On some aspects of duality principle

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Abstract This paper is devoted to the study of the relation between Osserman algebraic curvature tensors and algebraic curvature tensors which satisfy the duality principle. We give a short overview of the duality principle in Osserman manifolds and extend this notion to null vectors. Here, it is proved that a Lorentzian totally Jacobi-dual curvature tensor is a real space form. Also, we find out that a Clifford curvature tensor is Jacobi-dual. We provide a few examples of Osserman manifolds which are totally Jacobi-dual and an example of an Osserman manifold which is not totally Jacobi-dual.

1. Introduction

Let (M,g) be a pseudo-Riemannian manifold of signature $(\nu, n-\nu)$ assigned with the Levi–Civita connection ∇. The curvature operator $\mathcal{R}$ of (M,g) is defined by the equation $\mathcal{R}(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$. For a point $p\in M$, on the tangent vector space $\mathcal{V}=T_pM$, the equation $R(X,Y,Z,W)=g(\mathcal{R}(X,Y)Z,W)$ defines an algebraic curvature tensor $R\in\bigotimes^4\mathcal{V}^*$, which satisfies the usual $\mathbb{Z}_2$-symmetries and the first Bianchi identity.

Since $\mathcal{V}$ is equipped with an indefinite metric g of the signature $(\nu, n-\nu)$ there are various types of vectors depending on the norm $\varepsilon_X = g(X,X)$. The vector $X\in\mathcal{V}$ can be timelike (if $\varepsilon_X<0$), spacelike ($\varepsilon_X>0$), or null ($\varepsilon_X=0$). We can say that $X\in\mathcal{V}$ is nonnull ($\varepsilon_X\neq 0$) or unit ($\varepsilon_X\in\{-1,1\}$).

The Jacobi operator $\mathcal{J}_X:\mathcal{V}\to\mathcal{V}$ is a natural operator associated to a curvature operator by $\mathcal{J}_X(Y)=\mathcal{R}(Y,X)(X)$. In the case of nonnull $X\in\mathcal{V}$, $\mathcal{J}_X$ preserves the nondegenerate hyperplane $\{X\}^\perp=\{Y\in\mathcal{V}: g(X,Y)=0\}$, and we have a reduced Jacobi operator $\tilde{\mathcal{J}}_X:\{X\}^\perp\to\{X\}^\perp$, given as the restriction of $\mathcal{J}_X$.

We say that a curvature tensor R is Osserman if the characteristic polynomial of $\mathcal{J}_X$ is constant on both pseudospheres $\mathcal{S}^\pm=\{X\in\mathcal{V}\mid \varepsilon_X=\pm 1\}$. In a pseudo-Riemannian setting, the Jordan normal form plays a crucial role, since the characteristic polynomial does not determine the eigenstructure of a symmetric linear operator. We say that R is a Jordan Osserman curvature tensor if

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the Jordan normal form of $\mathcal{J}_X$ is constant on $\mathcal{S}^{\pm}$. A curvature tensor is Jacobi-diagonalizable if its Jacobi operator $\mathcal{J}_X$ is diagonalizable for all nonnull X. In this article we put things in a purely algebraic setting, while relations with the global differential geometry can be found in [13].

Our paper is organized as follows. In Section 1 we give some basic notions and notation that we use throughout the paper. Section 2 is devoted to the motivation for our investigations, which has been building from studying Osserman-type problems in the last two decades. As a result of those investigations, in Section 3, we give the most general definition of the duality principle, and therefore we slightly modify terminology to make it more precise. Sections 4 and 5 consist of new results: in Section 4 we prove that only Lorentzian manifolds which satisfy the totally Jacobi-dual condition are real space forms, and in Section 5 we show that the Clifford curvature tensor is Jacobi-dual. Also, we find necessary conditions for when a Clifford curvature tensor is totally Jacobi-dual. In this chapter we give a few important examples of Clifford curvature tensors which are totally Jacobi-dual. Section 6 deals with a certain Walker $(2,2)$-manifold whose curvature tensor is Osserman, but it is not totally Jacobi-dual. At the end of this section we give a short conclusion to our investigations.

2. Motivation

In the Riemannian setting, it is known that a local 2-point homogeneous space has a constant characteristic polynomial on the unit sphere bundle. Osserman [19] wondered if the converse held, and this question has been called the Osserman conjecture. In the proofs of some particular cases of the conjecture, the implication that

$$\mathcal{J}_X(Y) = \lambda Y \Longrightarrow \mathcal{J}_Y(X) = \lambda X \tag{2.1}$$

appeared naturally, and it can significantly simplify some calculations. The first results in this topic were given by Chi [9], who proved the conjecture in all dimensions, except the cases of dimensions $n = 4k$ for $k > 1$. In his work he used the statement that (2.1) holds if λ is an extremal (minimum or maximum) eigenvalue of the Jacobi operator.

The second author [20] used implication (2.1) to formulate the duality principle for an Osserman curvature tensor (or Osserman manifolds), and he proved it in the Riemannian setting. Moreover, the best results in this topic were given by Nikolayevsky [15]–[17], who used the duality principle [16] to prove the Osserman conjecture in all dimensions, except some possibilities in dimension $n = 16$.

It is interesting to investigate the connection between the Osserman condition and the duality principle. The natural question is whether being Osserman and satisfying the duality principle are equivalent properties for an algebraic curvature tensor. Recently, affirmative answers to the above question in the Riemannian setting were obtained in the following cases: in dimension $n = 3$ (see [1]), in dimension $n = 4$ (see [8]), and later for any dimension (see [18]).

The generalization of the Osserman conjecture has appeared in a pseudo-Riemannian setting. For example, in the Lorentzian setting, an Osserman manifold necessarily has a constant sectional curvature (see [5]). The investigation of the Osserman curvature tensor of signature $(2,2)$ has become very popular, and it is worth noting results from [6], which are based on the discussion of possible Jordan normal forms of a Jacobi operator. Many authors have worked on this topic, and a lot information about it could be found, for example, in monographs by Gilkey [12], [13] and Garcia-Rio, Kupeli, and Vázquez-Lorenzo [11].

The previous facts provide us good motivation to examine the duality principle for Osserman manifolds in a pseudo-Riemannian setting and to examine the relation between the duality principle and the Osserman condition of an algebraic curvature tensor.

3. The duality principle extension

Since $g(\mathcal{J}_Y(X),X)=g(\mathcal{J}_X(Y),Y)$, the implication (2.1) in a pseudo-Riemannian setting is inaccurate when X and Y belong to different unit pseudospheres. This is why we corrected it with the following implication (see [4]):

$$\mathcal{J}_X(Y)=\varepsilon_X\lambda Y \Longrightarrow \mathcal{J}_Y(X)=\varepsilon_Y\lambda X. \tag{3.1}$$

If we deal with the converse problem, then it is important to examine an optimal extension for our (X,Y)-domain, starting with the original (see [20]) where X and Y are mutually orthogonal units. In the case in which R is Jacobi-diagonalizable our domain can be equivalently extended to all $X,Y\in\mathcal{V}$ with $\varepsilon_X\neq 0$ (see [4]). The diagonalizability of a Jacobi operator is a natural Riemannian-like condition (a Jacobi operator, as a self-adjoint operator on a definite vector space, is diagonalizable in the Riemannian setting); moreover, it is known that every Jordan Osserman curvature tensor of nonneutral signature $(n\neq 2\nu)$ is necessarily Jacobi-diagonalizable (see [14]).

DEFINITION 3.1

We say that an algebraic curvature tensor R is Jacobi-dual (or that it satisfies the duality principle) if (3.1) holds for all $\lambda\in\mathbb{R}$ and $X,Y\in\mathcal{V}$ with $\varepsilon_X\neq 0$.

The concept with no restrictions on X and Y can be reformulated with the following definition.

DEFINITION 3.2

We say that an algebraic curvature tensor R is totally Jacobi-dual if the equivalence

$$(Y \text{ belongs to an eigenspace of } \mathcal{J}_X) \Longleftrightarrow (X \text{ belongs to an eigenspace of } \mathcal{J}_Y)$$

holds for all $X,Y\in\mathcal{V}$.

This definition is a natural generalization of the notion of a Jacobi-dual algebraic curvature tensor. Due to the property $g(\mathcal{J}_Y(X),X)=g(\mathcal{J}_X(Y),Y)$ we excluded λ from the definition, and we allowed that the null vector X can be an eigenvector for $\mathcal{J}_Y$ with nonnull Y.

4. Lorentzian totally Jacobi-dual curvature tensors

Since in the Riemannian case it was shown that the duality principle implies the Osserman condition, the next natural step should be an investigation of the same converse problem in the Lorentzian setting. We know that a Lorentzian Osserman curvature tensor has a constant sectional curvature, so we need the following theorem.

THEOREM 4.1

A Lorentzian totally Jacobi-dual curvature tensor is a real space form.

Proof

Let $T\in\mathcal{V}$ be a unit timelike vector ($\varepsilon_T=-1$). In the Lorentzian setting, $\mathcal{V}$ has the signature $(1,n-1)$; hence, $T^{\perp}$ has the signature $(0,n-1)$, and therefore the restriction $\tilde{\mathcal{J}}_T=\mathcal{J}_T|_{T^\perp}$ is diagonalizable as a self-adjoint operator on a definite space. Let $S_1,\dots,S_{n-1}$ be orthonormal ($\varepsilon_{S_i}=-\varepsilon_T=1$) eigenvectors of $\tilde{\mathcal{J}}_T$. Then $\mathcal{J}_T(S_i)=\varepsilon_T\lambda_iS_i$ and Jacobi duality gives $\mathcal{J}_{S_i}(T)=\varepsilon_{S_i}\lambda_iT$ for all $1\le i\le n-1$.

Let us define subspaces $\mathcal{U}_i$ $(1\le i\le n-1)$ of $\mathcal{V}$ by

$$\mathcal{U}_i=\operatorname{Span}\{T,S_i\},\qquad \mathcal{U}_i^{\perp}=\operatorname{Span}\Big(\bigcup_{j\neq i}\{S_j\}\Big),$$

and we shall show that subspaces $\mathcal{U}_i$ and $\mathcal{U}_i^\perp$ are invariant for each operator $\mathcal{J}_X$ where $X\in\mathcal{U}_i$.

We need the following lemma, which is a consequence of straightforward calculations (see [1]), so we omit its proof.

LEMMA 4.1

If $\mathcal{J}_X(Y)=\varepsilon_X\lambda Y$ and $\mathcal{J}_Y(X)=\varepsilon_Y\lambda X$ hold with $X\perp Y$, then $\mathcal{J}_{\alpha X+\beta Y}(\varepsilon_Y\beta X-\varepsilon_X\alpha Y)=\varepsilon_{\alpha X+\beta Y}\lambda(\varepsilon_Y\beta X-\varepsilon_X\alpha Y)$ holds for all $\alpha,\beta\in\mathbb{R}$.

In the case of nonnull $\alpha T+\beta S_i$, vectors $\alpha T+\beta S_i$ and $\beta T+\alpha S_i$ create an orthogonal basis for $\mathcal{U}_i$. According to Lemma 4.1, $\mathcal{J}_{\alpha T+\beta S_i}(\beta T+\alpha S_i)=\varepsilon_{\alpha T+\beta S_i}\lambda_i(\beta T+\alpha S_i)$, and with addition, $\mathcal{J}_{\alpha T+\beta S_i}(\alpha T+\beta S_i)=0$, so we can conclude that $\mathcal{J}_X(\mathcal{U}_i)\subseteq\mathcal{U}_i$ holds for all nonnull $X\in\mathcal{U}_i$. Since the Jacobi operator is self-adjoint for all nonnull $X\in\mathcal{U}_i$ we have $g(\mathcal{J}_X(\mathcal{U}_i^\perp),\mathcal{U}_i)=g(\mathcal{U}_i^\perp,\mathcal{J}_X(\mathcal{U}_i))\subseteq g(\mathcal{U}_i^\perp,\mathcal{U}_i)=\{0\}$, and therefore $\mathcal{J}_X(\mathcal{U}_i^\perp)\subseteq\mathcal{U}_i^\perp$ holds. A direct calculation for $\mathcal{J}_{\alpha T+\beta S_i}$ gives

$$\mathcal{J}_{\alpha T+\beta S_i}=\alpha^2\mathcal{J}_T+\beta^2\mathcal{J}_{S_i}+\alpha\beta\mathcal{K}(T,S_i), \tag{4.1}$$

where $\mathcal{K}(T,S_i)(X)=\mathcal{R}(X,T)S_i+\mathcal{R}(X,S_i)T$. If we choose α and β such that $\alpha T+\beta S_i$ is nonnull ($\alpha^2\neq\beta^2$), then (4.1) implies that $\mathcal{K}(T,S_i)(\mathcal{U}_i)\subseteq\mathcal{U}_i$ and

$\mathcal{K}(T,S_i)(\mathcal{U}_i^\perp)\subseteq\mathcal{U}_i^\perp$. Let us get back to (4.1), and set α and β such that $\alpha T+\beta S_i$ is null ($\alpha^2=\beta^2$) to conclude that they have the same invariance. Thus, $\mathcal{J}_X(\mathcal{U}_i)\subseteq\mathcal{U}_i$ and $\mathcal{J}_X(\mathcal{U}_i^\perp)\subseteq\mathcal{U}_i^\perp$ hold for all $X\in\mathcal{U}_i$.

The space $\mathcal{U}_i^\perp$ has the signature $(0,n-2)$, so the restriction $\mathcal{J}_{T\pm S_i}|_{\mathcal{U}_i^\perp}$ is diagonalizable as a self-adjoint operator on a definite space. Since $\mathcal{J}_N M=\mu M$ imlies $\mathcal{J}_M N-\nu M$, we have $\mu\varepsilon_M=R(M,N,N,M)=\nu\varepsilon_N$, and therefore for null N and nonull M mus be $\mu=0$ and thus $\mathcal{J}_N M=0$. Since $T\pm S_i$ is null and $\mathcal{J}_{T\pm S_i}|_{\mathcal{U}_i^\perp}$ is diagonalizable we can conclude that $\mathcal{J}_{T\pm S_i}(\mathcal{U}_i^\perp)=\{0\}$ and therefore

$$\mathcal{J}_{T\pm S_i}(S_j)=0 \tag{4.2}$$

holds for all $1\le i\ne j\le n-1$.

Then the relation $\mathcal{J}_{T+S_i}(S_j)=0$ and total Jacobi-duality implies that $T+S_i$ is an eigenvector of $\mathcal{J}_{S_j}$. Since T is an eigenvector of $\mathcal{J}_{S_j}$ and $g(T,T+S_i)=-1\ne 0$, they have the same eigenvalue, and therefore S_i is an eigenvector with the same eigenvalue and R is a real space form.

Alternatively, we can express $\mathcal{J}_{T\pm S_i}=\mathcal{J}_T+\mathcal{J}_{S_i}\pm\mathcal{K}(T,S_i)$ and get $\mathcal{J}_{T+S_i}+\mathcal{J}_{T-S_i}=2(\mathcal{J}_T+\mathcal{J}_{S_i})$. From (4.2) we have that $\mathcal{J}_{T\pm S_i}(S_j)=0$ and therefore $\mathcal{J}_{S_i}(S_j)=-\mathcal{J}_T(S_j)=-\varepsilon_T\lambda_j S_j=\varepsilon_{S_i}\lambda_j S_j$. Finally

$$\mathcal{J}_{S_i}(S_j)=\varepsilon_{S_i}\lambda_j S_j$$

holds for all $1\le i\ne j\le n-1$. Comparing this equation after (i,j)-symmetry $\mathcal{J}_{S_j}(S_i)=\varepsilon_{S_j}\lambda_i S_i$ and after the Jacobi-dual property $\mathcal{J}_{S_j}(S_i)=\varepsilon_{S_j}\lambda_j S_i$, we easily conclude that $\lambda_i=\lambda_j$ for $1\le i\ne j\le n-1$, which proves that R is a real space form. □

5. Clifford curvature tensors and duality

Let us recall the very first example of an Osserman curvature tensor, a tensor of constant sectional curvature 1, which has the expression

$$\mathcal{R}^0(X,Y)Z=g(Y,Z)X-g(X,Z)Y.$$

Any skew-adjoint endomorphism J on $\mathcal{V}$ with $J^2=\pm\mathrm{Id}$ generates another basic example of an Osserman curvature operator via

$$\mathcal{R}^J(X,Y)Z=g(JX,Z)JY-g(JY,Z)JX+2g(JX,Y)JZ.$$

The Clifford family of rank k is a set $\{J_1,J_2,\dots,J_k\}$ of skew-adjoint endomorphisms on $\mathcal{V}$ with the properties

$$J_iJ_j+J_jJ_i=2\varepsilon_i\delta_{ij}\mathrm{Id}, \tag{5.1}$$

for $\varepsilon_i\in\{-1,1\}$ and $1\le i,j\le k$.

If a curvature operator $\mathcal{R}$ can be represented as a linear combination of such operators $\mathcal{R}^{J_i}$ (for J_i from the Clifford family, $1\le i\le k$) including $\mathcal{R}^0$, then we say that $\mathcal{R}$ (or assigned curvature tensor R) is Clifford (or has a Clifford structure). Any Clifford curvature tensor is Osserman, and according to Nikolayevsky

[15], [16], any Riemannian Osserman curvature tensor with dimension $n \neq 16$ is Clifford. Since Osserman and Clifford algebraic curvature tensors are closely related we shall investigate Jacobi-dual and totally Jacobi-dual properties for Clifford curvature tensors.

If a curvature operator $\mathcal{R}$ is Clifford, then it can be written as

$$\mathcal{R} = \alpha_0 \mathcal{R}^0 + \sum_{i=1}^{k} \alpha_i \mathcal{R}^{J_i},$$

with $\alpha_j \in \mathbb{R}$ for $0 \leq j \leq k$. Skew-adjoint endomorphisms J_i have the property that $g(J_i X, X) = 0$, which simplifies our calculation of the Jacobi operator

$$\mathcal{J}_X(Y) = \mathcal{R}(Y,X)X = \alpha_0 \big(g(X,X)Y - g(Y,X)X\big) + \sum_{i=1}^{k} 3\alpha_i g(J_i Y, X) J_i X,$$

and therefore

$$\mathcal{J}_X(Y) = \alpha_0 \big(\varepsilon_X Y - g(Y,X)X\big) - 3\sum_{i=1}^{k} \alpha_i g(Y, J_i X) J_i X. \tag{5.2}$$

Interchanging the roles of X and Y in the previous relation immediately gives

$$\mathcal{J}_Y(X) = \alpha_0 \big(\varepsilon_Y X - g(X,Y)Y\big) - 3\sum_{i=1}^{k} \alpha_i g(X, J_i Y) J_i Y. \tag{5.3}$$

Let us suppose that Y belongs to an eigenspace of $\mathcal{J}_X$. Thus $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y$, and from (5.2) we have that

$$\varepsilon_X(\lambda - \alpha_0)Y = -\alpha_0 g(Y,X)X - 3\sum_{i=1}^{k} \alpha_i g(Y, J_i X) J_i X. \tag{5.4}$$

The right-hand side of (5.4) belongs to $\mathrm{Span}\{X, J_1 X, \dots, J_k X\}$. Using (5.1) for $i \neq j$ we have $g(J_i X, J_j X) = 0$, and thus the set $\{X, J_1 X, \dots, J_k X\}$ is orthogonal.

Let us suppose that X is nonnull. Since

$$\varepsilon_{J_i X} = g(J_i X, J_i X) = -g(X, J_i J_i X) = -g(X, \varepsilon_i X) = -\varepsilon_i \varepsilon_X,$$

the vector $J_i X$ is nonnull. Moreover, for unit X we have unit $J_i X$, so the set $\{X, J_1 X, \dots, J_k X\}$ is orthonormal and, consequently, linearly independent.

Unless $\lambda - \alpha_0 = 0$, (5.4) allows us to express Y. In the case in which $\lambda = \alpha_0$, the left-hand side of (5.4) is equal to zero, so the linearly independent set $\{X, J_1 X, \dots, J_k X\}$ gives $\alpha_0 g(Y,X) = 0$ and $-\alpha_i g(J_i Y, X) = \alpha_i g(Y, J_i X) = 0$; thus, by (5.3) we have $\mathcal{J}_Y(X) = \varepsilon_Y \alpha_0 X = \varepsilon_Y \lambda X$, and therefore X belongs to an eigenspace of $\mathcal{J}_Y$.

Otherwise from (5.4) we have that

$$Y = \frac{-\alpha_0 g(Y,X)}{\varepsilon_X(\lambda - \alpha_0)} X - 3\sum_{i=1}^{k} \frac{\alpha_i g(Y, J_i X)}{\varepsilon_X(\lambda - \alpha_0)} J_i X, \tag{5.5}$$

and after a substitution in (5.3),

$$\mathcal{J}_Y(X) = \alpha_0\Big(\varepsilon_Y X - g(X,Y)\Big(\frac{-\alpha_0 g(Y,X)}{\varepsilon_X(\lambda-\alpha_0)}X - 3\sum_{j=1}^{k}\frac{\alpha_j g(Y,J_jX)}{\varepsilon_X(\lambda-\alpha_0)}J_jX\Big)\Big)$$

$$- 3\sum_{i=1}^{k}\alpha_i g(X,J_iY)J_i\Big(\frac{-\alpha_0 g(Y,X)}{\varepsilon_X(\lambda-\alpha_0)}X - 3\sum_{j=1}^{k}\frac{\alpha_j g(Y,J_jX)}{\varepsilon_X(\lambda-\alpha_0)}J_jX\Big).$$

It gives

$$\mathcal{J}_Y(X) = \alpha_0\Big(\varepsilon_Y + \frac{\alpha_0(g(X,Y))^2}{\varepsilon_X(\lambda-\alpha_0)}\Big)X$$
$$+ \frac{3\alpha_0 g(X,Y)}{\varepsilon_X(\lambda-\alpha_0)}\sum_{i=1}^{k}\alpha_i\big(g(Y,J_iX)+g(X,J_iY)\big)J_iX$$
$$+ \frac{9}{\varepsilon_X(\lambda-\alpha_0)}\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j g(X,J_iY)g(Y,J_jX)J_iJ_jX.$$

Because $g(X,J_iY) = -g(Y,J_iX)$, the middle term on the right-hand side is cancelled out. The last term contains $\sum_{i,j=1}^{k}\alpha_i\alpha_j g(Y,J_iX)g(Y,J_jX)J_iJ_jX$, so we split it into three sums ($i<j$, $i>j$, and $i=j$) and then use symmetries and (5.1) to get that

$$\sum_{i,j=1}^{k}\alpha_i\alpha_j g(Y,J_iX)g(Y,J_jX)J_iJ_jX$$
$$= \sum_{1\le i<j\le k}\alpha_i\alpha_j g(Y,J_iX)g(Y,J_jX)(J_iJ_j+J_jJ_i)X + \sum_{i=1}^{k}\alpha_i^2\big(g(Y,J_iX)\big)^2J_i^2X$$
$$= \sum_{i=1}^{k}\alpha_i^2\big(g(Y,J_iX)\big)^2\varepsilon_iX.$$

Finally, we have that

$$\mathcal{J}_Y(X) = \Big(\alpha_0\varepsilon_Y + \frac{\alpha_0^2(g(X,Y))^2}{\varepsilon_X(\lambda-\alpha_0)} - \frac{9}{\varepsilon_X(\lambda-\alpha_0)}\sum_{i=1}^{k}\alpha_i^2\big(g(Y,J_iX)\big)^2\varepsilon_i\Big)X$$

and therefore X belongs to an eigenspace of $\mathcal{J}_Y$, which proves the following theorem.

THEOREM 5.1
A Clifford curvature tensor is Jacobi-dual.

To examine if a Clifford curvature tensor is totally Jacobi-dual we check the case in which $\varepsilon_X = 0$ in duality equation (3.1). Everything works fine for $X=0$, so let us start with null $X \neq 0$. If we suppose that Y belongs to an eigenspace of

$\mathcal{J}_X$, by (5.4) with $\varepsilon_X = 0$ we have

$$\alpha_0 g(Y,X)X + 3\sum_{i=1}^{k} \alpha_i g(Y, J_iX)J_iX = 0. \tag{5.6}$$

If the set $\{X, J_1X, \ldots, J_kX\}$ is linearly independent, then we have $\alpha_0 g(Y,X) = 0$ and $\alpha_i g(Y, J_iX) = 0$; thus, $\alpha_0 g(X,Y) = 0$ and $\alpha_i g(X, J_iY) = 0$, so by (5.3) we get $\mathcal{J}_Y(X) = \varepsilon_Y \alpha_0 X$, just like in the case $\lambda = \alpha_0$. So, we proved the following proposition.

PROPOSITION 5.2

Let R be a Clifford algebraic curvature tensor. If the set $\{X, J_1X, \ldots, J_kX\}$ is linearly independent for any null vector $X \neq 0$, then R is totally Jacobi-dual.

REMARK 1

Let us notice that, for $0 \neq X$ a null vector, all vectors from $\operatorname{Span}\{X, J_1X, \ldots, J_kX\}$ are null because $\varepsilon_{J_iX} = -\varepsilon_i\varepsilon_X = 0$. Since $g(J_iX, X) = 0$ and $g(J_iX, J_jX) = 0$, for all $i, j = 1, \ldots, k$, $i \neq j$, because of (5.2) we have $\operatorname{Ker}\mathcal{J}_X \supseteq \operatorname{Span}\{X, J_1X, \ldots, J_kX\} \supseteq \operatorname{Im}\mathcal{J}_X$. Then it follows that $\mathcal{J}_X^2 = 0$, that is, $\mathcal{J}_X$ is two-step nilpotent.

EXAMPLE 5.1 (REAL SPACE FORM)

The curvature operator of a pseudo-Riemannian manifold of constant sectional curvature c (real space form) is given by $\mathcal{R} = c\mathcal{R}^0$. By Theorem 5.1 it is Jacobi-dual; moreover, $\{X\}$ is linearly independent for any nonzero null X, so R is totally Jacobi-dual.

EXAMPLE 5.2 (COMPLEX SPACE FORM)

The curvature operator of a Kähler manifold of constant holomorphic sectional curvature c (complex space form) is given by $\mathcal{R} = (c/4)\mathcal{R}^0 - (c/4)\mathcal{R}^J$, where J is a skew-adjoint endomorphism with $J^2 = -\mathrm{Id}$. Since 1 and -1 are not square roots of -1, they are not eigenvalues of J; thus X and JX are linearly independent, and therefore R is totally Jacobi-dual.

EXAMPLE 5.3 (PARACOMPLEX SPACE FORM)

The curvature operator of a para-Kähler manifold of constant paraholomorphic sectional curvature c (paracomplex space form) is given by $\mathcal{R} = (c/4)\mathcal{R}^0 + (c/4)\mathcal{R}^J$ (see [10]), where J is a skew-adjoint endomorphism with $J^2 = \mathrm{Id}$. It is possible here to have linearly dependent X and JX. In this case $JX = \theta X$, and therefore our equation (5.6) $(c/4)g(Y,X)X + 3(c/4)g(Y, JX)JX = 0$ becomes $(1 + 3\theta^2)(c/4)g(Y,X)X = 0$. Because of $X = J^2X = \theta^2 X$ we have $\theta \in \{-1, 1\}$; thus $1 + 3\theta^2 = 4 \neq 0$, and therefore $g(Y,X) = 0$. Hence $g(Y, JX) = \theta g(Y,X) = 0$, so coefficients from (5.6) are zero, and finally (5.3) implies that R is totally Jacobi-dual.

EXAMPLE 5.4 (QUATERNIONIC SPACE FORM)
The curvature operator of a quaternionic Kähler manifold of constant quaternionic sectional curvature c (quaternionic space form) is given by $\mathcal{R} = (c/4)\mathcal{R}^0 - (c/4)\sum_{i=1}^{3} \mathcal{R}^{J_i}$, where $\{J_1, J_2, J_3\}$ is a canonical local basis, which means skew-adjoint endomorphisms J_i with $J_i^2 = -\mathrm{Id}$ for all $1 \le i \le 3$, where $J_1 J_2 = J_3$ holds.

Let us start with $\beta_0 X + \beta_1 J_1 X + \beta_2 J_2 X + \beta_3 J_3 X = 0$ to check the linear independence of a set $\{X, J_1X, J_2X, J_3X\}$ for nonzero null X. We can act with J_1, J_2, and J_3 on our equation to get the following matrix equation:

$$\begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ -\beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ -\beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{pmatrix} \begin{pmatrix} X \\ J_1X \\ J_2X \\ J_3X \end{pmatrix} = 0.$$

A computation of the matrix determinant gives $\Delta = (\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2)^2$. It is impossible to have $\beta_i \neq 0$ for some i, because in that case $\Delta > 0$ and our homogeneous matrix equation has a unique solution with $X = 0$. Hence $\beta_i = 0$ for all $0 \le i \le 3$, and therefore $\{X, J_1X, J_2X, J_3X\}$ is linearly independent for all nonzero X. Consequently, R is totally Jacobi-dual.

6. A counterexample and conclusion

Let us consider the following example of a pseudo-Riemannian manifold $(\mathbb{R}^4, g)$ with metric given in [11]:

$$\begin{aligned} g = x_2x_3\, dx_1 \otimes dx_1 - x_1x_4\, dx_2 \otimes dx_2 + dx_1 \otimes dx_2 + dx_2 \otimes dx_1 \\ + dx_1 \otimes dx_3 + dx_3 \otimes dx_1 + dx_2 \otimes dx_4 + dx_4 \otimes dx_2. \end{aligned}$$

Metrics of this type are well known as Walker metrics (for more details on Walker metrics, see [7]). Since the characteristic and minimal polynomials of $\mathcal{J}_X$ (for a unit vector X) are λ^4 and λ^3, it is a globally Jordan Osserman manifold. A straightforward calculation for this manifold gives

$$\mathcal{J}_{\frac{\partial}{\partial x_3}}\Big(\frac{\partial}{\partial x_1}\Big) = 0 \qquad \text{and} \qquad \mathcal{J}_{\frac{\partial}{\partial x_1}}\Big(\frac{\partial}{\partial x_3}\Big) = -\frac{1}{2}\frac{\partial}{\partial x_4}.$$

If we substitute $X = \frac{\partial}{\partial x_3}$ and $Y = \frac{\partial}{\partial x_1}$, then we can see that Y is an eigenvector of $\mathcal{J}_X$ (for the eigenvalue 0), but the null vector X is not an eigenvector of $\mathcal{J}_Y$. Consequently, our pseudo-Riemannian manifold $(\mathbb{R}^4, g)$ is Osserman but is not totally Jacobi-dual.

Conclusions

In this paper we study the relation between Osserman algebraic curvature tensors and Jacobi-dual algebraic curvature tensors. Every known example of an Osserman curvature tensor is Jacobi-dual; however, we failed to prove it in general. In our previous work, we gave affirmative answers only for the conditions of small index ($\nu \le 1$), low dimension ($n \le 4$), or some specific examples with small numbers of eigenvalues of a reduced Jacobi operator.

The Riemannian case works after the original proof (see [20]) and our extensions in [4] and [2]. The Lorentzian Osserman curvature tensor has a constant sectional curvature (see [5]), so it is totally Jacobi-dual (see Example 5.1). In the dimension $n = 4$ the problem is solved in [2], and some results have been given for the case when a reduced Jacobi operator has exactly two eigenvalues (see [3]). Let us recount our main result from [4]: a Jacobi-diagonalizable Osserman curvature tensor such that $\mathcal{J}_X$ has no null eigenvectors for all nonnull X is Jacobi-dual.

According to our counterexample (Osserman but not totally Jacobi-dual), the main converse question should be whether being Jacobi-dual necessarily implies being Osserman. In [1] this is proved for dimension $n = 3$ (any signature). In Riemannian settings this equivalence is proved in dimension 4 in [8] and is given an affirmative answer for any dimension in [18]. Also, the authors announced the extension to any Jacobi-diagonalizable case.

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