regular for $\hat{\mathfrak{R}}$. We shall interpret the equation when ν is the natural right capacitary measure of F discussed in §19. Let X be a process starting at an arbitrary point, R the terminal time, T the time it hits F, and S a positive random variable having $\lambda e^{-\lambda\sigma}$ as density function for positive σ and independent of R and T. The equation states that the event $T < S$ has the same probability as the event $R < S$ *and* $X(\tau) \in F$ *for some* τ *between* R *and* S. It follows that the event $R \leqq T$ has probability 1. Consequently F is negligible relative to $\mathfrak{R}$, and $\mathcal{K} - \hat{\mathfrak{K}}$ is also negligible.

Clearly Ψ_E remains the same if E is replaced by $E \cap \mathcal{K}$, and also if E is replaced by $E \cap \hat{\mathcal{K}}$ according to the foregoing proposition. So no generality is lost by assuming E to be included in $\mathcal{K} \cap \hat{\mathcal{K}}$. If (G^*) holds and if E is included in a left special set, then Ψ_E is a potential $V\pi_E$, with π_E a bounded measure concentrated on the union of E with the set of points left regular for E; this follows from Proposition 21.5 with φ taken identically 1 on $\mathcal{K}$. Under (G^*) again, the same statement is true if E is included in a right special set; this follows quickly from Proposition 12.1. Whenever Ψ_E is a potential $V\pi_E$, we shall speak of π_E as the right projective measure of E.

The left projective potential $\hat{\Psi}_E$ and measure $\hat{\pi}_E$ are defined similarly. It is easy to see that there exists, for each analytic set E, an increasing sequence of compact subsets, each included in some intersection $\mathcal{K}_\beta \cap \hat{\mathcal{K}}_\beta$, such that Ψ_F and $\hat{\Psi}_F$ increase to Ψ_E and $\hat{\Psi}_E$ as F runs through the sequence.

Let (G^*) hold, and let E be an analytic set whose closure is a compact subset of some intersection $\mathcal{K}_\beta \cap \hat{\mathcal{K}}_\beta$. Then both π_E and $\hat{\pi}_E$ are bounded measures, and we shall prove that they have the same mass. Choose a right special set D and a left special set $\hat{D}$ including E. The potential $\hat{\pi}_{\hat{D}}\, V$ is identically 1 on $\hat{D}$, therefore 1 at every point belonging to E or left regular for E so that

$$\hat{\Psi}_E(s) = \int \hat{\pi}_{\hat{D}}\, V.(r)\hat{K}_E(dr, s) = \int \hat{\pi}_{\hat{D}}\, K_E(dr)V(r, s).$$

Hence $\hat{\pi}_{\hat{D}}\, K_E$ is the left projective measure of E, and similarly $\hat{K}_E\, \pi_D$ is the right projective measure of E. We now have

$$\hat{\pi}_E(\mathcal{K}) = \int \hat{\pi}_E(dr)V\pi_D.(r) = \int \hat{\pi}_{\hat{D}}(dr)K_E\, V.(r, s)\pi_D(ds).$$

Since the last integral is unchanged under the replacement of $K_E\, V$ by $V\hat{K}_E$, the two projective measures of E have the same mass.

Under (G^*) the projective mass of an arbitrary analytic set E is defined to be the supremum of the mass of π_F as F ranges over the compact subsets of E that are included in left special sets. It is easily proved that the dual definition gives the same number, that F may be restricted to the sequence mentioned two paragraphs above, and that the mass of π_E or $\hat{\pi}_E$ is precisely the projective mass of E when one of the measures exists. It follows that a set is negligible for $\mathfrak{N}$ if and only if it is negligible for $\hat{\mathfrak{N}}$.

Under (G^*) a measure ν belongs to $\mathfrak{N}$ if and only if $V\nu$ is integrable with respect to $\hat{\pi}_{\hat{D}}$ for every left special set $\hat{D}$. The proof is like the one in §18.

Fix an increasing sequence of open sets G_n so that the closure of G_n is a compact subset of G_{n+1} and the union of the G_n is $\mathcal{K}$. Let $\hat{D}_n$ be $G_n \cap \hat{\mathcal{K}}_{1-1/n}$, and let F_n be the complement of $\hat{D}_n$. The sequences $\hat{D}_n$ and F_n take the part of the sequences G_n and F_n introduced before Theorem 18.8. We shall now extend that theorem to the relative theory.

THEOREM 21.7. *Let (G^*) hold, and let ν belong to $\mathfrak{N}$. Then $H_F V\nu$ decreases to 0 as F runs through the sequence F_n, except perhaps at the points where $V\nu$ is infinite and at the points of $\mathcal{K}$ which are right regular for every F_n. This exceptional set is negligible relative to $\mathfrak{R}$.*

Denote by E the set on which $H_F V\nu$ does not tend to 0, and consider a compact subset B that is included in a left special set. There is a bounded positive function g, vanishing outside some left special set, such that the integral $\int g(r)V(r, s)\, dr$ exceeds 1 for all s in B. The integral then dominates $\hat{\pi}_B V$, and consequently the inequality

$$\int \hat{\pi}_B(dr)H_F\, V\nu.(r) \leqq \int g(r)H_F V\nu.(r)\, dr$$

holds. The integral on the right decreases to 0 as F runs through the sequence F_n, by the dual of Theorem 14.7. As a consequence, $\hat{\pi}_B$ vanishes and B is negligible for $\hat{\mathfrak{R}}$, hence for $\mathfrak{R}$. This result and Proposition 21.6 prove E to be negligible for $\mathfrak{R}$. Suppose now that r is not right regular for F_k and that $V\nu$ is finite at r. The measure $K_{F_k}(r, ds)$ attributes no mass to a set negligible for $\mathfrak{R}$, and every point of F_n is left regular for F_k if n exceeds k. Therefore, by dominated convergence,

$$K_{F_n} V\nu.(r) \equiv K_{F_k}\, K_{F_n} V\nu.(r) \to 0, \qquad n \to \infty.$$

The proof of the second sentence of the theorem is now complete. To prove the third sentence, it is enough to show that a set C is negligible for $\mathfrak{R}$ if it is compact, if it comprises only points right regular for every F_n, and if it is included in some right special set. By Proposition 12.1, there is a measure ν of the form $f ds$, with f a bounded positive function that vanishes outside some right special set, whose potential $V\nu$ exceeds 1 at all points of C. It is clear that $H_F\, V\nu$ coincides with $V\nu$ at all points of C, as F runs through the sequence F_n; therefore C is negligible for $\mathfrak{R}$, by the first part of the proof.

The simplest example showing the need of exceptional points is given by uniform motion on a circle, taking a null, A reduced to a single point s, and ν a mass placed at s. Both $\mathcal{K}$ and $\hat{\mathcal{K}}$ are the full circle, $V\nu$ is a constant, and $H_F\, U\nu$ decreases to 0 except at s.

The analogue of Theorem 18.7 of course carries over in the obvious manner, but Theorem 18.9 requires some changes.

Under (G^*) the least determining set for $V\nu$ is the support of ν, provided ν is a measure in $\mathfrak{N}$. It may not be true, however, that under (G^*) an excessive function is a potential if it has a compact subset of some $\hat{\mathcal{K}}_\beta$ for a determining set. For these reasons we shall extend Theorem 18.9 in two parts, a strong version for potentials and a weaker one for excessive functions.

THEOREM 21.8. *Let (G^*) hold, let E be an analytic set, and let ν belong to $\mathfrak{N}$. Then, except perhaps at the points where $V\nu$ is infinite and at the points of $E \cap \mathcal{K}$ not right regular for E, the infimum of $K_G\, V\nu$, as G ranges over the neighborhoods* *of E, coincides with the supremum of $V\mu$, as μ ranges over the measures on $E \cap \hat{\mathcal{K}}$ whose potentials are dominated by $V\nu$.*

The theorem follows at once from Theorem 13.5, because an excessive function is a potential if it is dominated by $V\nu$. Observe that the infimum mentioned in the theorem may be replaced by the infimum of $H_B\, V\nu$ as B ranges over the analytic sets that include $E \cap \hat{\mathcal{K}}$ and are nearly open for the processes $\hat{X}$. For, the new infimum is dominated by the old one, and $H_B\, V\mu$ coincides with $V\mu$ if B is such a set and μ a measure on $E \cap \hat{\mathcal{K}}$. The supremum is easily seen to be the potential of $\nu_1 + \nu_2$, where ν_1 is the restriction of ν to E and ν_2 is $\hat{K}_E(\nu - \nu_1)$.

THEOREM 21.9. *Let (G^*) hold, let E be an analytic subset of $\hat{\mathcal{K}}$, and let φ be excessive for K_τ. Then, except at the points where φ is infinite and at the points of E not right regular for E, the infimum of $H_D\, \varphi$, as D ranges over the analytic sets that include E and are nearly open for the processes $\hat{X}$, coincides with the supremum of $V\mu$, as μ ranges over the measures on E whose potentials are dominated by φ.*

The proof is the same as that of Theorem 14.11, except that Theorem 21.8 takes the place of Proposition 14.10. The minor assertions of Theorem 18.9 are also extended without difficulty to the present circumstances.

The next proposition is needed in studying the function Θ_E.

PROPOSITION 21.10. *Let B be included in some $\mathcal{K}_\beta$. The points that are both regular for $\hat{\mathfrak{R}}$ and left regular for B then form a negligible set.*

In the proof we shall take λ strictly positive but omit it from notation. By the definition of $\mathcal{K}_\beta$, the mass of $H_{\mathfrak{R}}(r, ds)$ is then bounded away from 1, say by α, as r ranges over the closure of $\mathcal{K}_\beta$. Given a bounded measure ν, define ν_n recursively by taking ν_0 to be ν and setting

$$\nu_{2k+1} = \nu_{2k}\, H_B\,, \qquad \nu_{2k+2} = \nu_{2k+1}\, H_{\mathfrak{R}}\,.$$

The mass of ν_n decreases as n increases, and the mass of ν_{2k} does not exceed $\alpha^k \nu(\mathcal{K})$. The series $\sum \nu_n\, U$ therefore converges except on a negligible set, so that the relation

$$\begin{aligned} \nu U &= \sum (\nu_{2k}\, U - \nu_{2k+1}\, U) + \sum (\nu_{2k+1}\, U - \nu_{2k+2}\, U) \\ &= \sum \nu_{2k}(U - H_B\, U) + \sum \nu_{2k+1}(U - H_{\mathfrak{R}}\, U) \end{aligned}$$

holds except on a negligible set. Let us denote by W the kernel for potentials relative to the system of terminal times defined by B, by μ the sum of the ν_n of even index, and by μ' the sum of the ν_n of odd index. The preceding equation can be written

$$\nu U = \mu W + \mu' V,$$

and the right member obviously vanishes at every point that is regular for $\hat{\mathfrak{R}}$ and left regular for B. The set of such points is therefore negligible, for ν can be chosen so that νU is strictly positive everywhere.

Let E be an analytic subset of some $\hat{\mathfrak{K}}_\beta$, and let F be an analytic set which includes E, which includes every point right regular for E, and for which every point of E is left regular. We shall prove that $K_F\,\Theta_E$ coincides with Θ_E. This amounts to proving that the set Ω', defined by the relations

$$T = T' = R < \infty,$$

has probability null, T being the time a process X starting from a point of $\mathfrak{K}$ hits E, T' the time X hits F, and R the terminal time assigned to X. The event

$$\int_0^T a(X(\tau))\, d\tau = Z$$

has probability null, if Z is the auxiliary variable used in defining R, because the two members are independent. So we may assume R to be the time X hits the set A, provided we suppose all points regular for $\mathfrak{R}$ to be adjoined to A. For almost all ω in Ω' the point $X(T(\omega), \omega)$ belongs to $A \cap F$, because every point right regular for A belongs to A, and because every point that belongs to E or is right regular for E belongs to F. We may therefore assume further that A is included in E, for the replacement of A by $A \cap E$ leaves R unchanged on Ω', except on a set of probability null. Now, by Proposition 18.5, for almost all ω in Ω' the point $X(T(\omega), \omega)$ is right regular for both E and A, while the set of such points is negligible according to Proposition 21.10. Consequently Ω' is null, and $K_F\,\Theta_E$ coincides with Θ_E.

Now let E be an analytic set whose closure is included in a left special set $\hat{D}$, and let (G^*) hold. Then $K_{\hat{D}}\,\Theta_E$ is the same as Θ_E, by the preceding paragraph, so that Θ_E is a potential $V\kappa$, with κ a bounded measure. Denote by E' the union of E with the set of points right regular for E. If G is a neighborhood of E', then $\hat{K}_G\,\kappa$ coincides with κ since $K_G\,\Theta_E$ coincides with Θ_E; on letting G shrink to E' we find that κ is concentrated on the union of E with the set of points that are right regular or left regular for E. I do not know whether any of the mass of κ is borne by the points that are right regular for E but neither belong to E nor are left regular for E; such points are certainly required for systems of terminal times a little more general than the ones considered here. If E is included in a right special set, as well as a left special set, then there is a measure $\hat{\kappa}$ dual to κ; however, the two measures may not have the same mass.

Let (G^*) hold, and let ν be a measure concentrated on E whose potential is bounded by 1. If G is a neighborhood of E, then $V\nu$ is dominated by Ψ_G; on letting G run through the decreasing sequence of Proposition 2.2, we find that $V\nu$ is dominated by Θ_E. On the other hand, Θ_E is the limit of an increasing sequence of such potentials, provided E is included in $\hat{\mathfrak{K}}$ and every point right regular for E belongs to E. To see this, one considers a process X having for initial distribution $g(r)\, ds$, with g strictly positive. Let Ω' be the set where T is finite and not greater than R; here T is the time X hits E, and R is the terminal time. Choose an increasing sequence of compact subsets F_n of E so that each F_n is included in a left special set and $X(T(\omega), \omega)$ belongs to some F_n, for almost every ω in Ω'. The functions Θ_F increase almost everywhere to Θ_E as F runs through the sequence. Since Θ_E and the limit function are both excessive, the convergence takes place everywhere and gives the asserted approximation of Θ_E.

It is clear by this time that the functions Ψ_E and Θ_E are extreme extensions of the natural capacitary potential discussed in §19. There is an intermediate extension, not treated here, that stems from the algorithm used in proving Proposition 21.10 and has an interesting probabilistic interpretation. The functions Ψ_E and Θ_E coincide, of course, if the terminal time assigned to a process differs with probability 1 from the (finite) time the process hits E. If this condition is met for a sufficiently large class of sets, then the treatment of capacity in §19 extends to the relative theory with little alteration. It is enough to suppose, for example, that the system $\mathfrak{R}$ is determined by a function a having the property that the integrals

$$\int_0^\tau a(X(\sigma))\, d\sigma, \qquad \tau < \infty,$$

are finite, for every process and with probability 1. The next proposition gives a different criterion.

THEOREM 21.11. *Let (H) hold, and let E be an analytic set whose closure is included in $\mathcal{K} \cup \hat{\mathcal{K}}$. If R is the terminal time assigned to a process X and if T is the time X hits E, then the set Ω' on which T is finite and coincides with R has probability null.*

Let R be defined by a, A, and the auxiliary variable Z. The event

$$\int_0^T a(X(\tau))\, d\tau = Z$$

has probability null; we may therefore assume a to vanish identically, supposing every point regular for $\mathfrak{R}$ to be adjoined to A. The set $A \cap \mathcal{K}$ is negligible, under (H), for else $\mathcal{K}$ would contain a point regular for $\mathfrak{R}$; similarly, $A \cap \hat{\mathcal{K}}$ is negligible. Now, for almost all ω in Ω' the point $X(T(\omega), \omega)$ belongs to the union of these sets, because it belongs to A and to the closure of E. The probability of Ω' must therefore vanish.

Under (H), consequently, the theory of capacity can be treated as in §19.

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