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ON FINITENESS PROPERTIES OF LOCAL COHOMOLOGY MODULES OVER COHEN–MACAULAY LOCAL RINGS

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ABSTRACT. Let A be a Cohen–Macaulay local ring which contains a field k, and let $I \subseteq A$ be an ideal generated by polynomials in a system of parameters of A with coefficients in k. In this paper, we shall prove that all the Bass numbers of local cohomology modules are finite for all $j \in \mathbb{Z}$ provided that the residue field is separable over k. We also prove that the set of associated prime ideals of those is a finite set under the same hypothesis. Furthermore, we shall discuss finiteness properties of local cohomology modules over regular local rings.

We assume that all rings are commutative and noetherian with identity throughout this paper.

1. Introduction

In this paper, we shall prove the following theorems.

THEOREM 1. *Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of local rings, which is module-finite and flat. Let i be a nonnegative integer. Furthermore, let I be an ideal of A satisfied with the condition that if we set $I \cap R = J$ then $I = JA$.*

(a) *If the set of associated prime ideals of $H^i_J(R)$ is a finite set, then so is the set of associated prime ideals of $H^i_I(A)$;*
(b) *if all the Bass numbers of $H^i_J(R)$ are finite, then so are all the Bass numbers of $H^i_I(A)$.*

THEOREM 2. *Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of regular local rings, which is module-finite and flat, and I an ideal of A. Let*

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i,j be nonnegative integers. Set $I\cap R = J$. Suppose that $I = JA$ and R is an unramified regular local ring. Then the following assertions hold:

(i) $\operatorname{inj.dim}_A H^j_{\mathfrak{n}} H^i_I(A) \le 1$;
(ii) $\operatorname{inj.dim}_A H^i_I(A) \le \dim H^i_I(A) + 1$;
(iii) *the set of associated prime ideals of $H^i_I(A)$ is a finite set;*
(iv) *all the Bass numbers of $H^i_I(A)$ are finite.*

In 1993, Huneke and Sharp (cf. [10]) and Lyubeznik (cf. [18]) showed the following results.

Theorem 3 (Huneke, Sharp, and Lyubeznik). *Let $(R,\mathfrak{m})$ be a regular local ring containing a field, and I an ideal of R. Then the following assertions hold for all integers $i,j \ge 0$:*

(i) *$H^j_{\mathfrak{m}}(H^i_I(R))$ is an injective module;*
(ii) $\operatorname{inj.dim}_R(H^i_I(R)) \le \dim H^i_I(R)$;
(iii) *the set of associated prime ideals of $H^i_I(R)$ is a finite set;*
(iv) *all the Bass numbers of $H^i_I(R)$ are finite.*

Several authors extended these results to several directions (cf. [3], [4], [5], [11], [12], [13], [14], [15], [19], [20], [22], [25], [26], and [29], etc.). Lately, it has been known answers for the question: what is the Lyubeznik number (cf. [16], [17], [21], and [30]), which is a numerical invariant on local cohomology modules defined by Lyubeznik. Especially, Zhang gave the topological characterization on the top Lyubeznik number, whose statement and proofs are the characteristic free (cf. [31]).

Our aims in this paper are to develop results of Theorem 3 to those over Cohen–Macaulay local rings in Section 2, and to prove several results over regular local rings in Section 3.

2. Proof of Theorem 1

Definition 1. Let T be a module over a ring A and P a prime ideal of A. We define the jth Bass number $\mu_j(P,T)$ at P to be

$$\mu_j(P,T) = \dim_{\kappa(P)} \operatorname{Ext}^j_{R_P}(\kappa(P), T_P),$$

where $\kappa(P) = R_P/PR_P$ (cf. [2]).

Remark 1. Let $\phi : (R,\mathfrak{m}) \longrightarrow (A,\mathfrak{n})$ be a local ring homomorphism of local rings, which is module-finite and flat. Then such properties for an extension are preserved by localization: let P be a prime ideal and set $\mathfrak{p} = P\cap R$. Then the local ring homomorphism of local rings $\phi_P : R_{\mathfrak{p}} \longrightarrow A_P$ is module-finite and flat (cf. [24, Theorem 7.1, p. 46]).

Remark 2. Let $\phi : R \longrightarrow A$ be a ring homomorphism of rings, I an ideal of A and $J = I\cap R$. The condition $I = JA$ of Theorems 1 and 2 are preserved by localization, i.e., for any prime ideal $P \subset A$ we have $JR_{\mathfrak{p}} = IA_P \cap R_{\mathfrak{p}}$ where $\mathfrak{p} = P\cap R$.

Proof of Theorem 1.

(a) Since the map ϕ is a flat extension, it follows from [23, (9.B), Theorem 12, p. 58] that

$$\mathrm{Ass}_A(H \otimes A) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(H)} \mathrm{Ass}_A(A/\mathfrak{p}A),$$

where H is the local cohomology module $H^i_I(R)$. Of course, $\mathrm{Ass}_A(A/\mathfrak{p}A)$ is a finite set, since $A/\mathfrak{p}A$ is a noetherian A-module. Since $\mathrm{Ass}_R(H^i_I(R))$ is a finite set by assumption, the right-hand side of the above formula is a finite union of finite sets. We set $J = I \cap R$ and assume that $JA = I$. Since a local cohomology module commutes with a flat module, we have $H \otimes_R A = H^i_I(A)$. Thus, $\mathrm{Ass}_A(H^i_I(A))$ is a finite set for any integer $i \geq 0$.

(b) From Definition 1 on the Bass numbers, we only prove the finiteness of the Bass numbers of the local cohomology modules for the maximal ideal $\mathfrak{n}$, by localizing A at a prime ideal of A (observe Remark 1 and Remark 2). Since all the Bass numbers of $H^i_J(R)$ are assumed to be finite,

$$\mathrm{Ext}^p_R\big(R/\mathfrak{m}, H^i_J(R)\big)$$

is a finite dimensional $R/\mathfrak{m}$-vector space for all $p \geq 0$, that is, it is a finite R-module. Thus,

$$\mathrm{Ext}^p_R\big(R/\mathfrak{m}, H^i_J(R)\big) \otimes_R A = \mathrm{Ext}^p_A\big(A/\mathfrak{m}A, H^i_{JA}(A)\big)$$

is a finite A-module. Here, we note that A is R-flat.

Since $R \to A$ is a module-finite ring homomorphism (hence, an integral extension), $\mathfrak{m}A$ is equal to $\mathfrak{n}$ up to radicals. So, we have $\mathrm{Ext}^p_A(A/\mathfrak{n}, H^i_{JA}(A))$ is a finite A-module for all $p \geq 0$ by [9, Theorem 4.1, p. 426]. It holds that

$$\mathrm{Ext}^p_A\big(A/\mathfrak{n}, H^i_{JA}(A)\big)$$

is a finite dimensional $A/\mathfrak{n}$-vector space for all $p \geq 0$. Therefore, all the Bass numbers of $H^i_I(A) = H^i_{JA}(A)$ are finite. □

PROPOSITION 4. *Let $A \longrightarrow B$ be a local ring homomorphism of local rings, which is flat. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals of A. Then $\mathfrak{p}_1 = \mathfrak{p}_2$ if and only if $\mathrm{Ass}_B(B/\mathfrak{p}_1 B) = \mathrm{Ass}_B(B/\mathfrak{p}_2 B)$.*

Proof. We only prove that if $\mathrm{Ass}_B(B/\mathfrak{p}_1 B) = \mathrm{Ass}_B(B/\mathfrak{p}_2 B)$, then $\mathfrak{p}_1 = \mathfrak{p}_2$.

Suppose that $\mathrm{Ass}_B(B/\mathfrak{p}_1 B) = \mathrm{Ass}_B(B/\mathfrak{p}_2 B)$. Let x be an element of A not in $\mathfrak{p}_1$. Then we have an exact sequence $0 \longrightarrow A/\mathfrak{p}_1 \xrightarrow{x} A/\mathfrak{p}_1$. Since B is A-flat, it holds that the sequence $0 \longrightarrow B/\mathfrak{p}_1 B \xrightarrow{x} B/\mathfrak{p}_1 B$ is exact. Since x is $B/\mathfrak{p}_1 B$-regular, x is not in the union of all prime ideals of $\mathrm{Ass}_B(B/\mathfrak{p}_1 B) = \mathrm{Ass}_B(B/\mathfrak{p}_2 B)$. Hence, x is $B/\mathfrak{p}_2 B$-regular (cf. [24, Theorem 6.1, p. 38]). So, we have an exact sequence $0 \longrightarrow A/\mathfrak{p}_2 \otimes B \xrightarrow{x} A/\mathfrak{p}_2 \otimes B$. Since B is faithfully flat over A, the sequence $0 \longrightarrow A/\mathfrak{p}_2 \xrightarrow{x} A/\mathfrak{p}_2$ is exact. Then x is not in $\mathfrak{p}_2$. Therefore, we have the inclusion $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$.

The reverse inclusion follows from a similar argument. Therefore, we have $\mathfrak{p}_2 = \mathfrak{p}_1$, as required. $\square$

Thanks to Theorem 3, our theorem proposes the following corollary. One can find the collection of properties for the faithfully flat and flat local ring homomorphisms in [14].

COROLLARY 5. *Let $(A, \mathfrak{n})$ be a Cohen–Macaulay local ring containing a field k, of dimension d, and $x_1, x_2, \ldots, x_d$ a system of parameters. Let I be an ideal of A generated by polynomials over k in $x_1, x_2, \ldots, x_d$. Suppose that $A/\mathfrak{n}$ is separable over k (or rather, over the image of k in $A/\mathfrak{n}$ via the natural mapping $A \to A/\mathfrak{n}$). The following statements hold for all integers $i, j \geq 0$:*

(a) *the set of associated prime ideals of $H_I^i(A)$ is finite;*
(b) *all the Bass numbers of $H_I^i(A)$ are finite.*

Proof. Let B be the completion $\hat{A}$ of the ring A with respect to $\mathfrak{n}$-adic topology. Then $(B, \mathfrak{n}')$ is a local ring with the maximal ideal $\mathfrak{n}' = \mathfrak{n}B$ and $A \to B$ is a flat local extension. First, we shall prove that we can reduce the corollary to the case of a complete local ring.

Suppose that the set of associated prime ideals of $H_{IB}^i(B) = H \otimes B$ is finite, where we set $H = H_I^i(A)$. Then the right-hand side of Bourbaki's formula,

$$\mathrm{Ass}_B(H \otimes B) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(H)} \mathrm{Ass}_B(B/\mathfrak{p}B),$$

is a finite set. Since $\mathrm{Ass}_B(B/\mathfrak{p}B)$ is a finite set, it follows from Proposition 4 that $\mathrm{Ass}_A(H)$ is a finite set.

Suppose that all the Bass numbers of $H_{IB}^i(B)$ are finite. Let $\mathfrak{p}$ be any prime ideal of A and pick up a minimal prime ideal P of $\mathfrak{p}B$ in $\mathrm{Spec}(B)$. Then we have $\mathfrak{p} = P \cap A$ by the going-down theorem (cf. [24, Theorem 9.5, p. 68]) and it holds that the radical of $\mathfrak{p}B_P$ is equal to PB_P by the minimality of P. We suppose that

$$\mathrm{Ext}_B^l\big(B/P, H_{IB}^i(B)\big)_P = \mathrm{Ext}_{B_P}^l(\kappa(P), H_{IB_P}^i(B_P))$$

are finite dimensional $\kappa(P)$-vector spaces for all $l \geq 0$, so

$$\mathrm{Ext}_{B_P}^l\big(B_P/\mathfrak{p}B_P, H_{IB_P}^i(B_P)\big) = \mathrm{Ext}_{A_\mathfrak{p}}^l\big(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}, H_I^i(A_\mathfrak{p})\big) \otimes_{A_\mathfrak{p}} B_P$$

are finite B_P-modules for all $l \geq 0$ by [9, Theorem 4.1, p. 426]. Since the extension $A_\mathfrak{p} \to B_P$ is faithfully flat, $\mathrm{Ext}_{A_\mathfrak{p}}^l(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}, H_I^i(A_\mathfrak{p}))$ is a finite $A_\mathfrak{p}$-module for all $l \geq 0$.

Now, we prove the corollary. Completing the ring A with respect to $\mathfrak{n}$-adic topology, we may assume that A is a complete Cohen–Macaulay local ring containing a coefficient field K including the field k, since $A/\mathfrak{n}$ is a separable field extension over k (cf. [24, Theorem 28.3, (iii), (iv), p. 216]). Cohen's structure theorems for complete local rings imply that there is a complete regular local ring $R = K[[X_1, X_2, \dots, X_d]]$ such that $\phi : R \to A$ is a module-finite ring homomorphism, where $\phi(X_i) = x_i$ for all $i = 1, 2, \dots, d$. Then it holds that $(I \cap R)A = I$. Furthermore, the map $\phi : R \to A$ must be a flat extension by the Auslander–Buchsbaum–Serre formula (cf. [24, Theorem 19.1, p. 155]). Since R is a regular local ring containing the field K, the assertion follows from Theorem 1 and Theorem 3. $\square$

REMARK 3. We need not the condition $k \subset A$ in Corollary 5, provided that A is complete and $x_1 = p$ is the characteristic of the residue field $A/\mathfrak{n}$ (see Corollary 11 below).

EXAMPLE 1. *Singh* [28] *and Katzman* [11] *gave the examples of rings with respect to sets of infinite associated prime ideals of the top local cohomology modules (see also* [12] *for such examples involving nontop local cohomology modules). Especially Katzman's example states that even the second local cohomology module has an infinite set of distinct associated prime ideals. The local ring R and the local cohomology module are as follows:*

$$R = k[s,t,x,y,u,v]_{\mathfrak{m}}/\bigl(sx^2v^2 - (s+t)xyuv + ty^2u^2\bigr), \quad H^2_{(u,v)}(R),$$

where $\mathfrak{m}$ is the irrelevant maximal ideal (s,t,x,y,u,v). The ring R is a hypersurface, thus R is a Cohen–Macaulay local ring. The elements u, v have the relation $sx^2v^2 - (s+t)xyuv + ty^2u^2 = 0$ in R, so that the sequence u, v is not a part of system of parameters for R. Therefore, our claim (a) *in Corollary* 5 *avoids this example.*

On the other hand, our result states that $H^j_I(R)$ satisfies finiteness properties (a) *and* (b) *as in Corollary* 5*, for all $j \geq 0$ and for the ideal I generated by polynomials $f_1 = f_1(s,x,v,t-y,t-u)$, $f_2 = f_2(s,x,v,t-y,t-u), \dots, f_r = f_r(s,x,v,t-y,t-u)$ in $s,x,v,t-y,t-u$ over k. Now, we show that $s,x,v,t-y,t-u$ is a system of parameters of R.*

Let f be the polynomial $sx^2v^2-(s+t)xyuv+ty^2u^2$ in $S = k[s,t,x,y,u,v]_{\mathfrak{m}}$. Then we have an equality of ideals $(s,x,v,f) = (s,x,v,ty^2u^2)$ in S. So, the sequence s,x,v is a part of a system of parameters of R. Furthermore, since ideals $(t),(y)$ and (u) are minimal primes in $k[t,y,u]_{(t,y,u)}/(ty^2u^2)$, an elementary argument shows that the sequence $t-y, t-u$ is a system of parameters of $k[t,y,u]_{(t,y,u)}/(ty^2u^2) = S/(s,x,v,ty^2u^2) = R/(s,x,v)$. Therefore, the sequence $s,x,v,t-y,t-u$ is a system of parameters of R.

Since $t-u, v$ is a regular sequence on R, $H^2_{(t-u,v)}(R)$ is $(t-u,v)$-cofinite (see [6] *for the definition on cofiniteness). Hence,* $\mathrm{Hom}_R(R/(t-u,v), H^2_{(t-u,v)}(R))$ *is a finitely generated R-module. Therefore, the set of associated prime ideals of $H^2_{(t-u,v)}(R)$ is finite, while the set of associated prime ideals of $H^2_{(u,v)}(R)$ is infinite, as we mentioned above.*

REMARK 4. The converse statements of (a) and (b) in Theorem 1 also hold by Proposition 4 and faithfully flatness of ϕ.

3. Several results over regular local rings

In this section, we prove Theorem 2. To do so, we first establish some preliminary results.

DEFINITION 2. A regular local ring $(R, \mathfrak{m})$ is called unramified if R contains a field or if $p \notin \mathfrak{m}^2$ in the unequal characteristic case, where p is the characteristic of the residue field $R/\mathfrak{m}$ (cf. [8, p. 12, lines 4–5]). We note that if R contains a field, then the characteristic of R and its residue field are equal, and the converse also holds.

REMARK 5. Let $(R, \mathfrak{m})$ be an unramified regular local ring. If we localize R at a prime ideal, it remains unramified.

Indeed, let $(R, \mathfrak{m})$ be an unramified regular local ring, P a prime ideal of R. Of course, R_P is a regular local ring. In the case that R contains a field, R_P also contains a field, since we have the natural map $R \to R_P$. So, we may assume that the characteristic of R is unequal. Let p be the characteristic of $R/\mathfrak{m}$. If R_P is equi-characteristic, then R_P contains a field. So R_P is an unramified regular local ring. If the characteristic of R_P is unequal, then R_P/PR_P has a positive characteristic, which we denote by q. Since there is the natural inclusion $\mathbb{Z}/q\mathbb{Z} \subseteq R_P/PR_P$, the number q is in PR_P. Hence, we have $q \in P$. Here, we note that q is a prime number.

Consider the following commutative diagram consisting of natural mappings:

$$\begin{array}{ccccccccc} 0 & \to & P & \to & R & \to & R/P & \to & 0 \\ & & \cap & & \| & & \downarrow & & \\ 0 & \to & \mathfrak{m} & \to & R & \to & R/\mathfrak{m} & \to & 0, \end{array}$$

so we see that q is zero in $R/\mathfrak{m}$, for $q \in P$. Since p is the characteristic of $R/\mathfrak{m}$, p divides q. On the other hand, q is a prime number. So, we must have that p equals q.

Now, $p = q \in P$ and we assume that the characteristic of R is unequal and R is unramified. So, we have $p \notin \mathfrak{m}^2$. Therefore, q is not in P^2, for $P^2 \subseteq \mathfrak{m}^2$. It follows that q is not in P^2R_P, that is R_P is an unramified regular local ring.

LEMMA 6. *Let $(A, \mathfrak{n})$ be a local ring with the maximal ideal $\mathfrak{n}$, M be an (not necessarily finite) A-module with support $V(\mathfrak{n})$. Let l be a nonnegative integer. Suppose that M is an A-module of finite injective dimension.*

(i) *If there is an A-module N with finite length such that $\operatorname{Ext}_A^n(N, M) = 0$ for all $n \geq 1$, then M is an injective A-module.*

(ii) *If there is an A-module N with finite length such that $\operatorname{Ext}_A^n(N, M) = 0$ for all $n \geq l + 1$, then $\operatorname{inj.dim}_A M \leq l$.*

Proof. (i) We only prove that $\operatorname{Ext}_A^n(A/\mathfrak{n}, M) = 0$ for all $n \geq 1$, since the support of M is contained in $V(\mathfrak{n})$. Suppose that the injective dimension of

M is not greater than s for some $s \geq 1$. If the length of N is one, there is nothing to prove. We may assume that the length of N is greater than one. Then there is a finite filtration:

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_{r-1} \subset N_r = N,$$

such that N_u/N_{u-1} is isomorphic to $A/\mathfrak{n}$ as an A-module for all $u = 1, 2, \ldots, r$. Especially, N_1 is isomorphic to $A/\mathfrak{n}$ as an A-module.

Consider the exact sequence:

$$0 \longrightarrow N_{r-1} \longrightarrow N \longrightarrow A/\mathfrak{n} \longrightarrow 0.$$

Applying the derived functor $\mathrm{Ext}^*_A(-, M)$ of the contravariant functor $\mathrm{Hom}_A(-, M)$, it follows from the assumption of the lemma that

$$\mathrm{Ext}^j_A(N_{r-1}, M) \simeq \mathrm{Ext}^{j+1}_A(A/\mathfrak{n}, M),$$

for all $j \geq 1$. Especially, we have $\mathrm{Ext}^s_A(N_{r-1}, M) \simeq \mathrm{Ext}^{s+1}_A(A/\mathfrak{n}, M) = 0$. Therefore, we have $\mathrm{Ext}^j_A(N_{r-1}, M) = 0$ for all $j \geq s$, since $\mathrm{inj.dim}_A M \leq s$.

Next, we consider the exact sequence $0 \longrightarrow N_{r-2} \longrightarrow N_{r-1} \longrightarrow A/\mathfrak{n} \longrightarrow 0$, then we have $\mathrm{Ext}^j_A(N_{r-2}, M) \simeq \mathrm{Ext}^{j+1}_A(A/\mathfrak{n}, M) = 0$ for all $j \geq s$, since $\mathrm{Ext}^j_A(N_{r-1}, M) = 0$ for all $j \geq s$ and $\mathrm{inj.dim}_A M \leq s$. Repeating this argument for the short exact sequence: $0 \longrightarrow N_{u-1} \longrightarrow N_u \longrightarrow A/\mathfrak{n} \longrightarrow 0$ for all $u = 1, 2, \ldots, r$ decreasingly, we have $\mathrm{Ext}^j_A(A/\mathfrak{n}, M) = \mathrm{Ext}^j_A(N_1, M) \simeq \mathrm{Ext}^{j+1}_A(A/\mathfrak{n}, M) = 0$ for all $j \geq s$. Hence, it holds that $\mathrm{inj.dim}_A M \leq s - 1$. Eventually, we have showed that if $\mathrm{inj.dim}_A M \leq s$ then $\mathrm{inj.dim}_A M \leq s - 1$.

Inductively, we have finally $\mathrm{inj.dim}_A M = 0$. Therefore, M is an injective A-module.

(ii) Let I^* be a minimal injective resolution of M:

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} I^2 \xrightarrow{\partial^2} \cdots \longrightarrow I^{l-1} \xrightarrow{\partial^{l-1}} I^l \xrightarrow{\partial^l} I^{l+1} \longrightarrow \cdots,$$

so all injective A-modules in I^* have support in $V(\mathfrak{n})$. When we denote the image of ∂^i with P^{i+1} for all the non-negative integer i, P^{i+1} has support in $V(\mathfrak{n})$ for each i. So, we have the following isomorphism for all $n \geq 1$:

$$\mathrm{Ext}^{n+l}_A(N, M) \simeq \mathrm{Ext}^n_A(N, P^l).$$

It follows from the assumption of the lemma that $\mathrm{Ext}^n_A(N, P^l) = 0$ for all $n \geq 1$. Hence, P^l is already an injective A-module by the assertion (i). From the minimality of I^*, it holds that $P^l = I^l$ and $I^{l+1} = 0$. Therefore, the injective dimension of M is not greater than l. We complete the proof of part (ii) of the lemma. □

We shall prove several lemmas following referee's suggestions.

LEMMA 7. *Let A be a ring, and l a positive integer. Let*

$$(\sharp) \qquad 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A-modules.

(1) *If M' and M'' are injective A-modules, then M is an injective A-module;*
(2) *if* $\operatorname{inj.dim}_A M' \leq l-1$ *and* $\operatorname{inj.dim}_A M'' \leq l$, *then* $\operatorname{inj.dim}_A M \leq l$.

In addition, suppose that M is an injective A-module.

(3) *If M' is an injective A-module, then M'' is an injective A-module;*
(4) *if* $\operatorname{inj.dim}_A M' \leq l$, *then* $\operatorname{inj.dim}_A M'' \leq l-1$.

Proof. We note that an A-module T has injective dimension $\leq l$ if and only if $\operatorname{Ext}^i_A(A/\mathfrak{a}, T) = 0$ for all $i \geq l+1$ and for all ideals $\mathfrak{a} \subset A$. See [23, (18. A), Lemma 1, p. 127] for the proof of the fact that T is injective if and only if $\operatorname{Ext}^1_A(A/\mathfrak{a}, T) = 0$ for all ideals $\mathfrak{a} \subset A$. The proofs of the lemma are straightforward, by applying the derived functor $\operatorname{Ext}^i_A(A/\mathfrak{a}, -)$ to the sequences ($\natural$) and ($\sharp$). We leave the proofs to the reader. $\square$

LEMMA 8. *Let A be a ring, $\mathfrak{a}$ an ideal of A, and M an (not necessarily finite) A-module. Let l be a positive integer. Denote by I^* a minimal injective resolution of T:*

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{\partial^0} I^1\, \partial^1 \longrightarrow I^2 \xrightarrow{\partial^2} \cdots \longrightarrow I^{j-1} \xrightarrow{\partial^{j-1}} I^j \xrightarrow{\partial^j} I^{j+1} \xrightarrow{\partial^{j+1}} \cdots.$$

Furthermore, we denote by d^j the restriction of ∂^j to $\Gamma_\mathfrak{a}(I^j)$, after applying the functor $\Gamma_\mathfrak{a}(-)$ to I^*.

(1) *If $H^j_\mathfrak{a}(M)$ is an injective A-module for all $j \geq 0$, then* $\ker d^j$ *is an injective A-module for all $j \geq 0$, and* $\operatorname{Im} d^j$ *is an injective A-module for all $j \geq 0$;*
(2) *If* $\operatorname{inj.dim}_A H^j_\mathfrak{a}(M) \leq l$ *for all $j \geq 0$, then* $\operatorname{inj.dim}_A \ker d^j \leq l$ *for all $j \geq 0$, and* $\operatorname{inj.dim}_A \operatorname{Im} d^j \leq l-1$ *for all $j \geq 0$.*

Proof. First, we note that if I is an injective A-module, then $\Gamma_\mathfrak{a}(I)$ is an injective A-module (See [7, Lemma 3.2, p. 213]). Now, we decompose the complex $\Gamma_\mathfrak{a}(I^*)$:

$$\begin{aligned} 0 &\longrightarrow \Gamma_\mathfrak{a}(M) \longrightarrow \Gamma_\mathfrak{a}(I^0) \xrightarrow{d^0} \Gamma_\mathfrak{a}(I^1) \xrightarrow{d^1} \Gamma_\mathfrak{a}(I^2) \xrightarrow{d^2} \cdots \\ &\longrightarrow \Gamma_\mathfrak{a}(I^{j-1}) \xrightarrow{d^{j-1}} \Gamma_\mathfrak{a}(I^j) \xrightarrow{d^j} \Gamma_\mathfrak{a}(I^{j+1}) \xrightarrow{d^{j+1}} \cdots, \end{aligned}$$

to short exact sequences:

$$(\sharp)^0 \qquad 0 \longrightarrow \Gamma_\mathfrak{a}(M) \longrightarrow \Gamma_\mathfrak{a}(I^0) \longrightarrow \operatorname{Im} d^0 \longrightarrow 0,$$
$$(\natural)^1 \qquad 0 \longrightarrow \operatorname{Im} d^0 \longrightarrow \ker d^1 \longrightarrow H^1_\mathfrak{a}(M) \longrightarrow 0,$$
$$(\sharp)^1 \qquad 0 \longrightarrow \ker d^1 \longrightarrow \Gamma_\mathfrak{a}(I^1) \longrightarrow \operatorname{Im} d^1 \longrightarrow 0,$$
$$\cdots$$
$$(\natural)^{j+1} \qquad 0 \longrightarrow \operatorname{Im} d^j \longrightarrow \ker d^{j+1} \longrightarrow H^{j+1}_\mathfrak{a}(M) \longrightarrow 0,$$
$$(\sharp)^{j+1} \qquad 0 \longrightarrow \ker d^{j+1} \longrightarrow \Gamma_\mathfrak{a}(I^{j+1}) \longrightarrow \operatorname{Im} d^{j+1} \longrightarrow 0,$$
$$\cdots.$$

(1) Now, we prove part (1) of the lemma, that is both $\ker d^j$ and $\operatorname{Im} d^j$ are injective for all $j \geq 0$. We proceed by induction on j.

For $j=0$, $\ker d^0 = \Gamma_{\mathfrak{a}}(M) = H^0_{\mathfrak{a}}(M)$ is injective by assumption. Now, we shall show that $\operatorname{Im} d^0$ is injective. Indeed, $\Gamma_{\mathfrak{a}}(M) = H^0(M)$ is assumed to be injective. Furthermore, $\Gamma_{\mathfrak{a}}(I^0)$ is injective. From part (3) of Lemma 7, it follows that $\operatorname{Im} d^0$ is injective by the short exact sequence $(\sharp)^0$, as required.

Suppose that $\ker d^j$ and $\operatorname{Im} d^j$ are both injective for some $j \geq 0$ as the inductive hypothesis. Furthermore, $H^s_{\mathfrak{a}}(M)$ is assumed to be injective for all $s \geq 0$. From part (1) of Lemma 7, it follows that $\ker d^{j+1}$ is injective by the short exact sequence $(\natural)^{j+1}$, as required.

We proved that $\ker d^{j+1}$ is injective. Furthermore, $\Gamma_{\mathfrak{a}}(I^{j+1})$ is injective, so it follows from part (3) of Lemma 7 that $\operatorname{Im} d^{j+1}$ is injective by the short exact sequence $(\sharp)^{j+1}$. The proof is completed.

(2) Next, we prove part (2) of the lemma, that is both $\operatorname{inj.dim}_A \ker d^j \leq l$ and $\operatorname{inj.dim}_A \operatorname{Im} d^j \leq l-1$ for all $j \geq 0$. We also proceed by induction on j.

For $j=0$, $\ker d^0 = \Gamma_{\mathfrak{a}}(M) = H^0_{\mathfrak{a}}(M)$ has injective dimension $\leq l$ by assumption. Now, we shall show that $\operatorname{Im} d^0$ has injective dimension $\leq l-1$. Indeed, from part (4) of Lemma 7, it follows that $\operatorname{Im} d^0$ has injective dimension $\leq l-1$ by the short exact sequence $(\sharp)^0$, as required.

Suppose that $\operatorname{Im} d^j$ has injective dimension $\leq l-1$ and $\ker d^j$ has injective dimension $\leq l$ for some $j \geq 0$ as the inductive hypothesis. Furthermore, $H^s_{\mathfrak{a}}(M)$ has injective dimension $\leq l$ for all $s \geq 0$ by assumption. From part (2) of Lemma 7, it follows that $\ker d^{j+1}$ has injective dimension $\leq l$ by the short exact sequence $(\natural)^{j+1}$, as required.

We proved that $\ker d^{j+1}$ has injective dimension $\leq l$, so it follows from part (4) of Lemma 7 that $\operatorname{Im} d^{j+1}$ has injective dimension $\leq l-1$ by the short exact sequence $(\sharp)^{j+1}$. The proof is completed. $\square$

REMARK 6. Under the same hypothesis as in Lemma 8, if $\operatorname{inj.dim}_A H^j_{\mathfrak{a}}(M) \leq 1$ for all $j \geq 0$, then $\operatorname{inj.dim}_A \ker d^j \leq 1$ for all $j \geq 0$ and $\operatorname{Im} d^j$ is injective for all $j \geq 0$.

LEMMA 9. *Let $(A, \mathfrak{n})$ be a local ring and T an (not necessarily finite) A-module. Let l be a positive integer.*

(1) *If $\operatorname{inj.dim}_{A_P} T_P \leq \dim T_P$ for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \neq \mathfrak{n}$ and $H^j_{\mathfrak{n}}(T)$ is injective for all $j \geq 0$, then T has finite injective dimension and $\operatorname{inj.dim}_A T \leq \dim T$.*
(2) *If $\operatorname{inj.dim}_{A_P} T_P \leq \dim T_P + l$ for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \neq \mathfrak{n}$ and $\operatorname{inj.dim}_A H^j_{\mathfrak{n}}(T) \leq l$ for all $j \geq 0$, then T has finite injective dimension and $\operatorname{inj.dim}_A T \leq \dim T + 2l - 1$.*

Proof. Let d be the dimension of T. Let J^* be a minimal injective resolution of T:

$$(\star) \qquad 0 \to T \to J^0 \to J^1 \to J^2 \to \cdots \to J^{d-1} \xrightarrow{\partial^{d-1}} J^d \xrightarrow{\partial^d} J^{d+1} \to \cdots .$$

(1) Now, we shall prove part (1) of the lemma. Suppose that $\operatorname{inj.dim}_{A_P} T_P \le \dim T_P < d$ for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \ne \mathfrak{n}$ so we have $J_P^s = 0$ for all $s \ge d$, since the injective resolution remains minimal after localization by [27, Proposition 3.5, p. 26]. Hence, J^s has only support in $V(\mathfrak{n})$ for all $s \ge d$. Applying the functor $\Gamma_{\mathfrak{n}}$ to the above exact sequence, we have a complex

$$0 \to \Gamma_{\mathfrak{n}}(T) \to \Gamma_{\mathfrak{n}}(J^0) \to \cdots \to \Gamma_{\mathfrak{n}}(J^{d-1}) \xrightarrow{\mathrm{d}^{d-1}} J^d \xrightarrow{\mathrm{d}^d} J^{d+1} \to \cdots,$$

where d^{d-1} and d^d are the restriction maps $\partial^{d-1}|_{\Gamma_{\mathfrak{n}}(J^{d-1})}$ and $\partial^d|_{\Gamma_{\mathfrak{n}}(J^d)}$, respectively, and $\mathrm{d}^d = \partial^d$. Note that all kernels and images of all the differentials are injective in the last complex by part (1) of Lemma 8. Especially, $\operatorname{Im}\mathrm{d}^{d-1}$ is an injective A-module. Furthermore, an A-module $H_{\mathfrak{n}}^j(T)$ is assumed to be injective for all $j \ge 0$. Now, we consider the short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im}\mathrm{d}^{d-1} & \longrightarrow & \ker \mathrm{d}^d & \longrightarrow & H_{\mathfrak{n}}^d(T) & \longrightarrow & 0 \\ & & & & \| & & & & \\ & & \operatorname{Im}\partial^{d-1} & = & \ker\partial^d, & & & & \end{array}$$

from which it follows that $\operatorname{Im}\partial^{d-1}$ is an injective A-module.

On the other hand, the following exact sequence is just a minimal injective resolution of $\operatorname{Im}\partial^{d-1}$:

$$0 \longrightarrow \operatorname{Im}\partial^{d-1} \longrightarrow J^d \longrightarrow J^{d+1} \longrightarrow \cdots.$$

By the minimality of the resolution J^*, we must have $J^{d+1} = 0$ and $\operatorname{Im}\partial^{d-1} = J^d$, for $\operatorname{Im}\partial^{d-1}$ is already an injective A-module. Hence, we conclude that the injective resolution of T is as follows:

$$0 \longrightarrow T \longrightarrow J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots \longrightarrow J^{d-1} \xrightarrow{\partial^{d-1}} J^d \longrightarrow 0.$$

Therefore, we have $\operatorname{inj.dim}_A T \le d$, as required.

(2) Next, we prove part (2) of the lemma. Suppose that $\operatorname{inj.dim}_{A_P} T_P \le \dim T_P + l < d + l$ for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \ne \mathfrak{n}$, so we have $J_P^s = 0$ for all $s \ge d + l$, that is J^s has only support in $V(\mathfrak{n})$ for all $s \ge d + l$. Applying the functor $\Gamma_{\mathfrak{n}}$ to the above exact sequence, we have a complex

$$\begin{aligned} 0 &\to \Gamma_{\mathfrak{n}}(T) \to \Gamma_{\mathfrak{n}}(J^0) \to \cdots \to \Gamma_{\mathfrak{n}}(J^d) \xrightarrow{\mathrm{d}^d} \Gamma_{\mathfrak{n}}(J^{d+1}) \xrightarrow{\mathrm{d}^{d+1}} \Gamma_{\mathfrak{n}}(J^{d+2}) \to \cdots \\ &\to \Gamma_{\mathfrak{n}}(J^{d+l-1}) \xrightarrow{\mathrm{d}^{d+l-1}} J^{d+l} \xrightarrow{\mathrm{d}^{d+l}} J^{d+l+1} \to \cdots, \end{aligned}$$

where d^{d+k} is the restriction map $\partial^{d+k}|_{\Gamma_{\mathfrak{n}}(J^{d+l})}$ for $k \in \mathbb{Z}$. Note that $\Gamma_{\mathfrak{n}}(J^s) = J^s$ and $\mathrm{d}^s = \partial^s$ for all $s \ge d + l$, since J^s has only support in $V(\mathfrak{n})$ for all $s \ge d + l$. Especially, $\Gamma_{\mathfrak{n}}(J^{d+l}) = J^{d+l}$ and $\mathrm{d}^{d+l} = \partial^{d+l}$. Furthermore, from part (2) of Lemma 8, all kernels of all the differentials have injective dimension $\le l$ and all images of all the differentials have injective dimension $\le l - 1$ in the last complex. Especially, $\operatorname{Im}\mathrm{d}^{d+l-1}$ has injective dimension $\le l - 1$.

Now, we consider the short exact sequence and equalities:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} \mathrm{d}^{d+l-1} & \longrightarrow & \ker \mathrm{d}^{d+l} & \longrightarrow & H_{\mathfrak{n}}^{d+l}(T) & \longrightarrow & 0 \\ & & & & \| & & & & \\ & & \operatorname{Im} \partial^{d+l-1} & = & \ker \partial^{d+l}, & & & & \end{array}$$

from which it follows that $\operatorname{Im} \partial^{d+l-1}$ has injective dimension $\leq l-1$, since $\dim T = d$. Here, we note that we have the Grothendieck vanishing $H_{\mathfrak{n}}^{d+l}(T) = 0$.

On the other hand, the following exact sequence is just a minimal injective resolution of $\operatorname{Im} \partial^{d+l-1}$:

$$0 \longrightarrow \operatorname{Im} \partial^{d+l-1} \longrightarrow J^{(d+l)+0} \longrightarrow J^{(d+l)+1} \longrightarrow \cdots \longrightarrow J^{(d+l)+l-1} \longrightarrow 0.$$

Hence, the injective resolution $(\star)$ of T is finite and is expressed as follows:

$$\begin{aligned} &0 \to T \to J^0 \to J^1 \to J^2 \to \cdots \to J^{d+l-1} \xrightarrow{\partial^{d+l-1}} J^{d+l} \xrightarrow{\partial^{d+l}} J^{d+l+1} \to \\ &\quad \to J^{d+2l-2} \to J^{d+2l-1} \to 0. \end{aligned}$$

Therefore, we have $\operatorname{inj.dim}_A T \leq d + 2l - 1$, as required. □

Remark 7. When we consider the case of $l = 1$ in Lemma 9, part (2) of the lemma states that, if $\operatorname{inj.dim}_A T_P \leq \dim T_P + 1$ for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \neq \mathfrak{n}$ and $\operatorname{inj.dim}_A H_{\mathfrak{n}}^j(T) \leq 1$ for all $j \geq 0$, then T has finite injective dimension and $\operatorname{inj.dim}_A T \leq \dim T + 1$.

The following proposition is probably well known, but we could not find a proof for it.

Proposition 10. *Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of local rings, which is module-finite and flat, and T an R-module. Then we have* $\dim_R T = \dim_A T \otimes_R A$.

Proof. First, we shall prove that $\dim T \leq \dim T \otimes_R A$. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_{d-1} \subset \mathfrak{p}_d = \mathfrak{m}$ be a chain of prime ideals in $\operatorname{Supp}_R(T)$. For each chain $\mathfrak{p}_i \subset \mathfrak{p}_{i+1}$ for $i = 0, \ldots, d-1$, we have $\mathfrak{p}_i A \subseteq \mathfrak{p}_{i+1} A$. Let P_{i+1} be a minimal ideal of $\mathfrak{p}_{i+1} A$. Then the radical of $\mathfrak{p}_i A$ is in P_{i+1}. Let $Q_1 \cap \cdots \cap Q_s$ be the shortest primary decomposition of the radical of $\mathfrak{p}_i A$. Note that there is no inclusion among prime ideals lying over $\mathfrak{p}_i$, since $\phi : R \to A$ is an integral extension (cf. [24, Theorem 9.3(ii), p. 66]). Since ϕ is flat, the going down theorem holds, so $Q_j \cap R = \mathfrak{p}_i$ for all $j = 1, 2, \ldots, s$. Further, there is a prime ideal Q lying over $\mathfrak{p}_i$ such that Q is properly included in P_{i+1}. We can now see that Q is one of $Q_1, \ldots, Q_s$. Indeed, the inclusion $Q_1 \cap \cdots \cap Q_s = \sqrt{\mathfrak{p}_i A} \subseteq Q$ implies that there is an integer j such that $0 \leq j \leq s$ and $Q_j \subseteq Q$ by [1, Proposition 1.11, p. 8]. From the equality $Q \cap R = \mathfrak{p}_i$, it follows that $Q_j = Q$ as we noted above.

So, there is a chain of $\operatorname{Spec}(A)$ consisting proper inclusions as follows:

$$P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_{d-1} \subset P_d = \mathfrak{n},$$

with $P_i \cap R = \mathfrak{p}_i$ for all i such that $0 \le i \le d$. Then this chain of prime ideals in $\mathrm{Spec}(A)$ is just that in $\mathrm{Supp}_A(T)$. Indeed, if $\mathfrak{p}_i \in \mathrm{Supp}_R(T)$, then we have $T_{\mathfrak{p}_i} \neq 0$. If $T_{\mathfrak{p}_i} \neq 0$, then we must have $T_{\mathfrak{p}_i} \otimes_{R_{\mathfrak{p}_i}} A_{P_i} \neq 0$, since the local ring homomorphism $\phi_{P_i} : R_{\mathfrak{p}_i} \longrightarrow A_{P_i}$ is faithfully flat by Remark 1. Therefore, we have $(T \otimes_R A)_{P_i} = T_{\mathfrak{p}_i} \otimes_{R_{\mathfrak{p}_i}} A_{P_i} \neq 0$. It follows that $P_i \in \mathrm{Supp}_A(T \otimes_R A)$ for all i such that $0 \le i \le d$, as required.

Next, we shall prove that $\dim T \ge \dim T \otimes_R A$. Let $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_{d-1} \subset P_d$ be a chain of $\mathrm{Supp}_A(T \otimes_R A)$. Set $\mathfrak{p}_i = P_i \cap R$. Then we have a chain of $\mathrm{Spec}(R)$:

$$(\sharp) \qquad \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_{d-1} \subseteq \mathfrak{p}_d.$$

We claim that these inclusions in the chain ($\sharp$) are proper. Indeed, suppose that the inclusion $P_{i-1} \subset P_i$ with $\mathfrak{p}_i = P_{i-1} \cap R = P_i \cap R$ $(= \mathfrak{p}_{i-1})$. The Lying Over theorem implies that if P_{i-1} and P_i lie over the same prime, then they are incomparable.

Furthermore, we can check that the chain ($\sharp$) in $\mathrm{Spec}(R)$ is that in $\mathrm{Supp}_R(T)$. Indeed, suppose that $T_{\mathfrak{p}_i} = 0$ for some i, so $T_{\mathfrak{p}_i} \otimes_{R_{\mathfrak{p}_i}} A_{P_i} = 0$. Therefore, we have $(T \otimes_R A)_{P_i} = 0$. It follows that $P_i \notin \mathrm{Supp}_A(T \otimes_R A)$, which is a contradiction.

Therefore, we have the equality $\dim T = \dim T \otimes_R A$. The proof is completed. □

Now, we prove Theorem 2.

Proof of Theorem 2. We only prove (i) and (ii). By the results in [19], the assertions (iii) and (iv) in Theorem 2 follow from those of (a) and (b) in Theorem 1.

(i) First, we note that $H^j_{\mathfrak{n}} H^i_I(A)$ has a finite injective dimension as an A-module by the regularity condition of A. Since the support of $H^j_{\mathfrak{n}} H^i_I(A)$ is in $V(\mathfrak{n})$, we only prove that

$$\mathrm{Ext}^p_A\big(A/\mathfrak{n}, H^j_{\mathfrak{n}} H^i_I(A)\big) = 0$$

for all $p > 1$.

By the results of Zhou [32], $H^j_{\mathfrak{m}} H^i_J(R)$ has injective dimension ≤ 1. Thus, we have

$$\mathrm{Ext}^p_R\big(R/\mathfrak{m}, H^j_{\mathfrak{m}} H^i_J(R)\big) = 0$$

for all $p > 1$. Since the map $\phi : R \to A$ is a module-finite ring homomorphism (hence an integral extension), the radical of $\mathfrak{m}A$ is equal to $\mathfrak{n}$. Then it follows from flatness of ϕ that $H^j_{\mathfrak{n}} H^i_I(A) = H^j_{\mathfrak{m}A} H^i_{JA}(A) = H^j_{\mathfrak{m}} H^i_J(R) \otimes_R A$. Furthermore, we have

$$\mathrm{Ext}^p_A\big(A/\mathfrak{m}A, H^j_{\mathfrak{m}A} H^i_{JA}(A)\big) = \mathrm{Ext}^p_R\big(R/\mathfrak{m}, H^j_{\mathfrak{m}} H^i_J(R)\big) \otimes A = 0$$

for all $p > 1$ since ϕ is flat. Since $A/\mathfrak{m}A$ is an A-module of finite length and $H^j_{\mathfrak{n}} H^i_I(A)$ has a finite injective dimension as an A-module, it follows from part (ii) of Lemma 6 that $H^j_{\mathfrak{m}A} H^i_{JA}(A)$ has injective dimension ≤ 1. Therefore, the injective dimension of $H^j_{\mathfrak{n}} H^i_I(A)$ is not greater than one.

(ii) We shall show the assertion (ii) by induction on $d = \dim H^i_I(A) \geq 0$. Note that $d = \dim H^i_J(R)$ by Proposition 10.

Suppose that $d = 0$. Then the support of $H^i_I(A)$ is contained in $V(\mathfrak{n})$, so the injective dimension of $H^i_I(A) = H^0_{\mathfrak{n}}(H^i_I(A))$ is one by part (i) of the theorem.

Suppose that $d > 0$. Let $P \in \operatorname{Supp}_A(H^i_I(A))$ be a prime ideal such that P is not the maximal ideal. Set $\mathfrak{p} = P \cap R$; $\mathfrak{p}$ is not the maximal ideal of R, since the extension $R \to A$ is integral. Then the ring homomorphism $R_{\mathfrak{p}} \to A_P$ is a module finite extension and flat between regular local rings by Remark 1. The condition $I = JA$ is preserved by localization (cf. Remark 2), that is $IA_P = (JR_{\mathfrak{p}})A_P$ for a prime ideal P of A. Also, the property of a ring being unramified is preserved by localization. The dimensions of $H^i_I(A)_P$ and $H^i_J(R)_{\mathfrak{p}}$ are less than d and Proposition 10 implies that we can apply the inductive hypothesis for the local cohomology module $H^i_I(A)_P$ over A_P, that is

$$\operatorname{inj.dim}_{A_P} H^i_I(A)_P \leq \dim H^i_I(A)_P + 1 \leq d - 1 + 1 = d.$$

Furthermore, $H^j_{\mathfrak{n}} H^i_I(A)$ has injective dimension ≤ 1 for all $j \geq 0$ by part (i) of the theorem.

So, the assertion follows from Remark 7, that is $\operatorname{inj.dim}_A H^i_I(A) \leq d + 1$. The proof is completed. □

Corollary 11. *Let $(A, \mathfrak{n})$ be a complete ramified regular local ring, of dimension d, and $x_1, x_2, \ldots, x_d$ a system of parameters, where $x_1 = p$ is the characteristic of the residue field $A/\mathfrak{m}$. Suppose that I is an ideal of A generated by polynomials over $\mathbb{Z}$ in $x_2, \ldots, x_d$. Then we have the following assertions for integers $i, j \geq 0$:*

(i) $\operatorname{inj.dim}_A H^j_{\mathfrak{n}}(H^i_I(A)) \leq 1$;
(ii) $\operatorname{inj.dim}_A H^i_I(A) \leq \dim H^i_I(A) + 1$;
(iii) *the set of associated prime ideals of $H^i_I(A)$ is finite;*
(iv) *all the Bass numbers of $H^i_I(A)$ are finite.*

Proof. Since A is complete, we have a module-finite local extension:

$$\psi : R \longrightarrow A,$$

where $\psi(X_i) = x_i$, for $i = 2, 3, \ldots, d$ and $R = W[[X_2, \ldots, X_d]]$ is a complete and unramified regular local ring over a complete p-ring W (cf. [24, Theorem 29.8, p. 229]).

Set $J = I \cap R$. For any integer $n \in \mathbb{Z}$, consider its base-p expansion:

$$n = \pm a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots + a_m p^m,$$

where $0 \leq a_i < p$ for all $i = 0, 1, \ldots, d$. Here, W is complete and the maximal ideal of W is generated by p. Taking it into consideration, we have the natural inclusions $J \subset \mathbb{Z}[X_2, X_3, \ldots, X_d] \subset W[[X_2, X_3, \ldots, X_d]] = R$. So, we have $J = I \cap R \subset W[[X_2, \ldots, X_d]]$ and $JA = I$.

Furthermore, the map ψ is flat since A is an Eisenstein extension of R. The assertion follows from Theorem 2. □

Now, we propose the following questions below.

QUESTION 1. Let i, j be nonnegative integers. Let $(A, \mathfrak{n})$ be a regular local ring, I an ideal of A. Is $H^j_{\mathfrak{n}} H^i_I(A)$ injective?

QUESTION 2. Let i, j be nonnegative integers. Let $(R, \mathfrak{m})$ be an unramified regular local ring, J an ideal of R. Is $H^j_{\mathfrak{m}} H^i_J(R)$ injective?

If the above questions were answered affirmatively, we could prove that the upper bound of the injective dimension of local cohomology modules is its dimension over an unramified (and also any) regular local ring. Question 2 is suggested in Lyubeznik's paper [19]. We proceed to prove this as follows, modifying that of (iv) of Theorem 2.

PROPOSITION 12. *Let i be a nonnegative integer.*

(1) *If Question 1 has an affirmative answer for all $j \geq 0$, then the inequality $\operatorname{inj.dim}_A H^i_I(A) \leq \dim H^i_I(A)$ holds over a regular local ring $(A, \mathfrak{n})$ for all ideals I of A.*
(2) *If Question 2 has an affirmative answer for all $j \geq 0$, then the inequality $\operatorname{inj.dim}_R H^i_J(R) \leq \dim H^i_J(R)$ holds over an unramified regular local ring $(R, \mathfrak{m})$ for all ideal J of R.*

Proof. We only prove part (2) of the proposition. If A is a regular local ring, then A_Q is regular for all $Q \in \operatorname{Spec}(A)$. So, one can prove the assertion (1), repeating the proof below for a regular local ring $(A, \mathfrak{n})$.

Now, we prove part (2) of the proposition. We proceed by induction on $d = \dim H^i_J(R) \geq 0$.

Suppose that $d = 0$. Then the support of $H^i_J(R)$ is contained in $V(\mathfrak{m})$ so $H^i_J(R) = H^0_{\mathfrak{m}}(H^i_J(R))$ is an injective R-module by assumption.

Suppose that $d > 0$. Note that (R_P, PR_P) is an unramified regular local ring by Remark 5 and $\dim H^i_J(R)_P < \dim H^i_J(R) = d$ for each prime ideal $P \in \operatorname{Supp}_R(H^i_J(R))$ such that P is not the maximal ideal $\mathfrak{m}$. So we can apply the inductive hypothesis:

$$\operatorname{inj.dim}_{R_P} H^i_J(R)_P \leq \dim H^i_J(R)_P \leq d - 1,$$

for each prime ideal $P \in \operatorname{Supp}_R(H^i_J(R))$ with $P \neq \mathfrak{m}$, and $H^j_{\mathfrak{m}} H^i_J(R)$ is assumed be injective for all $j \geq 0$. Hence, the assertion follows from part (1) of Lemma 9. The proof is completed. □

Although the following assertions hold not only over a regular local ring but also over other rings, we concentrate rings on regular local rings.

EXAMPLE 2. *Let i, j be nonnegative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If the dimension of I is zero, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.*

EXAMPLE 3. *Let i, j be nonnegative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If the dimension of I is one, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.*

Proof. Since the local Lichtenbaum–Hartshorne vanishing theorem holds, we have $H^q_I(R) = 0$ for $q \neq \dim R - 1$. Then the spectral sequence $H^p_{\mathfrak{m}} H^q_I(R) \Longrightarrow H^{p+q}_{\mathfrak{m}}(R)$ degenerates. So, we have $H^t_{\mathfrak{m}} H^{\dim R-1}_I(R) = H^{t+\dim R-1}_{\mathfrak{m}}(R)$ for all $t \geq 0$. The local cohomology module $H^l_{\mathfrak{m}}(R)$ is an injective hull of $R/\mathfrak{m}$ for $l = \dim R$ and zero otherwise. Therefore, the assertion holds. □

EXAMPLE 4. *Let i, j be nonnegative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If I is a principal ideal, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.*

Proof. We may assume that I is generated by a nonzero and nonunit element of R. Since I is a principal ideal, we have $H^q_I(R) = 0$ for $q \neq 1$. So the spectral sequence $H^p_{\mathfrak{m}} H^q_I(R) \Longrightarrow H^{p+q}_{\mathfrak{m}}(R)$ degenerates, and we have $H^t_{\mathfrak{m}} H^1_I(R) = H^{t+1}_{\mathfrak{m}}(R)$ for all $t \geq 0$. Since $H^l_{\mathfrak{m}}(R)$ is an injective hull of $R/\mathfrak{m}$ for $l = \dim R$ and zero otherwise, the assertion holds. □

PROPOSITION 13. *Let i, j be nonnegative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R and f is a nonzero and nonunit element of R. If I is a subideal of a principal ideal (f) up to radicals, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.*

Proof. It is nothing to prove when I is the zero ideal. We may assume that I is a radical ideal of R and nonzero. So, we have an inclusion $I \subseteq (f)$. By Grothendieck's spectral sequence, we have the spectral sequence

$$H^p_f H^q_I(R) \Longrightarrow H^{p+q}_f(R).$$

Then since $H^p_f H^q_I(R) = 0$ for all $p > 1$ and $H^l_f(R) = 0$ for all $l \neq 1$, we have an exact sequence and equalities:

$$\begin{gathered} 0 \longrightarrow H^1_f H^0_I(R) \longrightarrow H^1_f(R) \longrightarrow H^0_f H^1_I(R) \longrightarrow 0, \\ H^0_f H^l_I(R) = 0 \qquad \text{for } l \neq 1, \\ H^i_f H^j_I(R) = 0 \qquad \text{for } i, j \geq 1. \end{gathered} \tag{$*$}$$

On the other hand, I is nonzero and R is an integral domain, so we have $H^1_f H^0_I(R) = 0$, since $\operatorname{depth}_I R \geq 1$. It follows from the above short exact sequence that

$$H^1_f(R) = H^0_f H^1_I(R). \tag{$**$}$$

Consider the short Čech complex:

$$0 \longrightarrow H^l_I(R) \longrightarrow H^l_I(R)_f \longrightarrow 0,$$

so we have an exact sequence:

$$0 \longrightarrow \Gamma_f H^l_I(R) \longrightarrow H^l_I(R) \longrightarrow H^l_I(R)_f \longrightarrow H^1_f H^l_I(R) \longrightarrow 0.$$

If $l \neq 1$, then $H^l_I(R) = H^l_I(R)_f$, since $\Gamma_f H^l_I(R) = H^1_f H^l_I(R) = 0$ by the equality $(*)$. Hence, we have $H^j_{\mathfrak{m}} H^l_I(R) = H^j_{\mathfrak{m}} H^l_I(R)_f = 0$ for all $j \geq 0$ and $l \neq 1$, since f is contained in $\mathfrak{m}$.

If $l = 1$, then we have short exact sequence

$$0 \longrightarrow \Gamma_f H^1_I(R) \longrightarrow H^1_I(R) \longrightarrow H^1_I(R)_f \longrightarrow 0,$$

since $H^1_f H^1_I(R) = 0$. From the equality $(**)$, the above exact sequence yields

$$0 \longrightarrow H^1_f(R) \longrightarrow H^1_I(R) \longrightarrow H^1_I(R)_f \longrightarrow 0.$$

Since f is contained in $\mathfrak{m}$, we have $H^j_{\mathfrak{m}} H^1_I(R)_f = 0$ for all $j \geq 0$. The short exact sequence above gives a long exact sequence of local cohomology modules from which we deduce that we have

$$H^j_{\mathfrak{m}} H^1_f(R) \simeq H^j_{\mathfrak{m}} H^1_I(R),$$

applying the derived functor of $\Gamma_{\mathfrak{m}}(-)$ to the last short exact sequence.

By Grothendieck's spectral sequence again, we have the spectral sequence:

$$H^p_{\mathfrak{m}} H^q_f(R) \Longrightarrow H^{p+q}_{\mathfrak{m}}(R).$$

Then we have an equality: $H^{n-1}_{\mathfrak{m}} H^1_f(R) = H^n_{\mathfrak{m}}(R) = E_R(R/\mathfrak{m})$, where n is the dimension of R and $E_R(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. On the other hand, $H^j_{\mathfrak{m}} H^i_f(R) = 0$ for all $j \geq 0$ and $i \neq 1$. Therefore, it holds that $H^j_{\mathfrak{m}} H^i_I(R)$ are injective modules for all $i, j \geq 0$, as required. □

Corollary 14. *Let i, j be nonnegative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If the height of I is one, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.*

Proof. We may assume that I is a radical ideal.

Assume first that I is equi-dimensional. Now, the minimal prime ideals of I have height one. Since the regular local ring is a unique factorization domain (cf. [24, Theorem 48, p. 142]), I is a principal ideal (cf. [24, Theorem 47, p. 141]). It follows from Example 4 that $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.

Suppose now that I is not equi-dimensional. We write $I = I_1 \cap I_2$, where all the minimal prime ideals of I_1 have height one and all the minimal prime ideals of I_2 have height ≥ 2. Since the regular local ring is a unique factorization domain, I_1 is a principal ideal (f) for some element f of $\mathfrak{m}$. Hence, $I = I_1 \cap I_2$ is a subideal of (f). It follows from Proposition 13 that $H^j_{\mathfrak{m}} H^i_I(R)$ is injective. □

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