Duality theorem for topological semigroups

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Abstract For topological semigroups S, we consider *Tannaka-type duality theorems*, which are extensions of the notion of *weak Tannaka duality theorem* for topological groups. In the case of topological semigroups, we must set as the *dual object* of S all isometric representations of S instead of all unitary representations. We define a property *T-type* for S. After arguments analogous to previous work from the author, we can prove that our *Tannaka-type duality theorem* is valid if and only if S is a T-type semigroup.

1. Tannaka-type duality theorem

A *topological semigroup* S is a semigroup with unit e which is simultaneously a topological space and whose semigroup operation is continuous. An *isometric representation* of S is a continuous homomorphism from $g \in S$ to the semigroup $\{T_g\}$ of isometric operators on a Hilbert space $\mathcal{H}$ with weak topology. In this paper, hereafter we call an isometric representation simply a representation.

For an isometric operator J on $\mathcal{H}$ and $\forall c \in \mathbf{C}$, $\forall v, u$, $v \perp u \in \mathcal{H}$,

$$\begin{aligned}\|v\|^2 + |c|^2\|u\|^2 = \|v + cu\|^2 &= \big\|J(v + cu)\big\|^2 \\ &= \|Jv\|^2 + |c|^2\|Ju\|^2 + 2\Re\big(\overline{c}\langle Jv, Ju\rangle\big) \\ &= \|v\|^2 + |c|^2\|u\|^2 + 2\Re\big(\overline{c}\langle Jv, Ju\rangle\big),\end{aligned}$$

where $\Re$ shows the real part. This implies that $(Jv, Ju) = 0$ and $Jv \perp Ju$. Hence, for an orthonormal system $\{v_\alpha\}$, $\{Jv_\alpha\}$ gives an orthonormal system too.

Let $\mathcal{H}^1$ and $\mathcal{H}^2$ be Hilbert spaces, and take complete orthonormal systems $\{v^1_\alpha\}$ in $\mathcal{H}^1$ and $\{v^2_\alpha\}$ in $\mathcal{H}^2$; $\{v^1_\alpha \otimes v^2_\beta\}$ is an orthonormal system in $\mathcal{H}^1 \otimes \mathcal{H}^2$. Therefore we can define the tensor product $J^1 \otimes J^2$ for any isometric operators J^k on $\mathcal{H}^k$ $(k = 1, 2)$ as an isometric operator in $\mathcal{H}^1 \otimes \mathcal{H}^2$.

Let $\Omega \equiv \{D = (\mathcal{H}^D, T^D_g)\}$ be the set of all representations of a given topological semigroup S whose dimensions are bounded by $\max(\aleph_0, \#S)$, and consider operations between elements of Ω as

(1) unitary equivalence: $D_1 \sim_W D_2$ (W: intertwining unitary operator),
(2) subrepresentation: $D_1 \succ D_2$,

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(3) tensor product: $D_1 \otimes D_2$,
(4) contragradient representation: $D \to \overline{D}$.†

Consider an operator field $\mathbf{A} \equiv \{A^D\}_{D\in\Omega}$ on Ω satisfying

(Cd-0) for each $D \in \Omega$, A^D is an isometric operator on the representation space $\mathcal{H}^D$,
(Cd-1) $D_1 \sim_W D_2 \Rightarrow W A^{D_1} W^{-1} = A^{D_2}$,
(Cd-2) $D_1 \succ D_2 \Rightarrow A^{D_1}|_{\mathcal{H}^{D_2}} = A^{D_2}$,
(Cd-3) $A^{D_1} \otimes A^{D_2} = A^{D_1 \otimes D_2}$,
(Cd-4) $\overline{A^D} = A^{\overline{D}}$.

We call such an operator field $\mathbf{A}$ a *birepresentation* of S, and we write $\mathcal{J}$ for the set of all birepresentations. On the space $\mathcal{J}$, induce a topology, the product of weak topologies τ^D $(D \in \Omega)$ on each component operator space on the Hilbert space $\mathcal{H}^D$. It is easy to see that, for any two birepresentations $\mathbf{A}_j \equiv \{A_j^D\}$ $(j = 1,2)$, their product $\mathbf{A}_1\mathbf{A}_2 \equiv \{A_1^D A_2^D\}$ is also a birepresentation. This product operation is continuous with respect to the above topology. So $\mathcal{J}$ is a topological semigroup.

Obviously for any $g \in S$ the operator field $\mathbf{T}_g \equiv \{T_g^D\}_{D\in\Omega}$ gives a birepresentation. Our weak Tannaka-type duality theorems assert the converses.

ASSERTION
For any birepresentation $\mathbf{A} \equiv \{A^D\}_{D\in\Omega}$, there exists a unique $g \in S$ such that $A^D = T_g^D$ $(\forall D \in \Omega)$. Moreover, the topology on $\mathcal{J}$ given above coincides with the original topology of S under the correspondence $g \mapsto \mathbf{T}_g$.

2. Separating system of isometric representations, completeness, and T-type semigroups

Hereafter, S is a Hausdorff (i.e., T_2-)topological semigroup.

DEFINITION 2.1
A set $\Omega_0 \equiv \{D_\alpha \equiv \{\mathcal{H}^{D_\alpha}, T_g^{D_\alpha}, v^{D_\alpha}\} \mid v^{D_\alpha}$ is a normalized cyclic vector, $\alpha \in A\}$ of cyclic isometric representations of S gives a *separating system of isometric representations (SSIR)* if and only if, for any neighborhood V of any element g_0 in S, there exist $D \in \Omega_0$ and $\varepsilon > 0$ such that

$$F(D,\varepsilon,g_0) \equiv \big\{g \in S \;\big|\; \big|1 - \langle T_g^D v^D, T_{g_0}^D v^D\rangle\big| < \varepsilon\big\} \subset V. \tag{2.1}$$

We denote by $\mathbf{J}(\mathcal{H})$ the space of all isometric operators on a Hilbert space $\mathcal{H}$ and introduce the weak topology on it.

For any $J_0, J \in \mathbf{J}(\mathcal{H})$,

$$\|Jv - J_0v\|^2 = -2\Re\big(\langle Jv - J_0v, J_0v\rangle\big). \tag{2.2}$$

† For definitions and some properties of contragradient representations, we refer the reader to [4, Section 1].

So, on $\mathbf{J}(\mathcal{H})$ the weak topology coincides with the strong topology. Moreover, $\mathbf{J}(\mathcal{H})$ becomes a topological semigroup with the multiplication of operators and this topology.

Let $D \equiv \{\mathcal{H}^D, T_g^D\}$ be any representation of S. The map $S \ni g \mapsto T_g^D \in \mathbf{J}(\mathcal{H}^D)$ is continuous for each D, by definition.

Construct $\mathbf{J}(\Omega) \equiv \prod_{D\in\Omega} \mathbf{J}(\mathcal{H}^D)$ with the natural product topology. The map

$$S \ni g \mapsto (T_g^D)_{D\in\Omega} \in \mathcal{J} \subset \prod_{D\in\Omega} \mathbf{J}(\mathcal{H}^D) = \mathbf{J}(\Omega) \tag{2.3}$$

is into-homomorphisms as topological semigroups.

Write S_J as the image of S in $\mathbf{J}(\Omega)$. The existence of an SSIR for a T_2-topological semigroup S shows that

(a) the map (2.3) is a one-to-one map from S to S_J,

(b) the inverse map of (2.3) from S_J (with restricted topology from $\mathbf{J}(\Omega)$) to S is continuous.

So S is embedded as a topological semigroup in $\mathbf{J}(\Omega)$. The following lemma is then obvious.

LEMMA 2.1

Let S be a T_2-topological semigroup with an SSIR. Our weak Tannaka-type duality theorem is equivalent to $S_J = \mathcal{J}$ and the map (2.3) being an isomorphism between S and its image $S_J = \mathcal{J}$ as topological semigroups.

On a T_2-topological semigroup S with an SSIR Ω_0, put

$$\mathcal{W} \equiv \left\{ W(D,\varepsilon) \equiv \left\{ (g_1,g_2) \in S\times S \;\middle|\; \|T_{g_1}^D v^D - T_{g_2}^D v^D\| < \varepsilon \right\} \ (D\in\Omega_0, \varepsilon>0) \right\}.$$

It is easy to see that $\mathcal{W}$ gives a fundamental system of *entourages* on $S\times S$, and defines a uniform structure on S (see [1]).

DEFINITION 2.2

A filter base $\mathcal{F} \equiv \{F_\alpha\}_{\alpha\in\Gamma}$ (where Γ is a partially ordered set) on S is called *Cauchy* if, for any entourage $W(D,\varepsilon)$, there exists an $\alpha\in\Gamma$ such that

$$\forall \beta \succ \alpha, \qquad \forall g_1, g_2 \in F_\beta, \qquad (g_1,g_2) \in W(D,\varepsilon).$$

We consider the topological semigroup $\mathbf{S} \equiv \mathbf{J}(\Omega) = \prod_{D\in\Omega} \mathbf{J}(\mathcal{H}^D)$. The identical representation of $\mathbf{J}(\mathcal{H}^D)$ is cyclic, so $\mathbf{S}$ has an SSIR. Let $\mathcal{F} \equiv \{F_\alpha\}$ be a Cauchy filter base on $\mathbf{S}$. The projection image $\mathcal{F}^D \equiv \{F_\alpha^D \equiv \mathrm{Proj}_{\mathcal{H}^D} F_\alpha\}$ for any $D\in\Omega$ gives a Cauchy filter base on $\mathbf{J}(\mathcal{H}^D)$. Conversely, for a filter base $\mathcal{F} \equiv \{F_\alpha\}_{\alpha\in\Gamma}$ on $\mathbf{J}(\Omega)$ to be Cauchy, it is enough that, for any D in Ω, $\mathcal{F}^D$ is Cauchy. Since on $\mathbf{J}(\Omega)$ the weak topology is equivalent to the strong topology, we can consider these Cauchy properties in the sense of strong topology on $\mathbf{J}(\Omega)$.

For any $v \in \mathcal{H}^D$ for a fixed D, a Cauchy filter base $\{F_\alpha^D v\}_{\alpha\in\Gamma}$ converges to a vector $u(v)$ in the Hilbert space $\mathcal{H}^D$; that is, for any $J_\alpha^D \in F_\alpha^D$ and any $v \in \mathcal{H}^D$,

$$\text{strong-}\lim_\alpha J_\alpha^D v = u(v).$$

Moreover, for any $a, b \in \mathbf{C}$,

$$\lim_\alpha J_\alpha^D(av_1 + bv_2) = au(v_1) + bu(v_2), \qquad \|u(v)\| = \lim_\alpha \|J_\alpha^D v\| = \|v\|. \tag{2.4}$$

Therefore, the map $\mathcal{H}^D \ni v \mapsto u(v) \in \mathcal{H}^D$ is linear and isometric. Thus there exists an isometric operator B^D such that $u(v) = B^D v$.

LEMMA 2.2

Any Cauchy filter base on $\mathbf{J}(\Omega) = \prod_{D\in\Omega} \mathbf{J}(\mathcal{H}^D)$ *converges to a* $\mathbf{B} \equiv (B^D)_{D\in\Omega} \in \mathbf{J}(\Omega)$, *where the* B^D*'s are isometric operators.*

For a topological semigroup S, any filter base $\mathcal{F}$ on it is mapped to a filter base $\mathcal{F}_J$ in S_J. And if $\mathcal{F}$ is Cauchy, then $\mathcal{F}_J$ in $\mathbf{J}(\Omega)$ is also Cauchy.

LEMMA 2.3

A Cauchy filter base $\mathcal{F}_J$ *on a semigroup* S_J *converges to an element* $\mathbf{B} \equiv (B^D)_{D\in\Omega}$ *in* $\mathbf{J}(\Omega)$.

DEFINITION 2.3

We say that a T_2-topological semigroup S is of *T-type* if

(T-1) S has an SSIR,
(T-2) S is complete.

3. Birepresentations of S

A birepresentation $\mathbf{A} \equiv \{A^D\}_{D\in\Omega}$ of S with an SSIR has analogous properties to a birepresentation in the case of groups. We argue similarly as in [4, Section 6] and [5, Section 2].

Consider the contragradient representation $\overline{D}$ of D, consider the vector $\overline{v}$ in $\mathcal{H}^{\overline{D}}$ corresponding to v in $\mathcal{H}^D$, and consider $\overline{A}$ the operator on $\mathcal{H}^{\overline{D}}$ corresponding to A on $\mathcal{H}^D$. By the condition (Cd-4) of the definition of birepresentation,

$$\overline{A^D} = A^{\overline{D}}. \tag{3.1}$$

From this, by the same calculations as in [5, Lemma 2.1 and Corollary 2.1.1], we can obtain the following.

(1) $\langle A^{D\oplus\overline{D}}(u \oplus \overline{u}), v \oplus \overline{v}\rangle$ is real valued.

(2) Denote by $I \equiv \{\mathbf{C}, I_g, v_0\}$ the trivial representation of S, and put $D_p \equiv I \oplus D \oplus \overline{D}$. Take vectors $w_0 \in \mathcal{H}^I$ and $w \in \mathcal{H}^D$ such that $2^{1/2}\|w_0\| = 2\|w\| = 1$, and put $v_p \equiv w_0 \oplus w \oplus \overline{w}$ in $\mathcal{H}^I \oplus \mathcal{H}^D \oplus \mathcal{H}^{\overline{D}}$. Then for any $g \in S$ the matrix

element

$$\langle T_g^{D_p} v_p, A^{D_p} v_p\rangle = \big\langle T_g^{D_p}(w_0 \oplus w \oplus \overline{w}), A^{D_p}(w_0 \oplus w \oplus \overline{w})\big\rangle \geq 0. \tag{3.2}$$

By an argument analogous to [4, Corollary 1.2.2], we get the following.

LEMMA 3.1

Let D and D_p be as above. Then $1 > \forall \varepsilon > 0$, $\forall g_0 \in S$, $\exists \delta > 0$,

$$F(D_p, \delta, g_0) \subset F(D, \varepsilon, g_0). \tag{3.3}$$

Proof

Put $\eta(g) \equiv \langle T_g^D v^D, T_{g_0}^D v^D\rangle$, and put $\eta_p(g) \equiv \langle T_g^{D_p} v^{D_p}, T_{g_0}^{D_p} v^{D_p}\rangle$. Then $1 - \eta_p(g) = 2^{-1}(1 - \Re\eta(g))$ and

$$\begin{aligned}\|1 - \eta(g)\|^2 &= \big(1 - \Re\eta(g)\big)^2 + \big(\Im\eta(g)\big)^2\\ &\leq \big(1 - \Re\eta(g)\big)^2 + \big(1 - \Re\eta(g)\big)\big(1 + \Re\eta(g)\big)\\ &\leq 3\big(1 - \Re\eta(g)\big) = 6\big(1 - \eta_p(g)\big).\end{aligned}$$

This shows that if $6\delta < \varepsilon^2$, then $F(D^p, \delta, g_0) \subset F(D, \varepsilon, g_0)$. □

We get also the following result.

COROLLARY 3.1.1

If $\{D\}$ gives an SSIR of S, then $\{D_p\}$ is also an SSIR of S.

Take $D = \{\mathcal{H}^D, T_g^D, v^D\}$ a cyclic representation of S, and put $K^D(g) \equiv \langle T_g^D v^D, A^D v^D\rangle$.

LEMMA 3.2

Let $\mathbf{A} \equiv \{A^D\}_{D\in\Omega}$ be a birepresentation of S with an SSIR. Then for any cyclic $D \equiv \{\mathcal{H}^D, T_g^D, v^D\}$ ($\|v^D\| = 1$) in Ω,

$$\sup_{g\in S}\big|K^D(g)\big| = 1. \tag{3.4}$$

Proof

The arguments are similar to the proof of [4, Lemma 2.2]. At first, obviously $|K^D(g)| \leq 1$. The relations for $\zeta^D(g) = \langle T_g^D v^D, u^D\rangle$,

$$\overline{\zeta^D(g)} = \zeta^{\overline{D}}(g), \tag{3.5}$$

$$\zeta^{D_1}(g) + \zeta^{D_2}(g) = \zeta^{D_1\oplus D_2}(g), \tag{3.6}$$

$$\zeta^{D_1}(g) \times \zeta^{D_2}(g) = \zeta^{D_1\otimes D_2}(g), \tag{3.7}$$

show that, when D runs over Ω and u, v run over any vectors in $\mathcal{H}^D$, the family $\mathfrak{F} \equiv \{\zeta^D(g)\}$ of matrix elements gives a $*$-algebra contained in the $*$-algebra $\mathcal{C}^b(S)$ of all bounded continuous functions on S with the norm $\|\zeta^D\| \equiv \sup_{g\in S}|\zeta^D(g)|$.

The completion $\mathfrak{F}^C$ of $\mathfrak{F}$ with respect to this norm is a C^*-algebra of continuous functions on S.

By Gelfand's representation theorem, $\mathfrak{F}^C$ is isomorphic to the space $\mathcal{C}^b(X)$ of all bounded continuous functions on a locally compact space X under the correspondence $\mathfrak{F}^C \ni f \mapsto f^\sim \in \mathcal{C}^b(X)$. A point x of X is a homomorphic map such that

$$\psi^x : \mathcal{C}^b(X) \to \mathbf{C}, \tag{3.8}$$

$$\psi^x(\varphi) \equiv \varphi(x) \quad \bigl(\varphi \in \mathcal{C}^b(X)\bigr). \tag{3.9}$$

For any element g in S and f in $\mathfrak{F}^C$,

$$f \mapsto f(g) \tag{3.10}$$

gives a homomorphic map from $\mathfrak{F}^C$ to $\mathbf{C}$. So there exists a unique element x_g in X as

$$f(g) = f^\sim(x_g).$$

The existence of an SSIR ensures that the map $g \mapsto x_g$ is one-to-one. So by this map, S is embedded into X. But $\mathcal{C}^b(X)$ is given as the space of $\{f^\sim \mid f \in \mathfrak{F}^C\}$ and $\mathfrak{F}^C \subset \mathcal{C}^b(S)$. This implies that the image of S is dense in X. So for any $x \in X$, $\delta > 0$, and $f^\sim \in \mathfrak{F}^C$, there exists $g_0 \in S$ such that

$$\bigl|f^\sim(g_0) - f^\sim(x)\bigr| < \delta. \tag{3.11}$$

For a given birepresentation $\mathbf{A} \equiv \{A^D\}_{D\in\Omega}$, consider the map

$$\zeta^D(g) = \langle T_g^D v^D, u^D\rangle \mapsto \langle A^D v^D, u^D\rangle \equiv \theta_{\mathbf{A}}(\zeta^D). \tag{3.12}$$

By considerations analogous to those in (3.5), (3.6), and (3.7), we get that

$$\begin{aligned}
\overline{\theta_{\mathbf{A}}(\zeta^D)} &= \overline{\langle A^D v^D, u^D\rangle} = \langle \overline{A^D} v^{\overline{D}}, u^{\overline{D}}\rangle = \theta_{\mathbf{A}}(\zeta^{\overline{D}}),\\
\theta_{\mathbf{A}}(\zeta^{D_1}) + \theta_{\mathbf{A}}(\zeta^{D_2}) &= \langle A^{D_1} v^{D_1}, u^{D_1}\rangle + \langle A^{D_2} v^{D_2}, u^{D_2}\rangle\\
&= \bigl\langle (A^{D_1} v^{D_1} \oplus A^{D_2} v^{D_2}), (u^{D_1} \oplus u^{D_2})\bigr\rangle = \theta_{\mathbf{A}}(\zeta^{D_1\oplus D_2}),\\
\theta_{\mathbf{A}}(\zeta^{D_1}) \times \theta_{\mathbf{A}}(\zeta^{D_2}) &= \langle A^{D_1} v^{D_1}, u^{D_1}\rangle \times \langle A^{D_2} v^{D_2}, u^{D_2}\rangle\\
&= \bigl\langle (A^{D_1} v^{D_1} \otimes A^{D_2} v^{D_2}), (u^{D_1} \otimes u^{D_2})\bigr\rangle = \theta_{\mathbf{A}}(\zeta^{D_1\otimes D_2}).
\end{aligned}$$

Consider the case in which $\sum_j \langle T_g^{D_j} v^{D_j}, u^{D_j}\rangle \equiv 0$ as a function on S for some countable set $\{D_j\} \subset \Omega$ and $\{v^{D_j}, u^{D_j} \in \mathcal{H}^{D_j}\}$ such that $\sum_j \|v^{D_j}\|^2$, $\sum_j \|u^{D_j}\|^2 < \infty$.

Put $D \equiv \sum_j^{\oplus} D_j$, put $v^D \equiv \sum_j^{\oplus} v^{D_j}$, and put $u^D \equiv \sum_j^{\oplus} u^{D_j}$. Then for any $g \in S$, $\langle T_g^D v^D, u^D\rangle \equiv 0$.

The condition (Cd-2) of the definition of birepresentation $\mathbf{A} \equiv \{A^D\}_{D\in\Omega}$ shows that the operator A^D keeps the invariant subspace H spanned by $\{T_g^D v^D (g \in G)\}$, that is, $A^D H \perp u^D$, and

$$0 = \langle A^D v^D, u^D\rangle = \sum_j \langle A^{D_j} v^{D_j}, u^{D_j}\rangle. \tag{3.13}$$

Therefore, map (3.12) generates a $*$-algebra homomorphism

$$f^{\sim}(g) \mapsto \theta_{\mathbf{A}}(f^{\sim}) \equiv f^{\sim}(x_{\mathbf{A}}) \tag{3.14}$$

of the space $\mathfrak{F}$ and of $\mathfrak{F}^C$ to $\mathbf{C}$; that is, it gives an element $x_{\mathbf{A}} \in X$ by the above equation.

Put $f^{\sim}(g) \equiv \langle T_g^D v^D, A^D v^D \rangle$, and apply (3.11). We obtain that

$$\begin{aligned} \left| f^{\sim}(g_0) - f^{\sim}(x_{\mathbf{A}}) \right| &= \left| \langle T_{g_0}^D v^D, A^D v^D \rangle - \langle A^D v^D, A^D v^D \rangle \right| \\ &= \left| \langle T_{g_0}^D v^D, A^D v^D \rangle - 1 \right| = \left| 1 - K^D(g_0) \right| < \delta. \end{aligned} \tag{3.15}$$

This proves (3.4). □

Let Ω_+ be the set of all cyclic representations $D = (\mathcal{H}^D, T_g^D, v^D)$ ($\|v^D\| = 1$) satisfying

$$K^D(g) = \langle T_g^D v^D, A^D v^D \rangle \geq 0 \quad (g \in S).$$

Then, by Lemma 3.2, for $D \in \Omega_+$,

$$\inf_{g \in S} \left(1 - K^{D_p}(g)\right) = 0. \tag{3.16}$$

And Ω_+ contains cyclic representations of type (D_p). Put

$$E(D, \varepsilon) \equiv \left\{ g \,\middle|\, 1 - K^D(g) < \varepsilon \right\}, \tag{3.17}$$

$$\mathrm{Z} \equiv \left\{ E(D, \varepsilon) \right\}_{D \in \Omega_+, \varepsilon > 0}. \tag{3.18}$$

LEMMA 3.3

For a birepresentation $\mathbf{A} = (A^D)_{D \in \Omega}$ of S with an SSIR, Z gives a Cauchy filter base on S.

Proof

Lemma 3.2 shows that $E(D, \varepsilon)$ is not empty and

$$\varepsilon_1 > \varepsilon_2 \Rightarrow E(D, \varepsilon_1) \supseteq E(D, \varepsilon_2). \tag{3.19}$$

Let $D^0 \equiv (D^1 \otimes D^2)$ (the cyclic part in $D^1 \otimes D^2$) with $D^1, D^2 \in \Omega_+$. Then

$$1 - K^{D^0}(g) \geq 1 - K^{D^1}(g), \qquad 1 - K^{D^0}(g) \geq 1 - K^{D^2}(g), \tag{3.20}$$

$$E(D^1, \varepsilon) \cap E(D^2, \varepsilon) \supseteq E(D^0, \varepsilon) \neq \phi. \tag{3.21}$$

So Z is a filter base.

Analogous calculations to [4, (7.8)] show that $1 - K^D(g) < \varepsilon$ leads to

$$\|A^D v^D - T_g^D v^D\| \leq (2\varepsilon)^{1/2}. \tag{3.22}$$

Hence, for any $g, h \in E(D, \varepsilon)$,

$$\|T_g^D v^D - T_h^D v^D\| \leq 2(2\varepsilon)^{1/2}. \tag{3.23}$$

For an arbitrary given entourage $W(D, 2(2\varepsilon)^{1/2})$ in $S \times S$, if we take the above $E(D,\varepsilon)$, then

$$\forall g, h \in E(D,\varepsilon), \quad (g,h) \in W\big(D, 2(2\varepsilon)^{1/2}\big) \tag{3.24}$$

that is, Z is Cauchy. □

4. Proof of Tannaka-type duality theorem for T-type semigroup

THEOREM 4.1

For a T-type semigroup S, the Tannaka-type duality theorem is valid.

Proof

For any given birepresentation $\mathbf{A} \equiv \{A^D\}$, we show that there exists a unique g in S such that

$$\{A^D\} = \{T_g^D\}. \tag{4.1}$$

The T-type semigroup S is complete by definition, and $\mathrm{Z} \equiv \{E(D, \varepsilon)\}_{D \in \Omega_+, \varepsilon > 0}$ is a Cauchy filter base. So there exists a limit point $g_{\mathbf{A}}$ and

$$\bigcap_{(D,\varepsilon)} \overline{E(D,\varepsilon)} = \{g_{\mathbf{A}}\}. \tag{4.2}$$

Therefore, $1 = K^D(g_{\mathbf{A}}) = \langle T_{g_{\mathbf{A}}}^D v^D, A^D v^D \rangle$, that is,

$$\forall D \in \Omega_+, \quad A^D v^D = T_{g_{\mathbf{A}}}^D v^D. \tag{4.3}$$

For a general cyclic representation D, consider $(D_p) \in \Omega_+$ as in Section 3; then we get from $A^{D_p} v_p = T_{g_{\mathbf{A}}}^{D_p} v_p$ that

$$I w_0 \oplus A^D w \oplus A^{\overline{D}} \overline{w} = I w_0 \oplus T_{g_{\mathbf{A}}}^D w \oplus T_{g_{\mathbf{A}}}^{\overline{D}} \overline{w}. \tag{4.4}$$

So we get, for any D in Ω, that $A^D w = T_{g_{\mathbf{A}}}^D w$. This concludes the proof. □

5. Converse of Theorem 4.1

LEMMA 5.1

For a T_2-topological semigroup S, if a weak Tannaka-type duality theorem holds, then S has an SSIR.

Proof

By Lemma 2.1, the inverse map of (2.3) from S_J to S must be continuous. A fundamental system of neighborhoods V of $(T_{g_0}^D)$ in S_J is given as the collection of

$$V_1 \equiv \bigcap_{1 \le j \le n} \big\{ \mathbf{T}_g = (T_g^D)_{D \in \Omega} \,\big|\, \|T_g^{D_j} v_j - T_{g_0}^{D_j} v_j\|^2 < \varepsilon_j \big\} \tag{5.1}$$

for a finite set $\{(D_j, v_j, \varepsilon_j)\}$, where $D_j \in \Omega$, $v_j \in \mathcal{H}^{D_j}$ ($\|v_j\| = 1$), $\varepsilon_j > 0$ ($j = 1, 2, \ldots, n$).

Consider the representation $D_0 \equiv \sum_j^{\oplus} D_j$, consider $v_0 = n^{-(1/2)} \sum_j^{\oplus} v_j$, and consider $\varepsilon_0 = \min_j \varepsilon_j$. Then

$$V_1 \supseteq V_2(\varepsilon_0) \equiv \{\mathbf{T}_g = (T_g^D)_{D\in\Omega} \mid \|v_0 - T_g^{D_0} v_0\|^2 < \varepsilon_0\}. \tag{5.2}$$

The evaluation

$$\begin{aligned} \|T_g^{D_0} v_0 - T_{g_0}^{D_0} v_0\|^2 &= 2\big(1 - \Re\big(\langle T_g^{D_0} v_0, T_{g_0}^{D_0} v_0\rangle\big)\big) \\ &\leq 2\big|1 - \langle T_g^{D_0} v_0, T_{g_0}^{D_0} v_0\rangle\big| \end{aligned} \tag{5.3}$$

shows that if we take $\delta < 2^{-1}\varepsilon_0$, then

$$V_2(\varepsilon_0) \supset V_\delta \equiv \{\mathbf{T}_g = (T_g^D)_{D\in\Omega} \mid \big|1 - \langle T_g^{D_0} v_0, T_{g_0}^{D_0} v_0\rangle\big| < \delta\}. \tag{5.4}$$

For any neighborhood V of g_0 in G, there exist V, V_1, $V_2(\epsilon_0)$, and V_δ such that

$$V \supseteq V_1 \supseteq V_2(\epsilon_0) \supseteq V_\delta. \tag{5.5}$$

This shows the separating condition of the existence of an SSIR in Definition 2.3. □

LEMMA 5.2

For a T_2-topological semigroup S, if the Tannaka-type duality theorem holds, then S must be complete.

Proof

For any Cauchy filter base $\mathcal{F}$ on S, its image $\mathcal{F}_J$ in $S_J \subset \mathbf{J}(\Omega)$ is also Cauchy. And by Lemma 2.2, it converges to an isometric operator field $\mathbf{A}_0 \equiv \{A_0^D\}$. We can easily confirm that $\mathbf{A}_0$ gives a birepresentation, that is,

$$\mathbf{A}_0 \in S_J. \tag{5.6}$$

From the assumption that the Tannaka-type duality theorem is valid, $\mathcal{F}$ must converge to a point in S, the inverse image of $\mathbf{A}_0$. □

Summarizing the results of Lemmas 5.1 and 5.2, we have the following.

THEOREM 5.1

For a T_2-topological semigroup S, if the Tannaka-type duality theorem holds, then S must be a T-type semigroup.

6. Main theorem and example

Summarizing Theorems 4.1 and 5.1, we obtain the following.

MAIN THEOREM

Let S be a T_2-topological semigroup. For S, the Tannaka-type duality theorem holds if and only if S is a T-type semigroup.

EXAMPLE 1

Let $\mathcal{H}$ be a Hilbert space of infinite dimension, and let $S \equiv \mathbf{J}(\mathcal{H})$ be the semigroup of all isometric operators on $\mathcal{H}$ with the weak (=strong) topology of operator space.

LEMMA 6.1

The semigroup $S \equiv \mathbf{J}(\mathcal{H})$ is a T_2-topological semigroup and has an SSIR.

Proof

As we showed in Section 2, S is a complete T_2-topological semigroup. Consider the identical representation $D_0 \equiv \{\mathcal{H}, T_J\}$,

$$S \ni J \mapsto T_J(\equiv J) \in \mathbf{J}(\mathcal{H}). \tag{6.1}$$

The family of all cyclic subrepresentations gives an SSIR of S. □

Lemma 2.3 claims that S is complete, so S is of T-type. And the Tannaka-type duality theorem holds for S.

7. Extension to a topological group

LEMMA 7.1

Let S_1 be a semigroup of isometric operators on a Hilbert space $\mathcal{H}$ with the weak (= strong) topology of operator space. Then S_1 is a topological semigroup. Moreover, if S_1 is a group of unitary operators, then S_1 is a topological group.

Proof

The relation for $T_1, T_2 \in S_1$,

$$\|T_1T_2v - v\| \leq \|T_1T_2v - T_1v\| + \|T_1v - v\| = \|T_2v - v\| + \|T_1v - v\|, \tag{7.1}$$

shows the continuity of multiplication on S_1 with respect to the strong topology. □

We denote by $\mathbf{U}(\mathcal{H})$ the space of all unitary operators on a Hilbert space $\mathcal{H}$, and introduce the weak topologies on it. Let $\Omega_0 \equiv \{D = (\mathcal{H}^D, T_g^D)\}$ be a set of *unitary* representations of some topological semigroup S. Construct $\mathbf{U}(\Omega_0) \equiv \prod_{D \in \Omega_0} \mathbf{U}(\mathcal{H}^D)$ with the natural product topology.

COROLLARY 7.1.1

We have that $\mathbf{U}(\Omega_0)$ is a topological group.

Proof

As the product of topological groups $\mathbf{U}(\mathcal{H}^D)$, $\mathbf{U}(\Omega_0)$ is a topological group. □

LEMMA 7.2

Any subgroup with the relative topology of a topological group is a topological group.

Proof

The proof is obvious. □

PROPOSITION 7.1

Let S be a T-type T_2-topological semigroup. If S has an SSIR $\Omega_1 \equiv \{D_\alpha \equiv \{\mathcal{H}^{D_\alpha}, T_g^{D_\alpha}, v^{D_\alpha}\} \mid v^{D_\alpha}$ is a normalized cyclic vector, $\alpha \in A\}$, all elements D_α of which are unitary representations, then there exists a topological group G which contains S as a topological subsemigroup.

Proof

Write Ω_0 as the set of all unitary representations of S. From the assumption, Ω_0 gives an SSIR of S.

For a given birepresentation $\mathbf{T}_g \equiv \{T_g^D\}_{D\in\Omega}$ of S, we consider the operator field $(\mathbf{T}_g)^{-1} \equiv \{(T_g^D)^{-1}\}_{D\in\Omega_0}$ on Ω_0 and take the group G generated by the family $\{\mathbf{T}_g, (\mathbf{T}_g)^{-1} \mid g \in S\}$. Then, G is in $\mathbf{U}(\Omega_0)$. So the above lemmata show that G is a topological group with the topology in $\mathbf{U}(\Omega_0)$, containing S.

But Ω_0 gives an SSIR of S. So the topology of S just coincides with the restricted one of G. □

COROLLARY

Let S be a T-type T_2-topological semigroup. If all representations of S are unitary representations, then S must be a topological group.

Proof

In this case, $\Omega = \Omega_0$. So G is the set of all birepresentations of S. Therefore, the Tannaka-type duality theorem claims that $S = G$. □

EXAMPLE 2

Consider the case where S is the additive semigroup $\mathbf{R}_+$ of all nonnegative real numbers. Let μ be the ordinary Lebesgue measure on S, and let $\mathcal{H}$ be the space $L^2(S,\mu)$.

The right translation operator T_{g_0} defined by

$$\begin{aligned} T_{g_0} f(g) &= 0 \quad (g \notin g_0 + S) \\ &= f(g_1) \quad (g = g_0 + g_1 \in g_0 + S) \end{aligned}$$

gives an isometric operator on $\mathcal{H}$. Let $\mathcal{R} \equiv \{\mathcal{H}, T_g\}$ define a nonunitary but isometric representation of S. It is easy to see that $\{\mathcal{R}\}$ is an SSIR, and S is complete. Thus, S is a T-type semigroup, and the Tannaka-type duality theorem is valid.

Now we consider the space $\mathcal{H}^0 \equiv L^2(\mathbf{R}, \mu^0)$ (where μ^0 is the ordinary Lebesgue measure on $\mathbf{R}$) and the representation $\mathcal{R}^0 \equiv (\mathcal{H}^0, T_g^0)$ (where T_g^0 is the translation by g). Note that $\mathcal{R}^0$ is a unitary representation. And S is embedded in the additive group of real numbers $\mathbf{R}$ which has a representation $\mathcal{R}^0$ on $\mathcal{H}^0$, extending $\mathcal{R}$. Thus, if we treat only unitary representations, then we get the whole additive group $\mathbf{R}$ as the set of *birepresentations* of S.

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