Socle filtrations of the standard Whittaker $(\mathfrak{g}, K)$-modules of $\mathrm{Spin}(r, 1)$

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Abstract Studied are the composition series of the standard Whittaker $(\mathfrak{g}, K)$-modules. For a generic infinitesimal character, the structures of these modules are completely understood, but if the infinitesimal character is integral, then there are not so many cases in which the structures of them are known. In this paper, as an example of the integral case, we determine the socle filtrations of the standard Whittaker $(\mathfrak{g}, K)$-modules when G is the group $\mathrm{Spin}(r, 1)$ and the infinitesimal character is regular integral.

1. Introduction

This paper is a continuation of the paper [6] in which the author defined and examined the standard Whittaker $(\mathfrak{g}, K)$-modules for real reductive linear Lie groups.

Let us review the definition of these modules. Let G be a real reductive linear Lie group in the sense of [7], and let $G = KAN$ be an Iwasawa decomposition of it. Let $\eta : N \longrightarrow \mathbb{C}^\times$ be a unitary character of N, and denote the differential representation $\mathfrak{n}_0 \to \sqrt{-1}\mathbb{R}$ of it by the same letter η. We assume that η is nondegenerate, that is, it is nontrivial on every root space corresponding to a simple root of $\Delta^+(\mathfrak{g}_0, \mathfrak{a}_0)$. Define

$$C^\infty(G/N; \eta) := \bigl\{ f : G \xrightarrow{C^\infty} \mathbb{C} \bigm| f(gn) = \eta(n)^{-1} f(g),\ g \in G, n \in N \bigr\}, \tag{1.1}$$

and call it the space of Whittaker functions on G. This is a representation space of G by the left translation, which is denoted by L. Let $C^\infty(G/N; \eta)_K$ be the subspace of $C^\infty(G/N; \eta)$ consisting of K-finite vectors. Let, as usual, M be the centralizer of A in K, and let M^η be the stabilizer of η in M. This subgroup acts naturally on $C^\infty(G/N; \eta)_K$ by the right translation. Consider the subspace of $C^\infty(G/N; \eta)_K$ consisting of those functions f which satisfy the following conditions:

(1) f is a joint eigenfunction of $Z(\mathfrak{g})$ (the center of the universal enveloping algebra $U(\mathfrak{g})$) with eigenvalue χ_Λ: $L(z)f = \chi_\Lambda(z) f$, $z \in Z(\mathfrak{g})$;

(2) for an irreducible representation $(\sigma, V_\sigma^{M^\eta})$ of M^η, f is in the σ^*-isotypic subspace (σ^* is the dual of σ) with respect to the right action of M^η;

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(3) f grows moderately at infinity (see [8]).

The space of functions which satisfy the above conditions (1)–(3) is isomorphic to

$$\begin{aligned} I_{\eta,\Lambda,\sigma} := \bigl\{ f : G \xrightarrow{C^\infty} V_\sigma^{M^\eta} \bigm| & f(gmn) = \eta(n)^{-1}\sigma(m)^{-1}f(g), g \in G, m \in M^\eta, n \in N; \\ & L(z)f = \chi_\Lambda(z)f, z \in Z(\mathfrak{g}); \text{ left } K\text{-finite}; \\ & f \text{ grows moderately at infinity}\bigr\}. \end{aligned}$$

We call this space the *standard Whittaker* $(\mathfrak{g}, K)$*-module.*

For generic infinitesimal character Λ, the structure of $I_{\eta,\Lambda,\sigma}$ is completely determined in [6]. On the other hand, if the infinitesimal character Λ is integral, its structure is not known except for the case $G = \mathrm{SL}(2,\mathbb{R})$ or $U(n,1)$, $n \geq 2$. In this paper, we examine the case $G = \mathrm{Spin}(r,1)$, $r \geq 3$, so that it will become a good example for the study of the case of other general groups.

The main results of this paper are as follows. For the notation of irreducible modules and the diagrammatic expression of the composition series, see Section 2.2.

THEOREM 1.1

Suppose that $G = \mathrm{Spin}(2n,1)$ *and that the infinitesimal character* $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ *(*$\Lambda_1 > \Lambda_2 > \cdots > \Lambda_n > 0$*) is regular integral. Let* σ *be an irreducible representation of* $M^\eta \simeq \mathrm{Spin}(2n-2)$.

(1) $I_{\eta,\Lambda,\sigma}$ *is not zero if and only if the highest weight* $\gamma = (\gamma_1, \dots, \gamma_{n-1})$ *of* σ *satisfies one of the following conditions:*

$$\Lambda_p - n + p + 1/2 \geq \gamma_p \geq \Lambda_{p+1} - n + p + 3/2, \quad p = 1, \dots, n-1, \tag{1.2}$$

$$\begin{cases} \Lambda_p - n + p + 1/2 \geq \gamma_p \geq \Lambda_{p+1} - n + p + 3/2, & p = 1, \dots, n-2, \\ -\Lambda_n - 1/2 \geq \gamma_{n-1} \geq -\Lambda_{n-1} + 1/2, \end{cases} \tag{1.3}$$

$$\begin{cases} i \in \{2, \dots, n\}, \\ \Lambda_p - n + p + 1/2 \geq \gamma_p \geq \Lambda_{p+1} - n + p + 3/2, & p = 1, \dots, i-2, \\ \Lambda_{p+1} - n + p + 1/2 \geq \gamma_p \geq \Lambda_{p+2} - n + p + 3/2, & p = i-1, \dots, n-2, \\ \Lambda_n - 1/2 \geq |\gamma_{n-1}|. \end{cases} \tag{1.4}$$

(2) *If (1.2) (resp., (1.3)) is satisfied, then* $I_{\eta,\Lambda,\sigma} \simeq \begin{array}{c} \pi_1 \\ \downarrow \\ \overline{\pi}_{0,n} \\ \downarrow \\ \pi_0 \end{array}$ *(resp.,* $\begin{array}{c} \pi_0 \\ \downarrow \\ \overline{\pi}_{0,n} \\ \downarrow \\ \pi_1 \end{array}$*).*

(3) *If* $i \in \{2, \dots, n-1\}$ *and (1.4) is satisfied, then* $I_{\eta,\Lambda,\sigma} \simeq \begin{array}{ccc} \overline{\pi}_{0,i-1} & & \overline{\pi}_{0,i+1} \\ & \searrow \quad \swarrow & \\ & \overline{\pi}_{0,i} & \end{array}.$

(4) *If* $i = n$ *and (1.4) is satisfied, then* $I_{\eta,\Lambda,\sigma} \simeq \begin{array}{ccc} \pi_0 & \overline{\pi}_{0,n-1} & \pi_1 \\ \searrow & \downarrow & \swarrow \\ & \overline{\pi}_{0,n} & \end{array}.$

THEOREM 1.2

Suppose that $G = \mathrm{Spin}(2n+1,1)$ and that the infinitesimal character $\Lambda = (\Lambda_1, \ldots, \Lambda_{n+1})$ $(\Lambda_1 > \Lambda_2 > \cdots > \Lambda_n > |\Lambda_{n+1}|)$ is regular integral. Let σ be an irreducible representation of $M^\eta \simeq \mathrm{Spin}(2n-1)$.

(1) *$I_{\eta,\Lambda,\sigma}$ is not zero if and only if the highest weight $\gamma = (\gamma_1, \ldots, \gamma_{n-1})$ of σ satisfies*

$$\begin{cases} \Lambda_p - n + p \geq \gamma_p \geq \Lambda_{p+1} - n + p + 1, & p = 1, \ldots, i-2, \\ \Lambda_{p+1} - n + p \geq \gamma_p \geq \Lambda_{p+2} - n + p + 1, & p = i-1, \ldots, n-2, \\ \Lambda_n - 1 \geq \gamma_{n-1} \geq |\Lambda_{n+1}| & (\text{if } i \leq n) \end{cases} \tag{1.5}$$

for some $i \in \{2, \ldots, n+1\}$.

(2) *If $i \in \{2, \ldots, n\}$ and (1.5) is satisfied, then* $I_{\eta,\Lambda,\sigma} \simeq \begin{tikzcd}[column sep=tiny] \overline{\pi}_{0,i-1} \arrow[dr] & & \overline{\pi}_{0,i+1} \arrow[dl] \\ & \overline{\pi}_{0,i} & \end{tikzcd}$.

(3) *If $i = n+1$ and (1.5) is satisfied, then* $I_{\eta,\Lambda,\sigma} \simeq \begin{tikzcd} \overline{\pi}_{0,n} \arrow[d] \\ \overline{\pi}_{0,n+1} \end{tikzcd}$.

This paper is organized as follows. In Section 2, we recall the structure of $\mathrm{Spin}(r,1)$ and the classification of irreducible Harish-Chandra modules of it. In Section 3, we first show that $I_{\eta,\Lambda,\sigma}$ has a unique irreducible submodule if it is nonzero. Also determined are the possible irreducible factors appearing in the composition series of it. In order to determine the socle filtration of $I_{\eta,\Lambda,\sigma}$, we need to use the explicit formulas of K-type shift operators. Such operators are obtained in Section 4. In Section 5, the socle filtration of $I_{\eta,\Lambda,\sigma}$ is completely determined. The key tools for our calculation are Lemmas 4.9 and 5.1 and Theorem 5.2.

Before going ahead, we introduce notation used in this paper. For a real Lie group L, the Lie algebra of it is denoted by $\mathfrak{l}_0$ and its complexification by $\mathfrak{l} = \mathfrak{l}_0 \otimes_{\mathbb{R}} \mathbb{C}$. This notation will be applied to any Lie groups. For a compact Lie group L, the set of equivalence classes of irreducible representations of L is denoted by $\widehat{L}$. The representation space of $\pi \in \widehat{L}$ is denoted by V_π^L. If L is connected and π is the irreducible representation whose highest weight is λ, we also denote it by V_λ^L. For $\pi \in \widehat{L}$, the contragredient representation is denoted by π^*, and if λ is the highest weight of π, then the highest weight of π^* is denoted by λ^*.

Suppose that K is a maximal compact subgroup of a real reductive group G. For a $(\mathfrak{g}, K)$-module π, the K-spectrum $\{\tau \in \widehat{K} \mid \tau \subset \pi|_K\}$ is denoted by $\widehat{K}(\pi)$.

2. The group $\mathrm{Spin}(r,1)$ and its irreducible Harish-Chandra modules

In the remainder of this paper, we put $G = \mathrm{Spin}(r,1)$, $r \geq 3$, and the infinitesimal character Λ is assumed to be regular integral.

2.1. Structure of $\mathrm{Spin}(r,1)$

Denote by E_{ij} the standard generators of $\mathfrak{gl}_{r+1}(\mathbb{C})$, and define $A_{ij}=E_{ij}-E_{ji}$. The group $\mathrm{Spin}(r,1)$ is the connected two-fold linear cover of $\mathrm{SO}_0(r,1)$. A maximal compact subgroup K of G is isomorphic to $\mathrm{Spin}(r)$. Set

$$\mathfrak{k}_0:=\left\{\begin{pmatrix} X & \mathbf{0} \\ {}^t\mathbf{0} & 0\end{pmatrix}\,\middle|\, X\in\mathfrak{so}(r)\right\},\qquad \mathfrak{s}_0:=\left\{\begin{pmatrix} O & \sqrt{-1}\boldsymbol{v} \\ -\sqrt{-1}{}^t\boldsymbol{v} & 0\end{pmatrix}\,\middle|\, \boldsymbol{v}\in\mathbb{R}^r\right\}.$$

Then $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{s}_0$ realizes the Lie algebra of G, and this is a Cartan decomposition of $\mathfrak{g}_0$. Let

$$h:=\sqrt{-1}A_{r+1,r},\qquad \mathfrak{a}_0:=\mathbb{R}h,$$

and define $f\in\mathfrak{a}_0^*$ by $f(h)=1$. Then $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{s}_0$. The restricted root system $\Delta(\mathfrak{g}_0,\mathfrak{a}_0)$ is $\{\pm f\}$. Choose a positive system $\Delta^+(\mathfrak{g}_0,\mathfrak{a}_0)=\{f\}$, and denote the corresponding nilpotent subalgebra $(\mathfrak{g}_0)_f$ by $\mathfrak{n}_0$. One obtains an Iwasawa decomposition

$$\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{a}_0+\mathfrak{n}_0,\qquad G=KAN,$$

where $A=\exp\mathfrak{a}_0$ and $N=\exp\mathfrak{n}_0$. Let

$$X_i:=A_{r,i}+\sqrt{-1}A_{r+1,i}\quad (1\le i\le r-1). \tag{2.1}$$

Then $\{X_i \mid 1\le i\le r-1\}$ is a basis of $\mathfrak{n}_0$.

In our $\mathrm{Spin}(r,1)$-case, M is isomorphic to $\mathrm{Spin}(r-1)$. It acts on the space of nondegenerate unitary characters of N by $\eta\mapsto\eta^m(n):=\eta(m^{-1}nm)$, $m\in M$. Therefore, we may choose a manageable unitary character when we calculate Whittaker modules. We use the nondegenerate character η defined by

$$\eta(X_i)=0,\quad i=1,\dots,r-2,\qquad \eta(X_{r-1})=\sqrt{-1}\xi,\quad \xi>0. \tag{2.2}$$

It is easy to see that M^η is isomorphic to $\mathrm{Spin}(r-2)$.

2.2. Classification of irreducible Harish-Chandra modules

We review the classification of irreducible Harish-Chandra modules of $G=\mathrm{Spin}(r,1)$ with regular integral infinitesimal character (for details, see, e.g., [1]). We use the notation $\pi_{0,i}$, $\overline{\pi}_{0,i}$, and so forth, in [1].

For an irreducible representation δ of M and an element $\nu\in\mathfrak{a}^*$, let $X_P(\delta,\nu)$ be the Harish-Chandra module of the principal series representation $\mathrm{Ind}_P^G(\delta\otimes e^{\nu+\rho_A})$. Here, $P=MAN$ is a minimal parabolic subgroup of G and $\rho_A:=\frac{1}{2}\operatorname{tr}(\mathrm{ad}_{\mathfrak{a}}|_{\mathfrak{n}})\in\mathfrak{a}^*$.

Firstly, consider the case $r=2n$, $n\ge 2$. There are two conjugacy classes of Cartan subgroups in G. One is compact and the other is maximally split. Let $\mathfrak{h}_c$ be the complexified Cartan subalgebra spanned by $\sqrt{-1}A_{2i,2i-1}$, $i=1,\dots,n$, and let H_c be the corresponding compact Cartan subgroup. Define a basis $\{\epsilon_i \mid i=1,\dots,n\}$ of $\mathfrak{h}_c^*$ by $\epsilon_i(\sqrt{-1}A_{2j,2j-1})=\delta_{ij}$ (Kronecker's delta). Choose a maximally split Cartan subgroup $H_s:=(H_c\cap M)A$. The complexified Lie algebra $\mathfrak{h}_s$ of it is the linear span of $\sqrt{-1}A_{2i,2i-1}$ $(i=1,\dots,n-1)$ and $h=\sqrt{-1}A_{2n+1,2n}$, so ϵ_i, $i=1,\dots,n-1$ and f form a basis of $\mathfrak{h}_s^*$.

Consider the irreducible Harish-Chandra modules with the regular integral infinitesimal character Λ, which is conjugate to

$$\begin{gathered}\sum_{p=1}^{n}\Lambda_p\epsilon_p \in \mathfrak{h}_c^*, \quad \Lambda_p \in \frac{1}{2}\mathbb{Z}, \\ \Lambda_p - \Lambda_{p+1} \in \mathbb{Z}_{>0} \text{ for } p=1,\dots,n-1, \text{ and } \Lambda_n > 0.\end{gathered} \tag{2.3}$$

There are two inequivalent discrete series representations π_i, $i=0,1$, whose Harish-Chandra parameters are

$$\sum_{p=1}^{n}\Lambda_p\epsilon_p, \qquad \sum_{p=1}^{n-1}\Lambda_p\epsilon_p - \Lambda_n\epsilon_n,$$

respectively.

Since $W_{\mathfrak{g}} \simeq S_n \ltimes \mathbb{Z}_2^n$ and $W(G,H_s) \simeq S_{n-1} \ltimes \mathbb{Z}_2^n$, and since H_s is connected, there are n equivalence classes of nontempered irreducible representations of $\mathrm{Spin}(2n,1)$. For $i=1,\dots,n$, define $\mu_{0,i} \in (\mathfrak{h}_s \cap \mathfrak{m})^*$ and $\nu_{0,i} \in \mathfrak{a}^*$ by

$$\mu_{0,i} := \sum_{p=1}^{i-1}\Lambda_p\epsilon_p + \sum_{p=i}^{n-1}\Lambda_{p+1}\epsilon_p - \rho_{\mathfrak{m}}, \qquad \nu_{0,i} := \Lambda_i f, \tag{2.4}$$

where $\rho_{\mathfrak{m}} := \frac{1}{2}\sum_{p=1}^{n-1}(2n-1-2p)\epsilon_p$. Let $\delta_{0,i}$ be the irreducible representation of M with the highest weight $\mu_{0,i}$, and let $\pi_{0,i} := X_P(\delta_{0,i}, \nu_{0,i})$. Then $\pi_{0,i}$ has the unique irreducible quotient, which we denote by $\overline{\pi}_{0,i}$.

Vogan classified irreducible $(\mathfrak{g}, K)$-modules in terms of regular characters. For a regular character γ, he defined the (integral) length $\ell(\gamma)$ of it (see [7, Definition 8.1.4]). In this paper, for an irreducible module π which corresponds to a regular character γ, we write $\ell(\pi) = \ell(\gamma)$ and call it the length of π.

The classification of irreducible $(\mathfrak{g}, K)$-modules of $\mathrm{Spin}(2n,1)$ is as follows (see [1], for example).

THEOREM 2.1

The irreducible Harish-Chandra modules of $\mathrm{Spin}(2n,1)$ *with the regular integral infinitesimal character* Λ *are parameterized by the set*

$$\{\pi_0, \pi_1\} \cup \{\overline{\pi}_{0,i} \mid i=1,\dots,n\}.$$

The lengths of π_0, π_1, *and* $\overline{\pi}_{0,i}$, $i=1,\dots,n$, *are* 0, 0, *and* $n-i+1$, *respectively.*

In order to state the composition series, we use diagrammatic expression.

DEFINITION 2.2

Suppose A_1, A_2 are distinct composition factors of a $(\mathfrak{g}, K)$-module V. If there exist elements $\{v_i\} \subset A_1$ and $\{X_i\} \subset \mathfrak{g}$ such that $\sum_i X_i v_i$ is nonzero and contained in A_2, then we connect A_1 and A_2 by an arrow $A_1 \to A_2$.

THEOREM 2.3 ([1])
The socle filtrations of $\pi_{0,i}$ are

$$\pi_{0,i} \simeq \begin{array}{c} \overline{\pi}_{0,i} \\ \downarrow \\ \overline{\pi}_{0,i+1} \end{array} \quad \text{if } i=1,\ldots,n-1, \qquad \text{and} \qquad \pi_{0,n} \simeq \begin{array}{ccc} & \overline{\pi}_{0,n} & \\ \swarrow & & \searrow \\ \pi_0 & & \pi_1 \end{array}.$$

The Blattner formula gives the K-spectra of the discrete series representations π_0, π_1. Starting from the discrete series, we obtain the K-spectrum of $\overline{\pi}_{0,i}$ inductively, by using Theorem 2.3. To state the theorem, let $\Lambda_0 := \infty$.

THEOREM 2.4
(1) *The K-spectra of π_0 and π_1 are*

$$\begin{aligned}\widehat{K}(\pi_0) = \Big\{(\tau_\lambda, V_\lambda^K) \,\Big|\, &\Lambda_{p-1} - n + p - \frac{1}{2} \geq \lambda_p \geq \Lambda_p - n + p + \frac{1}{2} \\ &(1 \leq p \leq n)\Big\},\end{aligned} \tag{2.5}$$

$$\begin{aligned}\widehat{K}(\pi_1) = \Big\{(\tau_\lambda, V_\lambda^K) \,\Big|\, &\Lambda_{p-1} - n + p - \frac{1}{2} \geq \lambda_p \geq \Lambda_p - n + p + \frac{1}{2} \\ &(1 \leq p \leq n-1); \\ &-\Lambda_n - \frac{1}{2} \geq \lambda_n \geq -\Lambda_{n-1} + \frac{1}{2}\Big\}.\end{aligned} \tag{2.6}$$

(2) *For $i = 1, \ldots, n$, the K-spectrum of $\overline{\pi}_{0,i}$ is*

$$\begin{aligned}\widehat{K}(\overline{\pi}_{0,i}) = \Big\{(\tau_\lambda, V_\lambda^K) \,\Big|\, &\Lambda_{p-1} - n + p - \frac{1}{2} \geq \lambda_p \geq \Lambda_p - n + p + \frac{1}{2} \ (1 \leq p \leq i-1); \\ &\Lambda_p - n + p - \frac{1}{2} \geq \lambda_p \geq \Lambda_{p+1} - n + p + \frac{1}{2} \ (i \leq p \leq n-1); \\ &\Lambda_n - \frac{1}{2} \geq |\lambda_n|\Big\}.\end{aligned} \tag{2.7}$$

In each case, every K-type occurs in $\overline{\pi}_{0,i}$ with multiplicity one.

Secondly, consider the case $r = 2n+1$, $n \geq 1$. There is only one conjugacy class of Cartan subgroups in G. Let $\mathfrak{h}_s$ be the complexified Cartan subalgebra spanned by $\sqrt{-1}A_{2i,2i-1}$, $i = 1, \ldots, n$ and $h = \sqrt{-1}A_{2n+2,2n+1}$, and let H_s be the corresponding Cartan subgroup. Define $\{\epsilon_i \mid i = 1, \ldots, n\}$ as in the $r = 2n$ case.

Consider the irreducible Harish-Chandra modules with the regular integral infinitesimal character Λ, which is conjugate to

$$\begin{aligned}&\sum_{p=1}^{n} \Lambda_p \epsilon_p + \Lambda_{n+1} f \in \mathfrak{h}_s^*, \quad \Lambda_p \in \frac{1}{2}\mathbb{Z}, \\ &\quad \text{with } \Lambda_p - \Lambda_{p+1} \in \mathbb{Z}_{>0} \text{ for } p = 1, \ldots, n, \text{ and } \Lambda_n + \Lambda_{n+1} \in \mathbb{Z}_{>0}.\end{aligned} \tag{2.8}$$

Since $W_{\mathfrak{g}} \simeq S_{n+1} \ltimes \mathbb{Z}_2^n$ and $W(G, H_s) \simeq S_n \ltimes \mathbb{Z}_2^n$, and since H_s is connected, there are $(n+1)$ equivalence classes of irreducible representations of $\mathrm{Spin}(2n+1, 1)$.

For $i = 1, \ldots, n+1$, define $\mu_{0,i} \in (\mathfrak{h}_s \cap \mathfrak{m})^*$ and $\nu_{0,i} \in \mathfrak{a}^*$ by

$$\mu_{0,i} := \sum_{p=1}^{i-1} \Lambda_p \epsilon_p + \sum_{p=i}^{n} \Lambda_{p+1} \epsilon_p - \rho_{\mathfrak{m}}, \qquad \nu_{0,i} := \Lambda_i f, \tag{2.9}$$

where $\rho_{\mathfrak{m}} := \sum_{p=1}^{n-1} (n-p)\epsilon_p$. Let $\delta_{0,i}$ be the irreducible representation of M with the highest weight $\mu_{0,i}$, and let $\pi_{0,i} := X_P(\delta_{0,i}, \nu_{0,i})$. Then $\pi_{0,i}$ has the unique irreducible quotient, which we denote by $\overline{\pi}_{0,i}$.

THEOREM 2.5

The irreducible Harish-Chandra modules of $\mathrm{Spin}(2n+1,1)$ *with the regular integral infinitesimal character* Λ *are parameterized by the set*

$$\{\overline{\pi}_{0,i} \mid i = 1, \ldots, n+1\}.$$

The lengths of $\overline{\pi}_{0,i}$, $i = 1, 2, \ldots, n+1$, *are* $n - i + 1$, *respectively.*

The socle filtrations of $\pi_{0,i}$ *are*

$$\pi_{0,i} \simeq \begin{matrix} \overline{\pi}_{0,i} \\ \downarrow \\ \overline{\pi}_{0,i+1} \end{matrix} \quad \text{if } i = 1, \ldots, n \qquad \text{and} \qquad \pi_{0,n+1} = \overline{\pi}_{0,n+1} \quad \text{is irreducible.}$$

Starting from $\pi_{0,n+1} = \overline{\pi}_{0,n+1}$, we obtain the K-spectrum of $\overline{\pi}_{0,i}$ inductively, by using Theorem 2.5. As before, let $\Lambda_0 := \infty$.

THEOREM 2.6

For $i = 1, \ldots, n+1$, *the* K*-spectrum of* $\overline{\pi}_{0,i}$ *is*

$$\begin{aligned} \widehat{K}(\overline{\pi}_{0,i}) = \big\{ (\tau_\lambda, V_\lambda^K) \;\big|\; & \Lambda_{p-1} - n + p - 1 \geq \lambda_p \geq \Lambda_p - n + p \ (1 \leq p \leq i-1); \\ & \Lambda_p - n + p - 1 \geq \lambda_p \geq \Lambda_{p+1} - n + p \ (i \leq p \leq n-1); \\ & \Lambda_n - 1 \geq \lambda_n \geq |\Lambda_{n+1}| \ (\text{if } i < n+1) \big\}. \end{aligned} \tag{2.10}$$

In each case, every K*-type occurs in* $\overline{\pi}_{0,i}$ *with multiplicity one.*

3. Composition factors of $I_{\eta,\Lambda,\sigma}$

In this section we first determine the submodules of $I_{\eta,\Lambda,\sigma}$. It is known by [3] that a $(\mathfrak{g}, K)$-module V can be a submodule of $C^\infty(G/N;\eta)_K$ if and only if the Gelfand–Kirillov dimension $\mathrm{Dim}\, V$ of it is equal to $\dim N$. For $G = \mathrm{Spin}(r,1)$, $\mathrm{Dim}\, \overline{\pi}_{0,1} = 0$, and the Gelfand–Kirillov dimensions of other irreducible modules are all $\dim N$ (see, e.g., [1]). Therefore, an irreducible submodule of $I_{\eta,\Lambda,\sigma}$ is isomorphic to one of π_0, π_1, or $\overline{\pi}_{0,i}$, $i = 2, \ldots, n+1$.

3.1. Unique simple submodule

By the discussion in [6, Section 4.2], the following lemma holds.

LEMMA 3.1

Let (π, V) *be an irreducible Harish-Chandra module with* $\mathrm{Dim}\, V = \dim N$. *Let*

$\{X_P(\delta_p,\nu_p) \mid p=1,\ldots,k\}$ be the set of principal series representations which contain (π,V) as a subquotient. If (π,V) is a submodule of $I_{\eta,\Lambda,\sigma}$, then σ is a submodule of $\delta_p|_{M^\eta}$ for every $p=1,\ldots,k$.

Conversely, for $\sigma \in \widehat{M^\eta}$, suppose that there exists a principal series $X_P(\delta,\nu)$ with infinitesimal character Λ which satisfies $\sigma \subset \delta|_{M^\eta}$. Then, $I_{\eta,\Lambda,\sigma}$ is nonzero.

By this lemma, we can determine the nonzero standard Whittaker $(\mathfrak{g},K)$-modules and their subrepresentations.

PROPOSITION 3.2

Suppose that $r=2n$ and the infinitesimal character Λ is regular integral. Let $\gamma=(\gamma_1,\ldots,\gamma_{n-1})$ be the highest weight of the irreducible representation σ of $M^\eta \simeq \mathrm{Spin}(2n-2)$.

(1) *The irreducible module π_0 (resp., π_1) is a submodule of $I_{\eta,\Lambda,\sigma}$ if and only if γ satisfies (1.2) (resp., (1.3)).*

(2) *The irreducible module $\overline{\pi}_{0,i}$, $i=2,\ldots,n$, is a submodule of $I_{\eta,\Lambda,\sigma}$ if and only if γ satisfies (1.4).*

Especially, $I_{\eta,\Lambda,\sigma}$ is nonzero if and only if the highest weight of σ satisfies one of the conditions (1.2), (1.3), or (1.4) for some $i=2,\ldots,n$. In these cases, π_0, π_1, or $\overline{\pi}_{0,i}$ is the unique simple submodule of $I_{\eta,\Lambda,\sigma}$.

Proof

We first show (2). By Theorem 2.3, $\overline{\pi}_{0,i}$, $i=2,\ldots,n$, is a composition factor of the principal series $\pi_{0,k}$ if and only if $k=i$ or $i-1$. Therefore, if $\overline{\pi}_{0,i}$ is a submodule of $I_{\eta,\Lambda,\sigma}$, then $\sigma \subset \delta_{0,i}|_{M^\eta}$ and $\sigma \subset \delta_{0,i-1}|_{M^\eta}$. Conversely, if σ satisfies this condition, then $I_{\eta,\Lambda,\sigma}$ is nonzero, by Lemma 3.1.

Recall the branching rule for the restriction of an irreducible representation of $\mathrm{Spin}(2n-1)$ to $\mathrm{Spin}(2n-2)$. For an irreducible representation δ_μ of $\mathrm{Spin}(2n-1)$ with the highest weight $\mu=(\mu_1,\ldots,\mu_{n-1})$, the restriction $\delta_\mu|_{\mathrm{Spin}(2n-2)}$ is a direct sum of $\sigma' \in \mathrm{Spin}(2n-2)^\wedge$, whose highest weight $\gamma=(\gamma_1,\ldots,\gamma_{n-1})$ satisfies

$$\mu_p \ge \gamma_p \ge \mu_{p+1}, \quad p=1,\ldots,n-2, \qquad \mu_{n-1} \ge |\gamma_{n-1}|, \qquad \mu_p-\gamma_p \in \mathbb{Z}.$$

It follows that the restriction of $\delta_{0,k} \in \widehat{M}$ to M^η is a direct sum of $\sigma' \in \widehat{M^\eta}$, whose highest weight $\gamma=(\gamma_1,\ldots,\gamma_{n-1})$ satisfies

$$\begin{cases} \Lambda_p - n + p + \frac{1}{2} \ge \gamma_p \ge \Lambda_{p+1} - n + p + \frac{3}{2}, \quad p=1,\ldots,k-2, \\ \Lambda_{k-1} - n + k - \frac{1}{2} \ge \gamma_{k-1} \ge \Lambda_{k+1} - n + k + \frac{1}{2}, \\ \Lambda_{p+1} - n + p + \frac{1}{2} \ge \gamma_p \ge \Lambda_{p+2} - n + p + \frac{3}{2}, \quad p=k,\ldots,n-2, \\ \Lambda_n - \frac{1}{2} \ge |\gamma_{n-1}|. \end{cases}$$

(If $k=n$, then the second and the third lines are omitted.) Therefore, when $2 \le i \le n$, $\sigma \in \widehat{M^\eta}$ satisfies $\sigma \subset \delta_{0,i}|_{M^\eta}$ and $\sigma \subset \delta_{0,i-1}|_{M^\eta}$ if and only if the highest weight γ of σ satisfies (1.4). This proves the *only if* part of the proposition.

The proofs of the *if* part and the uniqueness of the socle of $I_{\eta,\Lambda,\sigma}$ are the same as those of [6, Proposition 4.2], so we omit them here.

The proof of (1) is almost the same as that of (2). We can show that π_0 or π_1 is a submodule of $I_{\eta,\Lambda,\sigma}$ only if γ satisfies (1.2) or (1.3). But the method used here does not tell us the signature condition for γ_{n-1} in (1.2) and (1.3). To complete the proof, we need to write explicitly the Whittaker functions characterizing the submodule of $I_{\eta,\Lambda,\sigma}$. This will be done in Section 4 (see Lemma 4.8). □

Just in the same way, we can determine the submodules of $I_{\eta,\Lambda,\sigma}$ in the case $r = 2n + 1$.

PROPOSITION 3.3

Suppose that $r = 2n + 1$ and that the regular infinitesimal character Λ is integral. Let $\gamma = (\gamma_1, \dots, \gamma_{n-1})$ be the highest weight of the irreducible representation σ of $M^\eta \simeq \mathrm{Spin}(2n-1)$. Then the irreducible module $\overline{\pi}_{0,i}$, $i = 2, \dots, n+1$, is a submodule of $I_{\eta,\Lambda,\sigma}$ if and only if γ satisfies (1.5). Especially, $I_{\eta,\Lambda,\sigma}$ is nonzero if and only if the highest weight of σ satisfies the condition (1.5) for some $i = 2, \dots, n+1$. In these cases, $\overline{\pi}_{0,i}$ is the unique simple submodule of $I_{\eta,\Lambda,\sigma}$.

3.2. Composition factors

Hereafter, we denote $I_{\eta,\Lambda,\sigma}$ by $I_{\eta,\Lambda,\gamma}$ if the highest weight of σ is γ. We also denote by σ_γ the irreducible representation of M^η whose highest weight is γ. We determine the irreducible representations appearing in the composition series of $I_{\eta,\Lambda,\gamma}$.

PROPOSITION 3.4

(1) *Suppose that $r = 2n$ and that γ satisfies (1.2) or (1.3). Then, an irreducible composition factor of $I_{\eta,\Lambda,\gamma}$ is isomorphic to π_0, π_1, or $\overline{\pi}_{0,n}$.*

(2) *Suppose that $r = 2n$, $i \in \{2, \dots, n-1\}$ and that γ satisfies (1.4). Then, an irreducible composition factor of $I_{\eta,\Lambda,\gamma}$ is isomorphic to $\overline{\pi}_{0,i-1}$, $\overline{\pi}_{0,i}$, or $\overline{\pi}_{0,i+1}$.*

(3) *Suppose that $r = 2n$ and that γ satisfies (1.4) for $i = n$. Then, an irreducible composition factor of $I_{\eta,\Lambda,\gamma}$ is isomorphic to $\overline{\pi}_{0,n-1}$, $\overline{\pi}_{0,n}$, π_0, or π_1.*

(4) *Suppose that $r = 2n + 1$, $i \in \{2, \dots, n\}$, and that γ satisfies (1.5). Then, an irreducible composition factor of $I_{\eta,\Lambda,\gamma}$ is isomorphic to $\overline{\pi}_{0,i-1}$, $\overline{\pi}_{0,i}$, or $\overline{\pi}_{0,i+1}$.*

(5) *Suppose that $r = 2n + 1$ and that γ satisfies (1.5) for $i = n + 1$. Then, an irreducible composition factor of $I_{\eta,\Lambda,\gamma}$ is isomorphic to $\overline{\pi}_{0,n}$ or $\overline{\pi}_{0,n+1}$.*

Proof

We will show (2). The proofs of other statements are the same.

Assume that γ satisfies (1.4). Since $I_{\eta,\Lambda,\gamma}$ is induced from the representation $\sigma_\gamma \otimes \eta$ of $M^\eta N$, every K-type of a composition factor of $I_{\eta,\Lambda,\gamma}$ must contain the representation σ_γ. By (2.5)–(2.7) and (1.4), this is possible if and only if the composition factor is isomorphic to $\overline{\pi}_{0,i-1}$, $\overline{\pi}_{0,i}$, or $\overline{\pi}_{0,i+1}$. □

PROPOSITION 3.5

Suppose that one of the following conditions is satisfied:

(1) $r=2n$, γ *satisfies* (1.2) *or* (1.3), *and* π *is* π_0, π_1, *or* $\overline{\pi}_{0,n}$*;*
(2) $r=2n$, $i\in\{2,\dots,n\}$, γ *satisfies* (1.4), *and* π *is* $\overline{\pi}_{0,i-1}$, $\overline{\pi}_{0,i}$, *or* $\overline{\pi}_{0,i+1}$*;*
(3) $r=2n$, γ *satisfies* (1.4) *for* $i=n$, *and* π *is* $\overline{\pi}_{0,n-1}$, $\overline{\pi}_{0,n}$, π_0, *or* π_1*;*
(4) $r=2n+1$, $i\in\{2,\dots,n\}$, γ *satisfies* (1.5), *and* π *is* $\overline{\pi}_{0,i-1}$, $\overline{\pi}_{0,i}$, *or* $\overline{\pi}_{0,i+1}$*;*
(5) $r=2n+1$, γ *satisfies* (1.5) *for* $i=n+1$, *and* π *is* $\overline{\pi}_{0,n}$, *or* $\overline{\pi}_{0,n+1}$.

Then the multiplicity of π *in* $I_{\eta,\Lambda,\gamma}$ *is at least one.*

Proof
This can be shown just in the same way as [6, Proposition 4.4]. □

4. K-type shift operators

In order to determine the socle filtration of $I_{\eta,\Lambda,\gamma}$, we need to write the actions of elements in $\mathfrak{s}$ on this space explicitly. This is achieved by the K-type shift operators. The contents of this section are almost the same as those of [6, Section 5], so we do not repeat the explanation and refer the readers to this paper.

4.1. Gelfand–Tsetlin basis

In order to write the K-type shift operators explicitly, we realize the K-types by using the Gelfand–Tsetlin basis (see [2]).

DEFINITION 4.1

Let $\lambda=(\lambda_1,\dots,\lambda_{\lfloor r/2\rfloor})$ be a dominant integral weight of $\mathrm{Spin}(r)$. A λ-*Gelfand–Tsetlin pattern* is a set of vectors $Q=(\boldsymbol{q}_1,\dots,\boldsymbol{q}_{r-1})$ such that the following hold:

(1) $\boldsymbol{q}_i=(q_{i,1},q_{i,2},\dots,q_{i,\lfloor (i+1)/2\rfloor})$;
(2) the numbers $q_{i,j}$ are all integers or all half integers;
(3) $q_{2i+1,j}\geq q_{2i,j}\geq q_{2i+1,j+1}$, for any $j=1,\dots,i-1$;
(4) $q_{2i+1,i}\geq q_{2i,i}\geq |q_{2i+1,i+1}|$;
(5) $q_{2i,j}\geq q_{2i-1,j}\geq q_{2i,j+1}$, for any $j=1,\dots,i-1$;
(6) $q_{2i,i}\geq q_{2i-1,i}\geq -q_{2i,i}$;
(7) $q_{r-1,j}=\lambda_j$.

Here, $\lfloor a\rfloor$ is the largest integer not greater than a. The set of all λ-Gelfand–Tsetlin patterns is denoted by $\mathrm{GT}(\lambda)$.

NOTATION 4.2

For any set or number $*$ depending on $Q\in\mathrm{GT}(\lambda)$, we denote it by $*(Q)$ if we need to specify Q. For example, $q_{i,j}(Q)$ is the $q_{i,j}$ part of $Q\in\mathrm{GT}(\lambda)$.

THEOREM 4.3 ([2])

For a dominant integral weight λ of $\mathrm{Spin}(r)$, let $(\tau_\lambda, V_\lambda^{\mathrm{Spin}(r)})$ be the irreducible representation of $\mathrm{Spin}(r)$ with the highest weight λ. Then $\mathrm{GT}(\lambda)$ is identified with a basis of $(\tau_\lambda, V_\lambda^{\mathrm{Spin}(r)})$.

The action of the element $A_{p,q} \in \mathfrak{so}(r)$ is expressed as follows. For $j > 0$, let

$$l_{2i-1,j} := q_{2i-1,j} + i - j, \qquad l_{2i-1,-j} := -l_{2i-1,j},$$
$$l_{2i,j} := q_{2i,j} + i + 1 - j, \qquad l_{2i,-j} := -l_{2i,j} + 1,$$

and let $l_{2i,0} = 0$. Define $a_{p,q}(Q)$ by

$$a_{2i-1,j}(Q) = \operatorname{sgn} j \sqrt{-\frac{\prod_{1\le|k|\le i-1}(l_{2i-1,j} + l_{2i-2,k})\prod_{1\le|k|\le i}(l_{2i-1,j} + l_{2i,k})}{4\prod_{\substack{1\le|k|\le i,\\ k\ne\pm j}}(l_{2i-1,j} + l_{2i-1,k})(l_{2i-1,j} + l_{2i-1,k} + 1)}},$$

for $j = \pm 1, \dots, \pm i$, and

$$a_{2i,j}(Q) = \varepsilon_{2i,j}(Q)\sqrt{-\frac{\prod_{1\le|k|\le i}(l_{2i,j} + l_{2i-1,k})\prod_{1\le|k|\le i+1}(l_{2i,j} + l_{2i+1,k})}{(4l_{2i,j}^2 - 1)\prod_{\substack{0\le|k|\le i\\ k\ne\pm j}}(l_{2i,j} + l_{2i,k})(l_{2i,j} - l_{2i,k})}},$$

for $j = 0, \pm 1, \dots, \pm i$, where $\varepsilon_{2i,j}(Q)$ is $\operatorname{sgn} j$ if $j \ne 0$, and $\operatorname{sgn}(q_{2i-1,i}q_{2i+1,i+1})$ if $j = 0$.

Let $\sigma_{a,b}$ be the shift operator, sending $\boldsymbol{q}_a$ to $\boldsymbol{q}_a + (0, \dots, \overset{|b|}{\operatorname{sgn}(b)}, 0, \dots, 0)$.

THEOREM 4.4 (SEE [2])

Under the above notation, the action of the Lie algebra is expressed as

$$\tau_\lambda(A_{2i+1,2i})Q = \sum_{1\le|j|\le i} a_{2i-1,j}(Q)\sigma_{2i-1,j}Q,$$
$$\tau_\lambda(A_{2i+2,2i+1})Q = \sum_{0\le|j|\le i} a_{2i,j}(Q)\sigma_{2i,j}Q.$$

REMARK 4.5

The Gelfand–Tsetlin basis is compatible with the restriction to smaller groups $\mathrm{Spin}(k)$, $k = 1, \dots, r-1$. More precisely, the restriction of τ_λ to $\mathrm{Spin}(r-1)$ is multiplicity-free, and the highest weights of the irreducible representation appearing in $\tau_\lambda|_{\mathrm{Spin}(r-1)}$ are the above $\boldsymbol{q}_{r-2}$'s. Moreover, the vector $Q = (\boldsymbol{q}_1, \dots, \boldsymbol{q}_{r-2}, \boldsymbol{q}_{r-1})$ is contained in the $\boldsymbol{q}_{r-2}$-isotypic subspace of $\tau_\lambda|_{\mathrm{Spin}(r-1)}$.

REMARK 4.6

The highest weight λ^* of the contragredient representation $(\tau_{\lambda^*}, V_{\lambda^*}^K)$ of $(\tau_\lambda, V_\lambda^K)$ is $\lambda^* = (\lambda_1, \dots, (-1)^n\lambda_n)$ if $r = 2n$, and $\lambda^* = \lambda$ if $r = 2n+1$. In this case, $Q^* := (\boldsymbol{q}_1^*, \dots, \boldsymbol{q}_{r-1}^*) \in \mathrm{GT}(\lambda^*)$, $\boldsymbol{q}_{2i+1}^* := \boldsymbol{q}_{2i+1}$, $\boldsymbol{q}_{2i}^* := (q_{2i,1}, \dots, q_{2i,i-1}, (-1)^i q_{2i,i})$ is dual to $Q \in \mathrm{GT}(\lambda)$.

4.2. Shift operators

Choose a K-type $(\tau_\lambda, V_\lambda^K)$ of $I_{\eta,\Lambda,\gamma}$, whose highest weight is λ. By appropriately defining the action of $\mathfrak{g}$ on $C^\infty(A \to \mathrm{Hom}_{M^\eta}(V_\lambda^K, V_\gamma^{M^\eta}))$, the space $\mathrm{Hom}_K(V_\lambda^K, I_{\eta,\Lambda,\gamma})$ is identified with

$$\bigl\{\widetilde{\phi} \in C^\infty\bigl(A \to \mathrm{Hom}_{M^\eta}(V_\lambda^K, V_\gamma^{M^\eta})\bigr) \bigm| z \cdot \widetilde{\phi} = \chi_\Lambda(z)\widetilde{\phi}, z \in Z(\mathfrak{g}); \\ \widetilde{\phi} \text{ grows moderately at infinity}\bigr\}.$$

The space $\mathrm{Hom}_{M^\eta}(V_\lambda^K, V_\gamma^{M^\eta})$ is isomorphic to $(V_{\lambda^*}^K \otimes V_\gamma^{M^\eta})^{M^\eta}$, the space of M^η-invariants in $V_{\lambda^*}^K \otimes V_\gamma^{M^\eta}$. By Remark 4.5, a basis of this space is identified with the *partial Gelfand–Tsetlin patterns*

$$\mathrm{GT}\bigl((\lambda/\gamma)^*\bigr) := \bigl\{Q = (\boldsymbol{q}_{r-3}, \boldsymbol{q}_{r-2}, \boldsymbol{q}_{r-1}) \bigm| \boldsymbol{q}_{r-3} = \gamma^*, \boldsymbol{q}_{r-1} = \lambda^*; \\ \text{satisfies Definition 4.1(1)–(6)}\bigr\}.$$

Let $V_{(\lambda/\gamma)^*}^{K/M^\eta}$ be the vector space spanned by $\mathrm{GT}((\lambda/\gamma)^*)$, and let $C^\infty(A \to V_{(\lambda/\gamma)^*}^{K/M^\eta})$ be the space of $V_{(\lambda/\gamma)^*}^{K/M^\eta}$-valued C^∞-functions on A. By the above discussion, $\mathrm{Hom}_K(V_\lambda^K, I_{\eta,\Lambda,\gamma})$ is identified with a subspace of $C^\infty(A \to V_{(\lambda/\gamma)^*}^{K/M^\eta})$.

Denote by $\Delta_\mathfrak{s}$ the set of weights of the adjoint representation $(\mathrm{Ad}, \mathfrak{s})$ of K on $\mathfrak{s}$. For every $\alpha \in \Delta_\mathfrak{s}$, K-type shift operators

$$\begin{array}{rcl} P_\alpha : C^\infty(A \to V_{(\lambda/\gamma)^*}^{K/M^\eta}) & \to & C^\infty(A \to V_{(\lambda+\alpha/\gamma)^*}^{K/M^\eta}), \\ \cup & & \cup \\ \mathrm{Hom}_K(V_\lambda^K, I_{\eta,\Lambda,\sigma}) & \to & \mathrm{Hom}_K(V_{\lambda+\alpha}^K, I_{\eta,\Lambda,\sigma}) \end{array}$$

realize the action of elements in $\mathfrak{s}$ on $I_{\eta,\Lambda,\gamma}$ (see [6, Section 5]).

For notational convenience, let $\epsilon_{-k} := -\epsilon_k$ and $\epsilon_0 = 0$. Firstly, consider the case $r = 2n$. In this case, $\Delta_\mathfrak{s} = \{\epsilon_k \mid k = \pm 1, \ldots, \pm n\}$. By Remark 4.6, $(\lambda + \epsilon_k)^* = \lambda^* + \epsilon_k$ if $|k| < n$, and $(\lambda \pm \epsilon_n)^* = \lambda^* \pm (-1)^n \epsilon_n$. Secondly, consider the case $r = 2n+1$. In this case, $\Delta_\mathfrak{s} = \{\epsilon_k \mid k = 0, \pm 1, \ldots, \pm n\}$. By Remark 4.6, $(\lambda + \epsilon_k)^* = \lambda + \epsilon_k$.

For simplicity, denote by P_k the operator P_{ϵ_k} for $k = 0, \pm 1, \ldots, \pm n$. When $r = 2n$, the explicit forms of these operators are obtained in [4]. In the case $r = 2n+1$, they are obtained just in the same way.

PROPOSITION 4.7

Suppose $\phi(a) = \sum_{Q \in \mathrm{GT}((\lambda/\gamma)^*)} c(Q; a) Q \in C^\infty(A \to V_{(\lambda/\gamma)^*}^{K/M^\eta})$.

(1) *When* $r = 2n$, *the* K-*type shift operator* P_k, $k = \pm 1, \ldots, \pm(n-1)$ *is given by the following formula:*

$$P_k \phi(a) = -\sum_{Q \in \mathrm{GT}((\lambda/\gamma)^*)} a_{2n-1,k}(Q)\bigl(\partial_a - l_{2n-1,k}(Q) + n - 1\bigr) \\ \times c(Q; a) \sigma_{2n-1,k} Q \tag{4.1}$$

$$+\frac{\sqrt{-1}\xi}{a}\sum_{0\le|j|\le n-1}\sum_{\sigma_{2n-2,-j}Q\in \mathrm{GT}((\lambda/\gamma)^*)}\frac{a_{2n-2,j}(\sigma_{2n-2,-j}Q)a_{2n-1,k}(Q)}{l_{2n-2,j}(\sigma_{2n-2,-j}Q)-l_{2n-1,k}(Q)} \times c(\sigma_{2n-2,-j}Q;a)\sigma_{2n-1,k}Q.$$

Here, ∂_a is the differential operator $a(d/da)$. When $k=\pm n$, then the number k in the right-hand side is replaced by $(-1)^n k$.

(2) *When $r=2n+1$, the K-type shift operator P_k, $k=0,\pm1,\dots,\pm n$, is given by the following formula:*

$$P_k\phi(a)=-\sum_{Q\in \mathrm{GT}((\lambda/\gamma)^*)}a_{2n,k}(Q)\big(\partial_a-l_{2n,k}(Q)+n\big)c(Q;a)\sigma_{2n,k}Q$$
$$+\frac{\sqrt{-1}\xi}{a}\sum_{1\le|j|\le n}\sum_{\sigma_{2n-1,-j}Q\in \mathrm{GT}((\lambda/\gamma)^*)}\frac{a_{2n-1,j}(\sigma_{2n-1,-j}Q)a_{2n,k}(Q)}{l_{2n-1,j}-l_{2n,k}(Q)} \times c(\sigma_{2n-1,-j}Q;a)\sigma_{2n,k}Q. \tag{4.2}$$

First, we complete the proof of Proposition 3.2(1) by using these operators.

LEMMA 4.8

The irreducible module π_0 (resp., π_1) is a submodule of $I_{\eta,\Lambda,\gamma}$ if and only if γ satisfies (1.2) (resp., (1.3)).

Proof

Let λ be the minimal K-type of π_0 (resp., π_1). Suppose that γ satisfies (1.2) or (1.3). By [9, Theorem 2.4], an embedding of π_0 (resp., π_1) into $C^\infty(G/N;\eta)_K$ is characterized by the system of equations $P_{-1}\phi=0,\dots,P_{-n}\phi=0$ (resp., $P_{-1}\phi=0,\dots,P_{-n+1}\phi=0,P_n\phi=0$) for $\phi\in C^\infty(A\to V^{K/M^\eta}_{(\lambda/\gamma)^*})$. This system of equations is solved in [4] (though, notation is a little different).

Let $Q_0=(\gamma^*,\boldsymbol{q}_{2n-2},\lambda^*)$ be a vector in $\mathrm{GT}((\lambda/\gamma)^*)$ which satisfies $q_{2n-2,j}(Q_0)=\gamma_j^*$ $(j=1,2,\dots,n-2)$ and $q_{2n-2,n-1}=|\gamma^*_{n-1}|$. Then the function ϕ characterizing the embedding of π_0 (resp., π_1) into $I_{\eta,\Lambda,\gamma}$ is determined by the coefficient function $c(Q_0;a)$. This function is a solution of the differential equation

$$\Big(\partial_a+n-1+\sum_{p=1}^{n}\lambda_p-\sum_{p=1}^{n-2}\gamma_p-|\gamma_{n-1}|-(\operatorname{sgn}\gamma_{n-1})\frac{\xi}{a}\Big)c(Q_0;a)=0$$

$$\Big(\text{resp., }\Big(\partial_a+n-1+\sum_{p=1}^{n-1}\lambda_p-\lambda_n-\sum_{p=1}^{n-2}\gamma_p-|\gamma_{n-1}|+(\operatorname{sgn}\gamma_{n-1})\frac{\xi}{a}\Big)c(Q_0;a)=0\Big).$$

Since we set $\xi>0$, there exists a nonzero moderate growth solution if and only if $\gamma_{n-1}>0$ (resp., $\gamma_{n-1}<0$), that is, γ satisfies (1.2) (resp., (1.3)). $\square$

LEMMA 4.9

Let ϕ be an element of $C^\infty(A \to V_{(\lambda/\gamma)^*}^{K/M^\eta})$. If r, k, λ, and γ satisfy one of the following conditions, then $P_k\phi = 0$ implies $\phi = 0$, that is, P_k is injective:

(1) $r = 2n$, $k \in \{2, \dots, n-1\}$, $\gamma_{k-1} > \lambda_k$;
(2) $r = 2n$, $k \in \{-2, \dots, -(n-1)\}$, $\gamma_{|k|-1} < \lambda_{|k|}$;
(3) $r = 2n$, $\lambda_n^* = (-1)^n\lambda_n > 0$, $\lambda_n^* > |\gamma_{n-1}^*| = |(-1)^{n-1}\gamma_{n-1}|$, $k = -(-1)^n n$;
(4) $r = 2n$, $\lambda_n^* = (-1)^n\lambda_n < 0$, $-\lambda_n^* > |\gamma_{n-1}| = |(-1)^{n-1}\gamma_{n-1}|$, $k = (-1)^n n$;
(5) $r = 2n+1$, $k \in \{2, \dots, n\}$, $\gamma_{k-1} > \lambda_k$;
(6) $r = 2n+1$, $k \in \{-2, \dots, -n\}$, $\gamma_{|k|-1} < \lambda_{|k|}$.

Proof

Since the proofs of these are analogous, we show only (1). We will show $c(Q;a) = 0$ by induction on $\lambda_k^* - q_{2n-2,k}(Q)$.

Let Q_0 be an element of $\mathrm{GT}((\lambda/\gamma)^*)$ which satisfies $q_{2n-2,k}(Q_0) = \lambda_k^* = \lambda_k$, and let $Q_1 := \sigma_{2n-2,k}Q_0$. Then Q_1 is not in $\mathrm{GT}((\lambda/\gamma)^*)$, but $\sigma_{2n-2,-k}Q_1 = Q_0 \in \mathrm{GT}((\lambda/\gamma)^*)$ and $\sigma_{2n-1,k}Q_1 \in \mathrm{GT}((\lambda + \epsilon_k/\gamma)^*)$, because $\gamma_{k-1}^* = \gamma_{k-1} > \lambda_k = \lambda_k^*$ implies that $\sigma_{2n-1,k}Q_1$ satisfies the conditions in Definition 4.1(5): $q_{2n-2,k-1}(Q_1) \geq \gamma_{k-1}^* = q_{2n-3,k-1}(Q_1) \geq \lambda_k^* + 1 = q_{2n-1,k}(\sigma_{2n-1,k}Q_1) = q_{2n-2,k}(\sigma_{2n-2,k}Q_1)$. Therefore, the term $\sigma_{2n-2,k}Q_1$ appears in (4.1), and its coefficient in (4.1) is

$$\begin{aligned}
&\frac{a_{2n-2,k}(\sigma_{2n-2,-k}Q_1)a_{2n-1,k}(Q_1)}{l_{2n-2,k}(\sigma_{2n-2,-k}Q_1) - l_{2n-1,k}(Q_1)}c(\sigma_{2n-2,-k}Q_1;a) \\
&\quad = \frac{a_{2n-2,k}(Q_0)a_{2n-1,k}(\sigma_{2n-2,k}Q_0)}{l_{2n-2,k}(Q_0) - l_{2n-1,k}(Q_0)}c(Q_0;a).
\end{aligned} \tag{4.3}$$

In general,

$$\frac{a_{2n-2,j}(Q)a_{2n-1,k}(\sigma_{2n-2,j}Q)}{l_{2n-2,j}(Q) - l_{2n-1,k}(Q)} = \frac{a_{2n-2,j}(\sigma_{2n-1,k}Q)a_{2n-1,k}(Q)}{l_{2n-2,j}(Q) - l_{2n-1,k}(Q) - 1}.$$

Here, we used the definition of $a_{i,j}(Q)$. Therefore, (4.3) is

$$\frac{a_{2n-2,k}(\sigma_{2n-1,k}Q_0)a_{2n-1,k}(Q_0)}{(\lambda_k + n - k) - (\lambda_k + n - k) - 1}c(Q_0;a).$$

It is easy to check that $a_{2n-2,k}(\sigma_{2n-1,k}Q_0)a_{2n-1,k}(Q_0)$ is not zero. So if $P_k\phi = 0$, then $c(Q_0;a) = 0$. We have shown that $c(Q;a)$ is zero for those Q which satisfy $q_{2n-2,k}(Q) = \lambda_k^*$.

Assume that $c(Q;a) = 0$ is proved for those Q's which satisfy the condition $\lambda_k^* - q_{2n-2,k}(Q) = p$. Let Q_2 be an element of $\mathrm{GT}((\lambda/\gamma)^*)$ which satisfies $\lambda_k^* - q_{2n-2,k}(Q_2) = p+1$. Set $Q_3 := \sigma_{2n-2,k}Q_2$. This Q_3 is an element of $\mathrm{GT}((\lambda/\gamma)^*)$, and it satisfies $\lambda_k^* - q_{2n-2,k}(Q_3) = p$ and $\lambda_k^* - q_{2n-2,k}(\sigma_{2n-2,-i}Q_3) = p$ if $i \neq k$. Then by the hypothesis of induction, $c(Q_3;a) = 0$ and $c(\sigma_{2n-2,-i}Q_3;a) = 0$ for $i \neq k$. Consider the right-hand side of (4.1) for $Q = Q_3$. The terms other than

$c(Q_2;a) = c(\sigma_{2n-2,-k}Q_3;a)$ are zero. We can easily see that its coefficient

$$\frac{a_{2n-2,k}(\sigma_{2n-2,-k}Q_3)a_{2n-1,k}(Q_3)}{l_{2n-2,k}(\sigma_{2n-2,-k}Q_3) - l_{2n-1,k}(Q_3)}$$

is not zero. Therefore, if $P_k\phi(a) = 0$, then $c(Q_2;a) = 0$. This completes the proof. □

Choose a Cartan subalgebra $\mathfrak{h} := \bigoplus_{i=1}^{\lfloor (r+1)/2 \rfloor} \mathbb{C}A_{r-2i+3,r-2i+2}$ of $\mathfrak{g}$, and let $\gamma : Z(\mathfrak{so}_{r+1}) \to U(\mathfrak{h})^{W(\mathfrak{g},\mathfrak{h})}$ be the Harish-Chandra isomorphism. The following theorem is proved in [5].

THEOREM 4.10 ([5, THEOREM 1.1, LEMMA 3.2, PROPOSITION 5.3])

For $u \in \mathbb{C}$, let $C_{r+1}(u)$ be the element in $Z(\mathfrak{so}_{r+1})$ which satisfies

$$\gamma\big(C_{r+1}(u)\big) = \prod_{i=1}^{\lfloor (r+1)/2 \rfloor} (u^2 + A^2_{r-2i+3,r-2i+2}). \tag{4.4}$$

When $r = 2n+1$, let $\mathbb{PF}_{2n+2}$ be the element in $Z(\mathfrak{so}_{2n+2})$ which satisfies

$$\gamma(\mathbb{PF}_{2n+2}) = (-\sqrt{-1})^{n+1} A_{2,1}A_{4,2}\cdots A_{2n+2,2n+1}.$$

For $\tau_\lambda \in \widehat{K}$, define

$$u_k := \begin{cases} l_{2n-1,k} + 1/2 & \text{when } r = 2n \text{ and } |k| < n, \\ l_{2n-1,(-1)^n k} + 1/2 & \text{when } r = 2n \text{ and } k = \pm n, \\ l_{2n,k} & \text{when } r = 2n+1. \end{cases} \tag{4.5}$$

(1) *For $k = \pm 1, \ldots, \pm\lfloor r/2 \rfloor$, there exists a nonzero constant $d_{\lambda,k}$ such that*

$$P_{-k} \circ P_k\phi = d_{\lambda,k} L\big(C_{r+1}(u_k)\big)\phi, \quad \phi \in C^\infty(A \to V^{K/M^\eta}_{(\lambda/\gamma)^*}). \tag{4.6}$$

(2) *When $r = 2n+1$, there exists a nonzero constant d_λ such that*

$$P_0\phi = d_\lambda L(\mathbb{PF}_{2n+2})\phi, \quad \phi \in C^\infty(A \to V^{K/M^\eta}_{(\lambda/\gamma)^*}). \tag{4.7}$$

5. Determination of composition series

In this section, we determine the socle filtration of $I_{\eta,\Lambda,\gamma}$.

LEMMA 5.1

Let V_1 and V_2 be irreducible factors in $I_{\eta,\Lambda,\gamma}$ which satisfy one of the following conditions:

(1) *$V_1 \simeq V_2$, but they are different irreducible factors;*
(2) *$r = 2n$, γ satisfies (1.2), $V_1 \simeq \pi_0$, $V_2 \simeq \overline{\pi}_{0,n}$;*
(3) *$r = 2n$, γ satisfies (1.2), $V_1 \simeq \overline{\pi}_{0,n}$, $V_2 \simeq \pi_1$;*
(4) *$r = 2n$, γ satisfies (1.3), $V_1 \simeq \pi_1$, $V_2 \simeq \overline{\pi}_{0,n}$;*
(5) *$r = 2n$, γ satisfies (1.3), $V_1 \simeq \overline{\pi}_{0,n}$, $V_2 \simeq \pi_0$;*

(6) $r=2n$, $i\in\{2,\dots,n-1\}$, γ *satisfies (1.4),* $V_1\simeq\overline{\pi}_{0,i}$, $V_2\simeq\overline{\pi}_{0,i-1}$, *or* $\overline{\pi}_{0,i+1}$*;*

(7) $r=2n$, γ *satisfies (1.4) for* $i=n$, $V_1\simeq\overline{\pi}_{0,n}$, $V_2\simeq\overline{\pi}_{0,n-1}$, π_0, *or* π_1*;*

(8) $r=2n+1$, $i\in\{2,\dots,n\}$, γ *satisfies (1.5),* $V_1\simeq\overline{\pi}_{0,i}$, $V_2\simeq\overline{\pi}_{0,i-1}$, *or* $\overline{\pi}_{0,i+1}$*;*

(9) $r=2n+1$, γ *satisfies (1.5) for* $i=n+1$, $V_1\simeq\overline{\pi}_{0,n+1}$, $V_2\simeq\overline{\pi}_{0,n}$.

Then there is no nonzero $\mathfrak{g}$*-action in* $I_{\eta,\Lambda,\gamma}$ *which sends an element of* V_1 *to* V_2*.*

Proof

If there is a $\mathfrak{g}$-action sending an element of V_1 to V_2, then the K-spectra $\widehat{K}(V_1)$ and $\widehat{K}(V_2)$ should be adjacent, that is, there should be K-types $\tau_\lambda\in\widehat{K}(V_1)$ and $\tau_{\lambda'}\in\widehat{K}(V_2)$ such that $\lambda'-\lambda$ is a weight of $\mathfrak{s}$. We set $\epsilon_k=\lambda'-\lambda$.

First, we show (1). Recall the discussion in Section 4.2. The nonzero $\mathfrak{s}$-action which sends an element of $V_\lambda^K\subset V_1$ to $V_{\lambda'}^K\subset V_2$ is realized by the shift operator $P_k\phi\neq 0$, where ϕ is a nonzero element in $C^\infty(A\to V_{(\lambda/\gamma)^*}^{K/M^\eta})$ corresponding to an element in $\mathrm{Hom}_K(V_\lambda^K, I_{\eta,\Lambda,\gamma})$.

If $r=2n+1$ and $k=0$, then $P_0\phi=d_\lambda L(\mathbb{PF}_{2n+2})\phi$ by (4.7). Since $\mathbb{PF}_{2n+2}$ is a central element, $P_0\phi$ is a constant multiple of ϕ. So it realizes the K-type $V_\lambda^K\subset V_1$, or it is a zero element. In either case, $P_0\phi$ does not realize the K-type $V_{\lambda'}^K\subset V_2\neq V_1$.

Suppose that k is not zero. Consider the shift $P_{-k}\circ P_k\phi$. Theorem 4.10(1) asserts that $P_{-k}\circ P_k\phi=d_{\lambda,k}L(C_{r+1}(u_k))\phi$. Since $C_{r+1}(u_k)\in Z(\mathfrak{g})$, it acts by the scalar $\chi_\Lambda(C_{r+1}(u_k))$. By (4.4), the image of the Harish-Chandra map of $C_{r+1}(u_k)$ is

$$\chi_\Lambda\big(C_{r+1}(u_k)\big)=\prod_{i=1}^{\lfloor (r+1)/2\rfloor}(u_k^2-\Lambda_i^2).$$

By the definition (4.5) of u_k, the scalar $\chi_\Lambda(C_{r+1}(u_k))$ is zero if and only if one of the following conditions is satisfied:

$$r=2n, k>0,\quad \lambda_k=\Lambda_i-n+k-1/2\ \big(\exists i\in\{1,2,\dots,n\}\big),\tag{5.1}$$

$$r=2n, k<0,\quad \lambda_{|k|}=\Lambda_i-n+|k|+1/2\ \big(\exists i\in\{1,2,\dots,n\}\big),\tag{5.2}$$

$$r=2n, k=n,\quad \lambda_n=\pm\Lambda_i-1/2\ \big(\exists i\in\{1,2,\dots,n\}\big),\tag{5.3}$$

$$r=2n, k=-n,\quad \lambda_n=\pm\Lambda_i+1/2\ \big(\exists i\in\{1,2,\dots,n\}\big),\tag{5.4}$$

$$r=2n+1, k>0,\quad \lambda_k=\Lambda_i-n+k-1\ \big(\exists i\in\{1,2,\dots,n\}\big)\text{ or }|\Lambda_{n+1}|,\tag{5.5}$$

$$r=2n+1, k<0,\quad \lambda_{|k|}=\Lambda_i-n+|k|\ \big(\exists i\in\{1,2,\dots,n\}\big)\text{ or }|\Lambda_{n+1}|+1.\tag{5.6}$$

Recall the K-spectra of irreducible $(\mathfrak{g},K)$-modules (see Theorems 2.4, 2.6). If λ and k satisfy one of (5.1)–(5.6), then the K-type $V_{\lambda'}^K=V_{\lambda+\epsilon_k}^K$ is not a K-type of $V_1\simeq V_2$. Therefore, if $V_1\simeq V_2$ and $P_k\phi\neq 0$, then $P_{-k}\circ P_k\phi$ is a nonzero multiple of ϕ. This is impossible since V_1 and V_2 are different irreducible subquotients and there exist nonzero $\mathfrak{s}$-actions sending an element in V_1 to V_2 and vice versa. Therefore, (1) is proved.

Let us show the case (9). By Theorem 2.6, two K-types $\lambda \in \widehat{K}(V_1)$ and $\lambda' \in \widehat{K}(V_2)$ are adjacent if and only if $\lambda_n = \Lambda_n$, $\lambda'_n = \Lambda_n - 1$ and $\lambda_p = \lambda'_p$ for $p = 1, \ldots, n-1$. In this case, the shift operator sending V_λ^K to $V_{\lambda'}^K$ is P_{-n}. By (5.6), $P_n \circ P_{-n}\phi = 0$. We know that $\gamma_{n-1} \geq \Lambda_n$ since γ satisfies (1.5) for $i = n+1$. On the other hand, $\lambda'_n = \Lambda_n - 1$. Then the condition of Lemma 4.9(5) for $k = n$ (and λ is replaced by λ') is satisfied. So P_n is injective. It follows that $P_{-n}\phi = 0$, so that there is no nonzero $\mathfrak{g}$-action sending V_1 to V_2. This proves (9).

In the same way, we can show (2)–(8). □

In order to determine the second and higher floors of $I_{\eta,\Lambda,\gamma}$, the next theorem is very useful.

THEOREM 5.2 ([7, THEOREM 9.5.1])

In the setting of this paper, suppose that irreducible $(\mathfrak{g}, K)$-modules X and Y are not isomorphic. Then $\mathrm{Ext}^1_{\mathfrak{g},K}(X, Y) \neq 0$ only if $\ell(X) - \ell(Y) \equiv 1 \mod 2$.

This theorem imposes a parity structure on $I_{\eta,\Lambda,\gamma}$. By Yoneda's description of Ext^1, the group $\mathrm{Ext}^1_{\mathfrak{g},K}(X, Y)$ is nonzero if and only if there exists a $(\mathfrak{g}, K)$-module E, which is not isomorphic to $X \oplus Y$, such that

$$0 \to X \to E \to Y \to 0$$

is exact (see [7, Lemma 9.2.2]). We know that the socle of $I_{\eta,\Lambda,\gamma}$ consists of a single irreducible module. We also know that there is no self-extension in $I_{\eta,\Lambda,\gamma}$, by Lemma 5.1(1). Therefore, if the length of the module in the socle is even (resp., odd), then by this theorem, the lengths of the irreducible factors in the second floor are odd (resp., even), the third floor even (resp., odd), and so on.

LEMMA 5.3

Suppose that the unique irreducible submodule of $I_{\eta,\Lambda,\gamma}$ is π. Then the multiplicity of π in $I_{\eta,\Lambda,\gamma}$ is one.

Proof

Assume that there exists a composition factor V_1 which is isomorphic to π but is in the kth floor, $k > 1$. Then there exists an irreducible subquotient V_2 of $I_{\eta,\Lambda,\gamma}$ in the $(k-1)$st floor such that $V_1 \to V_2$. This V_2 must satisfy $\ell(V_1) - \ell(V_2) \equiv 1 \mod 2$ by Theorem 5.2 and Lemma 5.1(1). We know the candidates for V_2 (see Proposition 3.4), and the lengths of them (see Theorems 2.1, 2.5). These data imply that the pair (V_1, V_2) satisfies one of the conditions in Lemma 5.1. But this lemma tells us that there is no nonzero $\mathfrak{g}$-action sending an element of V_1 to V_2. Therefore, $V_1 \nrightarrow V_2$. This is a contradiction. □

Proofs of Theorems 1.1, 1.2

The statements (1) in these theorems are proved in Propositions 3.2 and 3.3.

Let us consider the case when $r = 2n$ and γ satisfies (1.2). In this case, Proposition 3.2(1) says that π_0 is the unique simple submodule of $I_{\eta,\Lambda,\gamma}$, and Proposition 3.4(1) says that only π_0, π_1, or $\overline{\pi}_{0,n}$ can be a composition factor of $I_{\eta,\Lambda,\gamma}$. By Lemma 5.3, the multiplicity of π_0 in $I_{\eta,\Lambda,\gamma}$ is just one. By Theorem 2.1, $\ell(\pi_0) = \ell(\pi_1) = 0$ and $\ell(\overline{\pi}_{0,n}) = 1$. Therefore, Theorem 5.2 implies that the second floor of $I_{\eta,\Lambda,\gamma}$ is a multiple of $\overline{\pi}_{0,n}$, and the third floor of it is a multiple of π_1, and so on. By Proposition 3.5, the multiplicities of $\overline{\pi}_{0,n}$ and π_1 in $I_{\eta,\Lambda,\gamma}$ are at least one. Therefore, the multiplicity of $\overline{\pi}_{0,n}$ (resp., π_1) in the second (resp., third) floor is at least one. We can show that the multiplicity of $\overline{\pi}_{0,n}$ (resp., π_1) in the second (resp., third) floor is just one, in the same way as the proof of [6, Lemma 5.14].

Assume that there exists a nonzero fourth floor in $I_{\eta,\Lambda,\gamma}$. Then there exists a $\mathfrak{g}$-action which sends an element in the fourth floor to the third floor. But the fourth floor is a multiple of $\overline{\pi}_{0,n}$, and the third floor is isomorphic to π_1. This contradicts Lemma 5.1(3). Therefore, there is no fourth floor in $I_{\eta,\Lambda,\gamma}$.

The proof of the case when γ satisfies (1.3) is just the same as above.

The proofs of Theorem 1.1(3), (4) and Theorem 1.2(2), (3) are almost the same as above (easier). For example, suppose that $r = 2n$ and γ satisfies (1.4), $i \in \{2, \ldots, n-1\}$. Then $\overline{\pi}_{0,i}$ is the unique simple submodule of $I_{\eta,\Lambda,\gamma}$. A composition factor of $I_{\eta,\Lambda,\gamma}$ is isomorphic to $\overline{\pi}_{0,i}$, $\overline{\pi}_{0,i-1}$, or $\overline{\pi}_{0,i+1}$. Since $\ell(\overline{\pi}_{0,k}) = n - k + 1$, $k = i-1, i, i+1$, and since the multiplicity of $\overline{\pi}_{0,i}$ in $I_{\eta,\Lambda,\gamma}$ is just one, the second floor of $I_{\eta,\Lambda,\gamma}$ is a direct sum of multiples of $\overline{\pi}_{0,i-1}$ and $\overline{\pi}_{0,i+1}$, and there is no higher floor. The multiplicity freeness of the factors in the second floor is proved in the same way as the proof of [6, Lemma 5.14]. $\square$

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