KO-theory of exceptional flag manifolds

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Abstract The KO-theory of the flag manifold G/T is determined by calculating the Atiyah–Hirzebruch spectral sequence when G is one of the exceptional Lie groups G_2, F_4, E_6, where T is a maximal torus of G.

1. Introduction

This work is a continuation of the work of [KH1], [KH2], [KKO], and [K] in which the KO-theory of various homogeneous spaces are calculated by the Atiyah–Hirzebruch spectral sequence. In [KKO], Kono and the authors calculated the KO-theory of the classical flag manifolds. Here, we mean by the classical (resp., exceptional) flag manifold the compact classical (resp., exceptional) group divided by its maximal torus. We will denote a maximal torus of a compact, connected Lie group G by T. We will calculate the KO-theory of the exceptional flag manifold G/T for $G=G_2,F_4,E_6$. Recently, a connection between Witt groups and KO-theory of homogeneous spaces such as Grassmannians and flag manifolds was found (see [Z], [Y1], [Y2]), and so our calculation has applications not only in topology but also in this direction. Our main result is the following.

THEOREM 1.1

The KO-theory of G/T for $G=G_2,F_4,E_6$ is given as

$$KO^{2n-1}(G/T)\cong(\mathbb{Z}/2)^{s_n}\qquad \text{and}\qquad KO^{2n}(G/T)\cong(\mathbb{Z}/2)^{s_{n+1}}\oplus\mathbb{Z}^t$$

for $n\in\mathbb{Z}/4$, where t,s_n are as in the following table:

G	t	s_0	s_{-1}	s_{-2}	s_{-3}
G_2	6	1	2	1	0
F_4	576	2	4	6	4
E_6	25920	2	4	6	4

The organization of the paper is as follows. In Section 2, we recall from [KH1] and [KH2] useful lemmas in calculating the Atiyah–Hirzebruch spectral sequence converging to the KO-theory. We also recall some basic facts on the self-conjugate K-theory. In Section 3, we consider the homotopy fiber of a certain cohomology

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class BT^6 studied in [KI1] and related spaces. Results in this section will be used in calculating the KO-theory of F_4/T and E_6/T. In Section 4, we determine the KO-theory of G_2/T. In Section 5, we first calculate the KO-theory of F_4/U for some maximal rank subgroup U of F_4. After this, we determine the KO-theory of F_4/T. In Section 6, we calculate the KO-theory of E_6/T by using a method similar to that for F_4/T.

2. Atiyah–Hirzebruch spectral sequence

2.1. KO-theory

Recall that the coefficient of KO-theory is given as

$$KO^* = \mathbb{Z}[\eta, \lambda, \beta, \beta^{-1}]/(2\eta, \eta^3, \eta\lambda, \lambda^2 - 4\beta)$$

for $|\eta| = -1$, $|\lambda| = -4$, $|\beta| = -8$. Let $(E_r(X), d_r)$ be the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q}(X) \cong H^p(X; KO^q) \Longrightarrow KO^*(X).$$

It is shown in [F] that the second differential d_2 is given as

$$d_2^{p,q} = \begin{cases} \mathrm{Sq}^2\pi_2, & q \equiv 0 \mod 8, \\ \mathrm{Sq}^2, & q \equiv -1 \mod 8, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where π_2 is the modulo 2 reduction. We now suppose the following condition of a space X.

(2.2) $\quad H^{2n}(X; \mathbb{Z})$ is a free abelian group, and $H^{2n+1}(X; \mathbb{Z}) = 0$ for $n \geq 0$.

Then for $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1 = 0$, $(H^*(X; \mathbb{Z}/2), \mathrm{Sq}^2)$ is a chain complex. We denote the cohomology of $(H^*(X; \mathbb{Z}/2), \mathrm{Sq}^2)$ by $H^*(X; \mathrm{Sq}^2)$ and call it the Sq^2-cohomology of X. It follows from (2.1) that there is an isomorphism

$$\iota : E_3^{p,-1}(X) \xrightarrow{\cong} H^p(X; \mathrm{Sq}^2). \tag{2.3}$$

The following useful lemma is proved in [KH1] and [KH2].

LEMMA 2.1

Let X be a CW-complex satisfying (2.2). Suppose that r is the smallest integer such that $d_r \neq 0$ for $r \geq 3$. Then the following hold.

(1) *We have $r \equiv 2 \mod 8$.*

(2) *If p is the smallest integer such that $d_r^{p,q} \neq 0$, there exists $x \in E_r^{p,0}(X)$ satisfying $d_r(\eta x) \neq 0$, and $\iota(\eta x)$ is indecomposable in $H^p(X; \mathrm{Sq}^2)$.*

(3) *Let x be as in (2). Suppose that there is a map $X \times X \to X$ by which $H^*(X; \mathrm{Sq}^2)$ becomes a Hopf algebra. Then $d_r x$ is primitive in $H^*(X; \mathrm{Sq}^2)$.*

Let us consider an extension of $E_\infty(X)$ to $KO^*(X)$.

LEMMA 2.2

Let X be a finite CW-complex satisfying (2.2). Then there exist integers s_n, t_n for $n \in \mathbb{Z}/4$ and isomorphisms

$$KO^{2n-1}(X) \cong (\mathbb{Z}/2)^{s_n} \qquad \text{and} \qquad KO^{2n}(X) \cong (\mathbb{Z}/2)^{s_{n+1}} \oplus \mathbb{Z}^{t_n}.$$

Proof

By assumption, the complex K-theory satisfies $K^{-1}(X)=0$, and by the Atiyah–Hirzebruch spectral sequence $(E_r(X), d_r)$, one sees that $KO^{2n-1}(X)$ is a torsion group. Then since the composite $KO^*(X) \xrightarrow{\mathbf{c}} K^*(X) \xrightarrow{\mathbf{r}} KO^*(X)$ is the 2-power map for the complexification $\mathbf{c}$ and the realization $\mathbf{r}$, it follows that $KO^{2n-1}(X) \cong (\mathbb{Z}/2)^{s_n}$ for some integer s_n. There is the Bott exact sequence

$$\cdots \to K^{*-1}(X) \to KO^{*+1}(X) \xrightarrow{\eta} KO^*(X) \xrightarrow{\mathbf{c}} K^*(X) \to \cdots.$$

Since $K^0(X)$ is a free abelian group and $K^{-1}(X)=0$ by assumption, $\eta : KO^{2n-1}(X) \to KO^{2n}(X)$ is an isomorphism on the torsion part. Thus the proof is completed. $\square$

We calculate integers s_n, t_n in Lemma 2.2. Define formal series $f_X(t)$ and $g_X(t)$ as

$$(2.4) \quad f_X(t) = \sum_{p\ge 0} \dim_{\mathbb{Q}} H^p(X;\mathbb{Q}) t^p \qquad \text{and} \qquad g_X(t) = \sum_{p \ge 0} \dim_{\mathbb{Z}/2} E_\infty^{p,-1}(X) t^p.$$

By [MT], the polynomial $f_X(t)$ for $G = G_2/T, F_4/T, E_6/T$ is given as

$$(2.5) \qquad f_X(t) = \begin{cases} \frac{(1-t^4)(1-t^{12})}{(1-t^2)^2}, & X = G_2/T, \\ \frac{(1-t^4)(1-t^{12})(1-t^{16})(1-t^{24})}{(1-t^2)^4}, & X = F_4/T, \\ \frac{(1-t^4)(1-t^{10})(1-t^{12})(1-t^{16})(1-t^{18})(1-t^{24})}{(1-t^2)^6}, & X = E_6/T. \end{cases}$$

LEMMA 2.3

Let X be a finite CW-complex satisfying (2.2), and let s_n, t_n be as in Lemma 2.2. Then it holds that

$$t_0 = t_{-2} = \frac{f_X(1) + f_X(\sqrt{-1})}{2}, \qquad t_{-1} = t_{-3} = \frac{f_X(1) - f_X(\sqrt{-1})}{2},$$

and

$$\begin{pmatrix} s_0 \\ s_{-1} \\ s_{-2} \\ s_{-3} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & -2 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} g_X(1) \\ g_X(\sqrt{-1}) \\ \mathrm{Re} g_X(\frac{1+\sqrt{-1}}{\sqrt{2}}) \\ \mathrm{Im} g_X(\frac{1+\sqrt{-1}}{\sqrt{2}}) \end{pmatrix}.$$

Proof

Since the Atiyah–Hirzebruch spectral sequences for rationalized cohomology theories are trivial, we have

$$t_0 = t_{-2} = \sum_{n\geq 0} \dim_{\mathbb{Q}} H^{4n}(X;\mathbb{Q}) \qquad \text{and} \qquad t_{-1} = t_{-3} = \sum_{n\geq 0} \dim_{\mathbb{Q}} H^{4n+2}(X;\mathbb{Q}),$$

and then the first two equalities follow. Notice that Lemma 2.2 implies that the extension of $\bigoplus_{p+q=2n-1} E_\infty^{p,q}(X)$ to $KO^{2n-1}(X)$ is trivial. Then by Bott periodicity and $E_\infty^{p,q}(X)=0$ for odd q with $q \not\equiv -1 \mod 8$, we have

$$KO^{2n-1}(X) \cong \bigoplus_{p+q=2n-1} E_\infty^{p,q}(X) \cong \bigoplus_{4k+n\geq 0} E_\infty^{8k+2n,-1}(X).$$

On the other hand, we have

$$g_X(t) = \sum_{n=0}^{3} \sum_{k\geq 0} \dim_{\mathbb{Z}/2} E_\infty^{8k+2n,-1}(X) t^{8k+2n}.$$

Then for $\omega = \frac{1+\sqrt{-1}}{\sqrt{2}}$, a primitive 8th root of unity, we get

$$g_X(\omega^\ell) = \sum_{n=0}^{3} \omega^{2\ell n} s_n = \begin{cases} s_0 + s_{-1} + s_{-2} + s_{-3}, & \ell = 0, \\ s_0 - \sqrt{-1}s_{-1} - s_{-2} + \sqrt{-1}s_{-3}, & \ell = 1, \\ s_0 - s_{-1} + s_{-2} - s_{-3}, & \ell = 2, \end{cases}$$

and thus the last equality follows. □

2.2. Self-conjugate K-theory

Let us next consider self-conjugate K-theory. Our basic reference is [A]. We denote the self-conjugate K-theory of a space X by $KSC^*(X)$. The coefficient of self conjugate K-theory is periodic by multiplication by a generator of KSC^{-4}. Moreover, there is an exact sequence

$$\cdots \to KO^{*+2}(X) \xrightarrow{\eta^2} KO^*(X) \xrightarrow{\mathbf{c}} KSC^*(X) \to KO^{*+3}(X) \to \cdots,$$

where $\mathbf{c}$ is the complexification. Then it follows that

$$KSC^* \cong \begin{cases} \mathbb{Z}, & * \equiv 0, -3 \mod 4, \\ \mathbb{Z}/2, & * \equiv -1 \mod 4, \\ 0, & * \equiv -2 \mod 4, \end{cases}$$

and $\mathbf{c}: KO^* \to KSC^*$ is an isomorphism for $* \equiv 0, -1 \mod 8$. Let $({}'E_r, {}'d_r)$ be the Atiyah–Hirzebruch spectral sequence

$${}'E_2^{p,q} \cong H^p(X; KSC^q) \Longrightarrow KSC^*(X).$$

LEMMA 2.4

Let X be a CW-complex satisfying (2.2).

(1) *The complexification*

$$\mathbf{c}: E_3^{p,q}(X) \to {}'E_3^{p,q}(X)$$

is an isomorphism for $q \equiv 0 \mod 8$ and a monomorphism for $q \equiv -1 \mod 8$.

(2) *If r is the least integer such that $'d_r \neq 0$ for $r \geq 3$, then*

$$r \equiv 2 \mod 8 \qquad \textit{and} \qquad 'd_r^{*,0} \neq 0.$$

Proof

(1) This follows from the above observation on $\mathbf{c} : KO^* \to KSC^*$. (2) Quite similarly to the proof of Lemma 2.1, we see that $r \equiv 2 \mod 4$ and $'d_r^{*,0} \neq 0$. By (1), we further see that $r \equiv 2 \mod 8$, completing the proof. □

REMARK 2.5

All results in this section hold if we localize at the prime 2 and will be used in the proof of Theorem 3.7 below.

3. KO-theory of a space related with a torus

In [KI1], the cohomology of BT^6 in connection with the Weyl group action of E_6 is given as

$$H^*(BT^6;\mathbb{Z}) = \mathbb{Z}[t, t_1, \ldots, t_6]/(t_1 + \cdots + t_6 - 3t), \quad |t| = |t_i| = 2.$$

Generalizing, we may put

$$H^*(BT^N;\mathbb{Z}) = \mathbb{Z}[t, t_1, \ldots, t_N]/(t_1 + \cdots + t_N - 3t), \quad |t| = |t_i| = 2,$$

for $N \geq 6$, which respects the above case of $N = 6$. Let c_i be the elementary symmetric function in $t_1, \ldots, t_N$, and let $y_4 = c_2 - 4t^2 \in H^4(BT^N;\mathbb{Z})$. Define $B\widetilde{T}^N$ as the homotopy fiber of

$$y_4 : BT^N \to K(\mathbb{Z}, 4),$$

where $B\widetilde{T}^6$ is the 4-connective cover of BT^6 in the sense of [KI1]. Let us calculate the mod 2 cohomology of $B\widetilde{T}^N$ following [KI1]. Define $\bar{c}_{2^i+1} \in \mathbb{Z}/2[t_1, \ldots, t_N]$ for $i \geq 0$ inductively as

$$\bar{c}_2 = c_2 \qquad \text{and} \qquad \bar{c}_{2^i+1} = \mathrm{Sq}^{2^i}\bar{c}_{2^{i-1}+1}.$$

PROPOSITION 3.1

The mod 2 cohomology of $B\widetilde{T}^N$ is given as

$$H^*(B\widetilde{T}^N;\mathbb{Z}/2) = \mathbb{Z}/2[t_1, \ldots, t_N, \gamma_{2^i+1} \mid i \geq 1]/(\bar{c}_{2^i+1} \mid i \geq 0)$$

for $* \leq 2N$, where $|\gamma_{2^i+1}| = 2(2^i + 1)$.

Proof

Let us consider the Serre spectral sequence of a homotopy fiber sequence

$$K(\mathbb{Z}, 3) \to B\widetilde{T}^N \to BT^N.$$

Recall that the mod 2 cohomology of $K(\mathbb{Z}, 3)$ is given as

$$H^*\big(K(\mathbb{Z}, 3);\mathbb{Z}/2\big) = \mathbb{Z}/2[u_{2^i+1} \mid i \geq 1],$$

where u_3 is the modulo 2 reduction of the fundamental class and $u_{2^i+1} = \mathrm{Sq}^{2^{i-1}} u_{2^{i-1}+1}$ for $i \geq 2$. By the definition of $B\widetilde{T}^N$, the transgression τ satisfies $\tau(u_3) = c_2$ $(= \bar{c}_2)$, and then $\tau(u_{2^i+1}) = \bar{c}_{2^i+1}$ for $i \geq 0$. Inductively, one sees that $\bar{c}_{2^i+1}$ includes the term c_{2^i+1}, implying that $\{\bar{c}_{2^i+1} \mid 2 \leq 2^i + 1 \leq n\}$ is a regular sequence in $\mathbb{Z}/2[t_1, \ldots, t_N]$. On the other hand, since u_3^2 is a permanent cycle, there exists $\gamma_3 \in H^6(B\widetilde{T}^N; \mathbb{Z}/2)$ which restricts to u_3^2. Put

$$\gamma_{2^i+1} = \mathrm{Sq}^{2^i} \gamma_{2^{i-1}+1}$$

for $i \geq 2$. By the Cartan formula, we have that γ_{2^i+1} restricts to $u_{2^i+1}^2$. Summarizing the above calculation, we obtain the desired result, where we need the condition $* \leq N$ for regularity of $\{\bar{c}_{2^i+1} \mid i \geq 0\}$. □

There is a sequence of natural maps

$$B\widetilde{T}^N \to B\widetilde{T}^{N+1} \to B\widetilde{T}^{N+2} \to \cdots.$$

We denote the colimit of this sequence by $B\widetilde{T}^\infty$. Then by Proposition 3.1, the Milnor exact sequence shows the following. Let R be a graded algebra over $\mathbb{Z}/2$ consisting of finite sums of homogeneous formal power series in $t_1, t_2, \ldots$ with $|t_i| = 2$.

COROLLARY 3.2

The mod 2 cohomology $B\widetilde{T}^\infty$ is given as

$$H^*(B\widetilde{T}^\infty; \mathbb{Z}/2) = R \otimes \mathbb{Z}/2[\gamma_{2^i+1} \mid i \geq 1]/(\bar{c}_{2^i+1} \mid i \geq 0).$$

In particular, for $n \geq 0$, $H^{2n}(B\widetilde{T}^\infty; \mathbb{Z}_{(2)})$ is a free $\mathbb{Z}_{(2)}$-module and $H^{2n+1}(B\widetilde{T}^\infty; \mathbb{Z}_{(2)}) = 0$.

Let us next calculate the Sq^2-cohomology of $B\widetilde{T}^N$ up to a certain dimension. To this end, we recall from [KH1] a special cohomology calculation.

LEMMA 3.3

Let (A, d) be a differential graded algebra over a field.

(1) *Suppose that for $a \in A^n$, da is a nonzero divisor and $a^2 = db$ for some $b \in A^{2n-1}$. Then it holds that*

$$H^*\bigl(A/(da)\bigr) \cong \Lambda(a) \otimes H^*(A).$$

(2) *Suppose that for $a \in A^n$, $\{a, da\}$ is a regular sequence and $a^2 = db, b^2 = dc$ for some $b \in A^{2n-1}, c \in A^{4n-3}$. Then it holds that*

$$H^*\bigl(A/(a, da)\bigr) \cong \Lambda(b) \otimes H^*(A).$$

Proof

(1) Since da is a nonzero divisor, there is a short exact sequence

$$0 \to A \xrightarrow{\cdot da} A \to A/(da) \to 0$$

which induces a long exact sequence

$$\cdots \to H^*(A) \xrightarrow{\cdot H^*(da)} H^{*+n+1}(A) \to H^{*+n+1}\big(A/(da)\big) \xrightarrow{\delta} H^{*+1}(A) \to \cdots,$$

where $A/(da)$ is, of course, a differential graded algebra. Obviously, $H^*(da) = 0$ and $\delta(a) = 1$. Then it follows that $H^*(A/(da))$ is a free $H^*(A)$-module with a basis $\{1, a\}$. Since $a^2 = db$, we obtain the desired result.

(2) Since $\{a, da\}$ is a regular sequence, there is an exact sequence

$$\cdots \to H^*\big(A/(da)\big) \xrightarrow{\cdot H^*(a)} H^{*+n}\big(A/(da)\big)$$
$$\to H^{*+n}\big(A/(a,da)\big) \xrightarrow{\delta} H^{*+1}\big(A/(da)\big) \to \cdots$$

as well as that in (1), in which $\delta(b) = a$. Since $H^*(A/(da)) \cong \Lambda(a) \otimes H^*(A)$ by (1), we see that $H^*(A/(a,da))$ is a free $H^*(A)$-module with a basis $\{1, b\}$. For $b^2 = dc$, the proof is completed. □

PROPOSITION 3.4

For $* \leq 2N-2$,

$$H^*(B\widetilde{T}^N; \mathrm{Sq}^2) = \Lambda(x_3, x_7, x_{2^i} \mid i \geq 3), \quad |x_j| = 2j,$$

where N *can be* ∞.

Proof

Put $A = \mathbb{Z}/2[t_1, \ldots, t_N]$ (or the above R for $N = \infty$). Notice that since A is acyclic under Sq^2, for any $x \in A^+$, there exists $y \in A$ satisfying $x^2 = dy$.

By Lemma 3.3, we have

$$H^*\big(A/(\bar{c}_2, \bar{c}_3)\big) = \Lambda(x_3),$$

where $x_3 = \sum_{i<j} t_i t_j^2$ satisfying $\mathrm{Sq}^2 x_3 = c_2^2$. The Adem relation $\mathrm{Sq}^2\mathrm{Sq}^{2^i} = \mathrm{Sq}^{2^i+2} + \mathrm{Sq}^{2^i+1}\mathrm{Sq}^1$ implies that

$$\mathrm{Sq}^2 \bar{c}_{2^i+1} = \bar{c}_{2^{i-1}+1}^2 \tag{3.1}$$

for $i \geq 2$. On the other hand, as is noted in the proof of Proposition 3.1, $\{\bar{c}_{2^i+1} \mid 2 \leq 2^i + 1 \leq N\}$ is a regular sequence in A. Then, applying Lemma 3.3 repeatedly, one gets

$$H^*\big(A/(\bar{c}_{2^i+1} \mid i \geq 0)\big) = \Lambda(x_3, x_{2^i} \mid i \geq 2)$$

for $* \leq 2N$, where $\mathrm{Sq}^2 x_{2^i} \equiv \bar{c}_{2^i+1} \mod (\bar{c}_{2^j+1} \mid 0 \leq j \leq i-1)$. Notice here that since $H^{2(2^{i+1}+1)}(A/(\bar{c}_{2^j+1} \mid j \geq 0)) = 0$, we can apply Lemma 3.3 repeatedly. Since $\mathrm{Sq}^2 c_4 = \bar{c}_5 \mod (\bar{c}_2, \bar{c}_3)$, we may take $x_4 = c_4$.

Put $F_0 = A/(\bar{c}_{2^i+1} \mid i \geq 0)$ and $F_n = A/(\bar{c}_{2^i+1} \mid i \geq 0) \otimes \mathbb{Z}/2[\gamma_{2^i+1} \mid i \leq n-1]$ for $n \geq 1$. It is proved in [KI1] that $\mathrm{Sq}^2 \gamma_3 = c_4$. Consider the spectral sequence associated with a filtration $F_0 \subset F_1$. Then we get

$$H^*(F_1) = \Lambda(x_3, x_7, x_{2^i} \mid i \geq 3) \otimes \mathbb{Z}/2[\gamma_3^2],$$

where $x_7 = \gamma_3 c_4 + d_7$ for $d_7 \in A$ with $\mathrm{Sq}^2 d_7 = c_4^2$. Similarly to (3.1), we have $\mathrm{Sq}^2 \gamma_{2^i+1} = \gamma_{2^{i-1}+1}$. Then by considering the spectral sequence associated with a filtration $F_n \subset F_{n+1}$ for $n \geq 1$ inductively, we obtain

$$H^*(F_{n+1}) = \Lambda(x_3, x_7, x_{2^i} \mid i \geq 3) \otimes \mathbb{Z}/2[\gamma_{2^n+1}^2].$$

Thus the proof is completed. □

Let us next consider the homotopy fiber F of the cohomology class $t : B\widetilde{T}^\infty \to K(\mathbb{Z}, 2)$. Let $\alpha : F \to B\widetilde{T}^\infty$ be the natural map.

PROPOSITION 3.5

For $n \geq 0$, $H^{2n}(F; \mathbb{Z}_{(2)})$ is a free $\mathbb{Z}_{(2)}$-module and $H^{2n+1}(F; \mathbb{Z}_{(2)}) = 0$.

Proof

By Proposition 3.1, for $* \leq 2N$, the same claim is true for $B\widetilde{T}^N$ and then also for $B\widetilde{T}^\infty$ by sending N to ∞. Since the map $t : B\widetilde{T}^\infty \to K(\mathbb{Z}, 2)$ is injective in the $\mathbb{Z}_{(2)}$-cohomology, $\alpha^* : H^*(B\widetilde{T}^\infty; \mathbb{Z}_{(2)}) \to H^*(F; \mathbb{Z}_{(2)})$ is surjective, and thus the proof is completed. □

Define a map $\mu : BT^\infty \times BT^\infty \to BT^\infty$ by the equations

$$\mu^*(t_{2i}) = 1 \otimes t_i \qquad \text{and} \qquad \mu^*(t_{2i-1}) = t_i \otimes 1$$

for $i \geq 1$ in cohomology. Then by an easy inspection we see that μ lifts to a map $\tilde{\mu} : F \times F \to F$.

PROPOSITION 3.6

The natural map $\alpha : F \to B\widetilde{T}^\infty$ induces an isomorphism in the Sq^2-cohomology. Moreover, $H^*(F; \mathrm{Sq}^2)$ becomes a Hopf algebra by $\tilde{\mu}$ in which $\alpha^*(x_{2^i})$ is not primitive for $i \geq 4$, where x_j is as in Proposition 3.4.

Proof

The first assertion easily follows from a direct calculation.

Computing the Sq^2-cohomology of the subring $\mathbb{Z}/2[c_1, c_2, c_3, \ldots]/(c_1, \bar{c}_2, \bar{c}_3, \ldots)$ of $H^*(F; \mathbb{Z}/2)$, we see that $\alpha^*(x_{2^i})$ can be chosen as an element of this subring for $i \geq 3$. Then for

$$\tilde{\mu}^*\big(\alpha^*(c_n)\big) = \sum_{i=0}^{n} \alpha^*(c_i) \otimes \alpha^*(c_{n-i}), \tag{3.2}$$

we obtain

$$\tilde{\mu}^*\big(\alpha^*(x_{2^i})\big) = \alpha^*(x_{2^i}) \otimes 1 + 1 \otimes \alpha^*(x_{2^i}) + \cdots .$$

Choose representatives of x_3, x_7 as in the proof of Proposition 3.4. As in [KKO], it is straightforward to see that $\tilde{\mu}^*(\alpha^*(x_3)) = x_3 \otimes 1 + 1 \otimes x_3$. By definition, we have $\tilde{\mu}^*(\alpha^*(\gamma_3)) = \alpha^*(\gamma_3) \otimes 1 + 1 \otimes \alpha^*(\gamma_3) + \cdots$. Then by an easy calculation analogous to $\alpha^*(x_3)$, we see that $\tilde{\mu}^*(\alpha^*(x_7)) = \alpha^*(x_7)\otimes 1 + 1\otimes \alpha^*(x_7)$. Thus we have obtained that $H^*(F;\mathrm{Sq}^2)$ is a Hopf algebra by the map $\tilde{\mu}$.

Since $\bar{c}_{2^i+1} = c_{2^i+1} + \cdots$ as above, we have $x_{2^i} = c_{2^i} + \cdots$ for $i \geq 3$. Then by (3.2), the last assertion follows. □

We now aim at proving the following.

THEOREM 3.7

The Atiyah–Hirzebruch spectral sequence $E_r(B\widetilde{T}^\infty)_{(2)}$ collapses at the E_3-term.

Proof

By Corollary 3.2, $B\widetilde{T}^\infty$ satisfies the condition (2.2) at the prime 2. Let $\bar{x}_j$ be an element of $\mathrm{Ker}\{\mathrm{Sq}^2 : H^*(B\widetilde{T}^\infty;\mathbb{Z}_{(2)}) \to H^*(B\widetilde{T}^\infty;\mathbb{Z}/2)\} \cong E_3^{*,0}(B\widetilde{T}^\infty)_{(2)}$ whose modulo 2 reduction is $x_j \in H^*(B\widetilde{T}^\infty;\mathrm{Sq}^2)$ for $j = 3, 7, 2^i$ $(i \geq 3)$. Then by Lemma 2.1, our aim is to prove that $\bar{x}_j$ is a permanent cycle for $j = 3, 7, 2^i$ $(i \geq 3)$.

Consider the natural map $\alpha : F \to B\widetilde{T}^\infty$. Then it follows from Lemma 2.1, Proposition 3.5, and Proposition 3.6 that it is sufficient to show that $\alpha^*(\bar{x}_3) \in \mathrm{Ker}\{\mathrm{Sq}^2 : H^*(F;\mathbb{Z}_{(2)}) \to H^*(F;\mathbb{Z}/2)\} \cong E_3^{*,0}(F)_{(2)}$ is a permanent cycle. We next consider the complexification $\mathbf{c} : E_r(F)_{(2)} \to {}'E_r(F)_{(2)}$. Then by Lemma 2.4, we only have to prove that $\mathbf{c}(\alpha^*(\bar{x}_3)) \in {}'E_3(F)_{(2)}$ is a permanent cycle.

Let u be a generator of $K^{-2}_{(2)}$ satisfying $(1-\mathbf{t})(u) = 0$ for the complex conjugation $\mathbf{t}$, and let H_i be the pullback of the Hopf bundle on BT^1 by the composite $F \to BT^\infty \to BT^1$ in which the first arrow is the natural map and the second arrow corresponds to the cohomology class t_i. Put $\xi_3 = u^{-3}\sum_{i<j} H_i H_j^2 \in K^6(B\widetilde{T}^\infty)_{(2)}$. Then for $(1-\mathbf{t})(\xi_3) = 0$, ξ_3 lies in $KSC^6(F)_{(2)}$. Obviously, ξ_3 corresponds to $\mathbf{c}(\alpha^*(\bar{x}_3))$, and thus $\mathbf{c}(\alpha^*(\bar{x}_3))$ is a permanent cycle, as is desired. □

4. KO-theory of G_2/T

The mod 2 cohomology of G_2/T including the action of the Steenrod operations is calculated as

$$H^*(G_2/T;\mathbb{Z}/2) = \mathbb{Z}/2[t_1, t_2, \gamma_3]/(\rho_2, \rho_3, \gamma_3^2), \quad |t_i| = 2, |\gamma_3| = 6, \mathrm{Sq}^2\gamma_3 = 0,$$

where

$$\rho_2 = t_1^2 + t_1t_2 + t_2^2 \qquad \text{and} \qquad \rho_3 = t_1^2t_2 + t_1t_2^2.$$

PROPOSITION 4.1

The Sq^2*-cohomology of* G_2/T *is given as*

$$H^*(G_2/T;\mathrm{Sq}^2) = \Lambda(x_3, \gamma_3),$$

where $x_3 = t_1^3 + t_1t_2^2 + t_2^3$.

Proof

Since $\mathrm{Sq}^2\rho_2 = \rho_3$, we obtain the desired result by Lemma 3.3. □

COROLLARY 4.2
The Atiyah–Hirzebruch spectral sequence $E_r(G_2/T)$ collapses at the E_3-term. In particular, we have

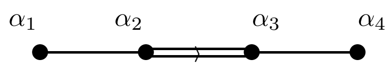
$$g_{G_2/T}(t) = (1+t^6)^2.$$

Proof
The result follows from Lemma 2.1 and Proposition 4.1. □

Proof of Theorem 1.1 for G_2
The result follows from (2.5), Lemma 2.2(1), and Corollary 4.2. □

5. KO-theory of F_4/T

Recall that the Dynkin diagram of F_4 is given as follows:



It is shown in [IT] that the centralizer of the circle in F_4 defined by $\alpha_2 = \alpha_3 = \alpha_4 = 0$ is isomorphic to $T^1 \cdot \mathrm{Sp}(3)$. Let U be the centralizer of the torus defined by $\alpha_2 = 0$. Then $U \cong T^3 \times \mathrm{Sp}(1)$ as a space, implying that the homology of U is torsion-free. Note that F_4/U satisfies the condition (2.2). Then we calculate the Atiyah–Hirzebruch spectral sequence converging to $KO^*(F_4/U)$ from which we deduce the one converging to $KO^*(F_4/T)$.

5.1. KO-theory of F_4/U

We first calculate the mod 2 cohomology of F_4/U. Let ω_i $(i = 1, 2, 3, 4)$ be the fundamental weight of F_4 as in [TW], and put

$$t = \omega_1, \qquad y_1 = \omega_2 - \omega_3, \qquad y_2 = \omega_3 - \omega_4, \qquad y_4 = \omega_4.$$

Then it is clear that

$$H^*(BT;\mathbb{Z}) = \mathbb{Z}[t, y_1, y_2, y_3].$$

As in [IT], the Weyl group of U is generated by a single element R satisfying

$$R(t) = t, \qquad R(y_1) = t - y_1, \qquad R(y_2) = y_2, \qquad R(y_3) = y_3.$$

Since $H^*(BU;\mathbb{Z})$ is torsion-free as noted above, $H^*(BU;\mathbb{Z})$ is the invariant ring of $H^*(BT;\mathbb{Z})$ under the action of the Weyl group of U. Then one gets

$$H^*(BU;\mathbb{Z}) = \mathbb{Z}[t, y_2, y_3, q], \quad q = y_1(t - y_1).$$

On the other hand, the mod 2 cohomology of F_4 is given as

$$H^*(F_4;\mathbb{Z}/2) = \mathbb{Z}/2[a_3]/(a_3^4) \otimes \Lambda(a_5, a_{15}, a_{23}), \quad |a_i| = i, \beta a_5 = a_3^2.$$

Then by a result of Toda [T], we can calculate the $\mathbb{Z}_{(2)}$-coefficient cohomology of F_4/U as follows.

PROPOSITION 5.1

There is a regular sequence $\bar\rho_2,\bar\rho_6,\bar\rho_8,\bar\rho_{12}$ in $\mathbb{Z}_{(2)}[t,y_2,y_3,q]$ with $|\bar\rho_i|=2i$ such that

$$H^*(F_4/U;\mathbb{Z}_{(2)})=\mathbb{Z}_{(2)}[t,y_2,y_3,q,\gamma_3]/(\bar\rho_2,\bar\rho_6,\bar\rho_8,\bar\rho_{12},2\gamma_3+\bar\rho_3),$$

where $\bar\rho_3$ is defined by the equation $\mathrm{Sq}^2\bar\rho_2=\bar\rho_3$.

We now determine the mod 2 cohomology of F_4/U. Define $q_i\in\mathbb{Z}[t,y_2,y_3,q]$ ($|q_i|=4i$) as

$$1+q_1+q_2+q_3=(1+q)\big(1+y_2(t-y_2)\big)\big(1+y_3(t-y_3)\big).$$

By definition, one has

$$\mathrm{Sq}^2q_1=tq_1,\qquad \mathrm{Sq}^2q_2=0,\qquad \mathrm{Sq}^2q_3=tq_3. \tag{5.1}$$

A calculation in [IT] implies that the rational cohomology of F_4/U is given as

$$H^*(F_4/U;\mathbb{Q})=\mathbb{Q}[t,y_2,y_3,q]/(\sigma_2,\sigma_6,\sigma_8,\sigma_{12}), \tag{5.2}$$

where

$$\begin{aligned}&\sigma_2=-t^2+q_1, &&\sigma_6=-t^6+4t^2q_2-8q_3,\\ &\sigma_8=3t^2q_3-q_2^2, &&\sigma_{12}=-q_2^3+27q_3^2.\end{aligned} \tag{5.3}$$

Let $\bar\rho_i$ ($i=2,6,8,12$) be as in Proposition 5.1. Then by (5.1) and (5.3), we may put

$$\bar\rho_2=-t^2+q_1\qquad\text{and}\qquad\bar\rho_3=tq_1.$$

Put

$$R=\mathbb{Z}_{(2)}[t,y_2,y_3,q,\gamma_3]/(\bar\rho_2,\bar\rho_3,-\gamma_3^2+t^2q_2-2q_3,\sigma_8,\sigma_{12}).$$

Since $\sigma_6\equiv4(-\gamma_3^2+t^2q_2-2q_3)\mod(\bar\rho_2,\bar\rho_3)$ and the natural map $H^*(F_4/U;\mathbb{Z}_{(2)})\to H^*(F_4/U;\mathbb{Q})$ is injective, there is a surjection $R\to H^*(F_4/U;\mathbb{Z}_{(2)})$ which induces a surjection

$$\phi:R/2\to H^*(F_4/U;\mathbb{Z}/2).$$

We now put

$$\begin{aligned}&\rho_2=t^2+q_1, &&\rho_3=tq_1, &&\rho_6=\gamma_3^2+t^2q_2,\\ &\rho_8=t^2q_3+q_2^2, &&\rho_{12}=q_2^3+q_3^2.\end{aligned} \tag{5.4}$$

Then since the Poincaré series of F_4/U over $\mathbb{Q}$ and $\mathbb{Z}/2$ are the same, we have

$$R/2=\mathbb{Z}/2[t,y_2,y_3,q,\gamma_3]/(\rho_2,\rho_3,\rho_6,\rho_8,\rho_{12}),$$

here in the Poincaré series, and γ_3 is cancelled by ρ_3. One can easily verify that $\rho_2,\rho_3,\rho_6,\rho_8,\rho_{12}$ is a regular sequence in $\mathbb{Z}/2[t,y_2,y_3,q,\gamma_3]$, implying that the Poincaré series of $R/2$ is $((1-t^{12})(1-t^{16})(1-t^{24}))/(1-t^2)^3$. On the other hand, the Poincaré series of $H^*(F_4/U;\mathbb{Z}/2)$ is equal to that of $H^*(F_4/U;\mathbb{Q})$

which is $((1-t^{12})(1-t^{16})(1-t^{24}))/(1-t^2)^3$ by (5.2). Then we conclude that Poincaré series of $R/2$ and $H^*(F_4/U;\mathbb{Z}/2)$ are the same, and thus the map ϕ is an isomorphism. Summarizing, we obtain the following.

PROPOSITION 5.2

The mod 2 cohomology of F_4/U is given as

$$H^*(F_4/U;\mathbb{Z}/2)=\mathbb{Z}/2[t,y_2,y_3,q,\gamma_3]/(\rho_2,\rho_3,\rho_6,\rho_8,\rho_{12}),$$

where $|t|=|y_2|=|y_3|=2, |q|=4, |\gamma_3|=6$, and ρ_i is as in (5.4).

COROLLARY 5.3

The Sq^2-cohomology of F_4/U is given as

$$H^*(F_4/U;\mathrm{Sq}^2)=\Lambda(x_7,x_{11},\bar{\gamma}_3),\quad |x_i|=2i, |\bar{\gamma}_3|=6,$$

where $\mathrm{Sq}^2x_7\equiv\rho_8 \mod (\rho_2,\rho_3)$, $\mathrm{Sq}^2x_{11}=\rho_{12}$, $\bar{\gamma}_3=\gamma_3+\delta_3$, and $\mathrm{Sq}^2\delta_3=q_2$ for $\delta_3\in\mathbb{Z}/2[t,y_2,y_3,q]$.

Proof

Considering the projection $F_4/T\to F_4/U$, one sees from [KI2] that

$$\mathrm{Sq}^2\gamma_3=q_2.$$

Let A be a differential graded algebra $\mathbb{Z}/2[t,y_2,y_3,q]$ with $|t|=|y_i|=2, |q|=4$, and $dt=t^2, dy_i=y_i^2, dq=tq$, where the degree of the differential is 2. Then by Proposition 5.2, our aim is to determine the cohomology of a differential graded algebra

$$A\otimes\mathbb{Z}/2[\gamma_3]/(\rho_2,\rho_3,\rho_6,\rho_8,\rho_{12}),$$

where $|\gamma_3|=6, d\gamma_3=q_2$, and ρ_i is as in (5.4). By definition, we have

$$A/(\rho_2,\rho_3)=\mathbb{Z}/2[y_2,y_3]\otimes\langle 1,t,t^2\rangle$$

as a $\mathbb{Z}/2[y_2,y_3]$-module, and then $H^*(A/(\rho_2,\rho_3))=0$. Hence for $d\rho_8\equiv 0 \mod (\rho_2,\rho_3)$ and $d\rho_{12}=0$, it follows from (3.3) that

$$H^*\big(A/(\rho_2,\rho_3,\rho_8,\rho_{12})\big)=\Lambda(x_7,x_{11}),\quad |x_i|=2i.$$

Since $dq_2=0$ and $H^*(A)=0$, there exists $\delta_3\in H^6(A)$ satisfying $d\delta_3=q_2$. Put $\bar{\gamma}_3=\gamma_3+\delta_3$. Then one has

$$A\otimes\mathbb{Z}/2[\gamma_3]/(\rho_2,\rho_3,\rho_8,\rho_{12})=A\otimes\mathbb{Z}/2[\bar{\gamma}_3]/(\rho_2,\rho_3,\rho_8,\rho_{12})$$

and $\rho_6\equiv\bar{\gamma}_3^2+d(t^2\delta_3+\delta_5) \mod (\rho_2,\rho_3)$, where $\delta_5\in H^{10}(A)$ is given by $d\delta_5=\delta_3^2$. Thus for $d\bar{\gamma}_3=0$, we obtain

$$H^*\big(A\otimes\mathbb{Z}/2[\gamma_3]/(\rho_2,\rho_3,\rho_6,\rho_8,\rho_{12})\big)=\Lambda(x_7,x_{11},\bar{\gamma}_3),$$

completing the proof. □

THEOREM 5.4

The Atiyah–Hirzebruch spectral sequence $E_r(F_4/U)$ collapses at the E_3-term. In *particular, we have*

$$g_{F_4/U}(t) = (1+t^6)(1+t^{14})(1+t^{22}).$$

Proof
The result follows from Lemma 2.1(1), (2) and Corollary 5.3. □

THEOREM 5.5
The KO-theory of F_4/U is given as

$$KO^{2n-1}(F_4/U) \cong (\mathbb{Z}/2)^{s_n} \qquad \text{and} \qquad KO^{2n}(F_4/U) \cong (\mathbb{Z}/2)^{s_{n+1}} \oplus \mathbb{Z}^t$$

for $n \in \mathbb{Z}/4$, where

$$t = 144, \qquad s_0 = s_{-3} = 1, \qquad s_{-1} = s_{-2} = 3.$$

Proof
As is noted above, we have $f_{F_4/U}(t) = ((1-t^{12})(1-t^{16})(1-t^{24}))/(1-t^2)^3$. Then the proof is completed by Lemma 2.2, 2.3, and Theorem 5.4. □

5.2. KO-theory of F_4/T

Let $\rho_i \in \mathbb{Z}/2[t, y_1, y_2, y_3, \gamma_3]$ be as in (5.4), where $q = y_1(t - y_1)$. In [KI2], the mod 2 cohomology of F_4/T is calculated as

$$H^*(F_4/T; \mathbb{Z}/2) = \mathbb{Z}/2[t, y_1, y_2, y_3, \gamma_3]/(\rho_2, \rho_3, \rho_6, \rho_8, \rho_{12})$$

and $\mathrm{Sq}^2\gamma_3 = q_2$. Then the induced map from the projection $\pi : F_4/T \to F_4/U$ in the mod 2 cohomology satisfies

$$\pi^*(t) = t \qquad \text{and} \qquad \pi^*(y_i) = y_i \quad (i = 1, 2, 3). \tag{5.5}$$

Define a map $\lambda : F_4/T \to BT^6$ by $\lambda^*(t_i) = t - y_{4-i}$ and $\lambda^*(t_{i+3}) = y_i$ for $i = 1, 2, 3$. Then $\lambda^*(c_2 - 4t^2) = -t^2 + q_1 = 0$, implying that there is a lift $\tilde{\lambda} : F_4/T \to B\widetilde{T}^6$ satisfying

$$\tilde{\lambda}^*(t_i) = t - y_{4-i}, \qquad \tilde{\lambda}^*(t_{i+3}) = y_i \quad (i = 1, 2, 3), \qquad \text{and} \qquad \tilde{\lambda}^*(\gamma_3) = \gamma_3, \tag{5.6}$$

where the last equality is shown in [KI2].

PROPOSITION 5.6
The Sq^2-cohomology of F_4/T is given as

$$H^*(F_4/T; \mathrm{Sq}^2) = \Lambda(x_3, x_7, x_{11}, \bar{\gamma}_3), \quad |x_i| = 2i, |\bar{\gamma}_3| = 6,$$

where $\tilde{\lambda}^*(x_3) = x_3$, $\pi^*(x_7) = x_7$, $\pi^*(x_{11}) = x_{11}$, and $\pi^*(\bar{\gamma}_3) = \bar{\gamma}_3$.

Proof
Let A be a differential graded algebra $\mathbb{Z}/2[t, y_1, y_2, y_3]$ with $|t| = |y_i| = 2$ and $dt = t^2, dy_i = y_i^2$. Then the desired Sq^2-cohomology is equal to the cohomology of

$$A \otimes \mathbb{Z}/2[\gamma_3]/(\rho_2, \rho_3, \rho_6, \rho_8, \rho_{12}),$$

where $d\gamma_3 = q_2$. Since $H^*(A) = 0$, $d\rho_2 = \rho_3$, $d\rho_8 \equiv 0 \mod (\rho_2, \rho_3)$, and $d\rho_{12} = 0$, it follows from Lemma 3.3 that

$$H^*\big(A/(\rho_2,\rho_3,\rho_8,\rho_{12})\big)=\Lambda(x_3,x_7,x_{11}),$$

where $dx_3=q_2$ and x_7,x_{11} are as in Proposition 5.3. Then by defining $\bar\gamma_3$ as in the proof of Proposition 5.3, the first assertion follows. The second assertion follows from (5.5) and (5.6). □

REMARK 5.7

Since $H^*(F_4/T;\mathrm{Sq}^2)$ is an exterior algebra generated by four generators of degree $-2 \mod 8$ as in Proposition 5.6, we cannot directly see that $E_r(F_4/T)$ collapses at the E_3-term by Lemma 2.1. On the other hand, $H^*(F_4/U;\mathrm{Sq}^2)$ can be thought of as a subalgebra of $H^*(F_4/T;\mathrm{Sq}^2)$ generated by three of its four generators, and then we can apply Lemma 2.1 to see that $E_r(F_4/U)$ collapses at the E_3-term as above.

THEOREM 5.8

The Atiyah–Hirzebruch spectral sequence $E_r(F_4/T)$ collapses at the E_3-term. In particular, we have

$$g_{F_4/T}(t)=(1+t^6)^2(1+t^{14})(1+t^{22}).$$

Proof

By Theorem 3.7 and Proposition 5.6, $\iota^{-1}(x_3)$ in the 2-localized spectral sequence $E_3^{6,-1}(F_4/T)_{(2)}$ is a permanent cycle. Then since the 2-localization $E_3^{p,q}(F_4/T)\to E_3^{p,q}(F_4/T)_{(2)}$ is injective, $\iota^{-1}(x_3)$ in the integral spectral sequence $E_3^{6,-1}(F_4/T)$ is also a permanent cycle. By Theorem 5.4 and Proposition 5.6, $\iota^{-1}(x_7),\iota^{-1}(x_{11})$, $\iota^{-1}(\bar\gamma_3)\in E_3^{*,-1}(F_4/T)$ are also permanent cycles. Thus the proof is completed by Lemma 2.1(2). □

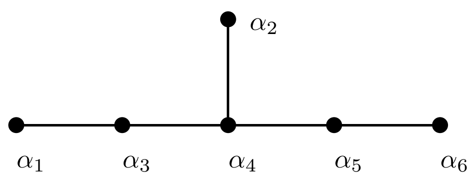
Proof of Theorem 1.1 for F_4

The result follows from (2.5), Lemma 2.2, and Corollary 5.8. □

6. KO-theory of E_6/T

Our method of computing the Atiyah–Hirzebruch spectral sequence $E_r(E_6/T)$ is similar to the case of F_4/T. Namely, we first calculate the Atiyah–Hirzebruch spectral sequence converging to $KO^*(E_6/U)$ for an appropriate maximal rank subgroup U and then deduce that of $KO^*(E_6/T)$.

We know that the Dynkin diagram of E_6 is given as follows:



In [IT], it is proved that the centralizer of the circle in E_6 defined by $\alpha_1=\alpha_3=\alpha_4=\alpha_5=\alpha_6=0$ is isomorphic to $T^1\cdot\mathrm{SU}(6)$. Then the identity component of the centralizer of the torus defined by $\alpha_5 = \alpha_6 = 0$ is isomorphic to $T^1 \cdot (T^2 \times \mathrm{U}(3))$ which we denote by U. It is clear that the homology of U is torsion-free and E_6/U satisfies the condition (2.2).

6.1. KO-theory of E_6/U

Let us calculate the $\mathbb{Z}_{(2)}$-coefficient cohomology of F_4/U. We set some notation. Let ω_i $(i = 1, \ldots, 6)$ be the fundamental weight of E_6 as in [TW]. Put

$$t_1 = -\omega_1 + \omega_2, \qquad t_2 = \omega_1 + \omega_2 - \omega_3, \qquad t_3 = \omega_2 + \omega_3 - \omega_4,$$
$$t_4 = \omega_4 - \omega_5, \qquad t_5 = \omega_5 - \omega_6, \qquad t_6 = \omega_6.$$

Then as in Section 2, we have

$$H^*(BT; \mathbb{Z}) = \mathbb{Z}_{(2)}[t, t_1, \ldots, t_6]/(c_1 - 3t).$$

As in [TW], the Weyl group of U is generated by two elements R_1, R_2 satisfying

$$R_1(t_i) = t_i \quad (i = 1, 2, 3, 6), \qquad R_1(t_4) = t_5, \qquad R_1(t_5) = t_4,$$
$$R_2(t_i) = t_i \quad (i = 1, 2, 3, 4), \qquad R_2(t_5) = t_6, \qquad R_2(t_6) = t_5.$$

Then it follows that

$$H^*(BU; \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}[t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3],$$

where $\hat{c}_1 = t_4 + t_5 + t_6$, $\hat{c}_2 = t_4t_5 + t_5t_6 + t_6t_4$, and $\hat{c}_3 = t_4t_5t_6$.

As in [MT], the mod 2 cohomology of E_6 is given as

$$H^*(E_6; \mathbb{Z}/2) = \mathbb{Z}/2[a_3]/(a_3^4) \otimes \Lambda(a_5, a_9, a_{15}, a_{17}, a_{23}), \quad |a_i| = i, \beta a_5 = a_3^2.$$

Then by [T], we obtain the following.

PROPOSITION 6.1

There is a regular sequence $\bar{\rho}_2, \bar{\rho}_5, \bar{\rho}_6, \bar{\rho}_8, \bar{\rho}_9, \bar{\rho}_{12}$ in $\mathbb{Z}_{(2)}[t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3]$ with $|\bar{\rho}_i| = 2i$ satisfying

$$H^*(E_6/U; \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}[t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3, \gamma_3]/(\bar{\rho}_2, \bar{\rho}_5, \bar{\rho}_6, \bar{\rho}_8, \bar{\rho}_9, \bar{\rho}_{12}, 2\gamma_3 + \bar{\rho}_3),$$

where $\bar{\rho}_3$ is defined by the equation $\mathrm{Sq}^2 \bar{\rho}_2 = \bar{\rho}_3$.

Let us compute the mod 2 cohomology of E_6/U. Let c_i be the ith symmetric function in $t_1, \ldots, t_6$ for $i = 1, \ldots, 6$. Obviously, c_i is a polynomial in $t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3$. A calculation in [TW] implies that the rational cohomology of E_6/U is given as

$$H^*(E_6/U; \mathbb{Q}) = \mathbb{Q}[t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3]/(\sigma_2, \sigma_5, \sigma_6, \sigma_8, \sigma_9, \sigma_{12}), \tag{6.1}$$

where

$$\sigma_2 = c_2 - \frac{4}{3^2}c_1^2, \qquad \sigma_5 = c_5 - \frac{1}{3}c_4c_1 + \frac{1}{3^2}c_3c_1^2 - \frac{2}{3^5}c_1^5,$$
$$\sigma_6 = 8c_6 + c_3^2 - \frac{4}{3^2}c_4c_1^2 - \frac{4}{3^6}c_1^6,$$
$$\sigma_8 = -3c_6c_1^2 + c_4^2 - c_4c_3c_1 + \frac{19}{3^4}c_4c_1^4 - \frac{5}{3^4}c_3c_1^5 + \frac{31}{3^8}c_1^8.$$

By Proposition 6.1, we may put

$$\bar{\rho}_2 = c_2 - \frac{4}{3^2}c_1^2 \qquad \text{and} \qquad \bar{\rho}_3 = c_3 + c_2c_1.$$

Put

$$R_1 = \mathbb{Z}_{(2)}[t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3, \gamma_3]/(\bar{\rho}_2, \bar{\rho}_5, 2\gamma_3 + \bar{\rho}_3).$$

Then since the natural map $H^*(E_6/U;\mathbb{Z}_{(2)}) \to H^*(E_6/U;\mathbb{Q})$ is injective, there is a surjection $R_1 \to H^*(E_6/U;\mathbb{Z}_{(2)})$ which reduces to a surjection

$$\phi_1 : R_1/2 \to H^*(E_6/U;\mathbb{Z}/2).$$

Put

$$\rho_2 = c_2, \qquad \rho_3 = c_3 + c_2c_1, \qquad \rho_5 = c_5 + c_4c_1. \tag{6.2}$$

Then ρ_2, ρ_3, ρ_5 is a regular sequence in $\mathbb{Z}/2[t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3]$ and

$$R_1/2 = \mathbb{Z}/2[t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3, \gamma_3]/(\rho_2, \rho_3, \rho_5),$$

implying that the Poincaré series of $R_1/2$ is $(1-t^{10})/((1-t^2)^4(1-t^6))$. On the other hand, the Poincaré series of $H^*(E_6/U;\mathbb{Z}/2)$ and $H^*(E_6/U;\mathbb{Q})$ are the same, which is $\frac{(1-t^{10})(1-t^{12})(1-t^{16})(1-t^{18})(1-t^{24})}{(1-t^2)^4(1-t^6)}$ by (6.1). Then ϕ_1 is an isomorphism in dimension ≤ 11.

Note that $\sigma_6 \equiv 4(2c_6 + \gamma_3^2 + \frac{4}{3^2}\gamma_3c_1^3 - \frac{1}{3^2}c_4c_1^2 + \frac{35}{3^6}c_1^6) \mod (\bar{\rho}_2, 2\gamma_3 + \bar{\rho}_3)$. Then since $H^*(E_6/U;\mathbb{Z}_{(2)}) \to H^*(E_6/U;\mathbb{Q})$ is injective, if we put

$$R_2 = R_1/\Big(2c_6 + \gamma_3^2 + \frac{4}{3^2}\gamma_3c_1^3 - \frac{1}{3^2}c_4c_1^2 + \frac{35}{3^6}c_1^6, \sigma_8\Big),$$

ϕ_1 induces a surjection

$$\phi_2 : R_2/2 \to H^*(E_6/U;\mathbb{Z}/2).$$

Put

$$\rho_6 = \gamma_3^2 + c_4c_1^2 + c_1^6, \qquad \rho_8 = c_6c_1^2 + c_4^2 + c_4c_1^4 + c_1^8. \tag{6.3}$$

Then one sees that

$$R_2/2 = \mathbb{Z}/2[t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3, \gamma_3]/(\rho_2, \rho_3, \rho_5, \rho_6, \rho_8).$$

Since $\rho_2, \rho_3, \rho_5, \rho_6, \rho_8$ is a regular sequence in $\mathbb{Z}/2[t_1, t_2, t_3, \hat{c}_1, \hat{c}_2, \hat{c}_3, \gamma_3]$, one can calculate the Poincaré series of $R_2/2$. Then comparing the Poincaré series as above, we obtain that ϕ_2 is an isomorphism in dimension ≤ 35.

Put

$$\rho_9 = c_6c_1^3, \qquad \rho_{12} = c_6^2 + c_6c_4c_1^2 + c_4^2c_1^4 + c_4c_1^8. \tag{6.4}$$

Since $\mathrm{Sq}^2\phi_2(\rho_8) = \phi_2(\rho_9)$ and $\mathrm{Sq}^8\phi_2(\rho_8) = \phi_2(\rho_{12})$, there is also a surjection

$$\phi_3 : R_3 \to H^*(E_6/U;\mathbb{Z}/2),$$

where

$$R_3 = R_2/(2, \rho_8, \rho_{12}).$$

Since $\rho_2,\rho_3,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{12}$ is a regular sequence in $\mathbb{Z}/2[t_1,t_2,t_3,\hat{c}_1,\hat{c}_2,\hat{c}_3,\gamma_3]$, one can calculate the Poincaré series of R_3. Comparing it with the Poincaré series of $H^*(E_6/U;\mathbb{Z}/2)$, we conclude that ϕ_3 is an isomorphism. Summarizing, we obtain the following.

PROPOSITION 6.2

The mod 2 cohomology of E_6/U is given as

$$H^*(E_6/U;\mathbb{Z}/2)=\mathbb{Z}/2[t_1,t_2,t_3,\hat{c}_1,\hat{c}_2,\hat{c}_3,\gamma_3]/(\rho_2,\rho_3,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{12}),$$

where $|t_i|=2,|\hat{c}_i|=2i,|\gamma_3|=6$ and ρ_i is as in (6.2), (6.3), and (6.4).

COROLLARY 6.3

The Sq^2-cohomology of E_6/U is given as

$$H^*(E_6/U;\mathrm{Sq}^2)=\Lambda(x_7,x_{11},x_{15}),\quad |x_i|=2i,$$

where $\mathrm{Sq}^2x_{11}\equiv\rho_{12} \mod (\rho_2,\rho_3,\rho_5,\rho_9)$, $\mathrm{Sq}^2x_{15}=\rho_8^2$, $x_7=\gamma_3c_4+\delta_7$, and $\mathrm{Sq}^2\delta_7=c_4^2$ for $\delta_7\in\mathbb{Z}/2[t_1,t_2,t_3,\hat{c}_1,\hat{c}_2,\hat{c}_3]$.

Proof

As in the proof of Corollary 5.3, we see that $\mathrm{Sq}^2\gamma_3=c_4$. Put $A=\mathbb{Z}/2[t_1,t_2,t_3,\hat{c}_1,\hat{c}_2,\hat{c}_3]$. Then our aim is to calculate the cohomology of a differential graded algebra

$$A\otimes\mathbb{Z}/2[\gamma_3]/(\rho_2,\rho_3,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{12}).$$

Obviously, $A/(\rho_2,\rho_3)\cong\mathbb{Z}/2[t_1,t_2,t_3]\otimes\langle 1,\hat{c}_1,\hat{c}_1^2\rangle$ as a $\mathbb{Z}/2[t_1,t_2,t_3]$-module, implying $H^*(A/(\rho_2,\rho_3))=0$. Then since $dc_4=\rho_5$ and $d\rho_8=\rho_9$, it follows from Lemma 3.3 that

$$H^*\big(A/(\rho_2,\rho_3,\rho_5,\rho_8,\rho_9)\big)=\Lambda(c_4,x_{15}),\quad |x_i|=2i,$$

where $\mathrm{Sq}^2x_{15}=\rho_8^2$. For $d\rho_{12}\equiv 0 \mod (\rho_5,\rho_9)$ and $H^{24}(A/(\rho_2,\rho_3,\rho_5,\rho_8,\rho_9))=0$, we get

$$H^*\big(A/(\rho_2,\rho_3,\rho_5,\rho_8,\rho_9,\rho_{12})\big)=\Lambda(c_4,x_{11},x_{15}),\quad |x_i|=2i,$$

where $\mathrm{Sq}^2x_{11}\equiv\rho_{12} \mod (\rho_2,\rho_3,\rho_5,\rho_9)$. By the spectral sequence associated with a filtration

$$A/(\rho_2,\rho_3,\rho_5,\rho_8,\rho_{12})\subset A\otimes\mathbb{Z}/2[\gamma_3]/(\rho_2,\rho_3,\rho_5,\rho_8,\rho_9,\rho_{12}),$$

we get

$$H^*\big(A\otimes\mathbb{Z}/2[\gamma_3]/(\rho_2,\rho_3,\rho_5,\rho_8,\rho_9,\rho_{12})\big)=\Lambda(x_7,x_{11},x_{15})\otimes\mathbb{Z}/2[\gamma_3^2],$$

where $x_7=\gamma_3c_4+\delta_7$ and $\delta_7\in\mathbb{Z}/2[t_1,t_2,t_3,\hat{c}_1,\hat{c}_2,\hat{c}_3]$ is given by $d\delta_7=c_4^2$. Since $\rho_6=\gamma_3^2+d(\gamma_3c_1^2+c_1^5)$, we obtain

$$H^*\big(A\otimes\mathbb{Z}/2[\gamma_3]/(\rho_2,\rho_3,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{12})\big)=\Lambda(x_7,x_{11},x_{15}),$$

completing the proof. $\square$

THEOREM 6.4

The Atiyah–Hirzebruch spectral sequence $E_r(E_6/U)$ collapses at the E_3-term. In particular, we have

$$g_{E_6/U}(t) = (1+t^{14})(1+t^{22})(1+t^{30}).$$

Proof

From Lemma 2.1 and Proposition 6.3, the result follows. □

THEOREM 6.5

The KO-theory of E_6/U is given as

$$KO^{2n-1}(E_6/U) \cong (\mathbb{Z}/2)^{s_n} \qquad \text{and} \qquad KO^{2n}(E_6/U) \cong (\mathbb{Z}/2)^{s_{n+1}} \oplus \mathbb{Z}^t$$

for $n \in \mathbb{Z}/4$, where

$$t = 4320, \qquad s_0 = s_{-3} = 1, \qquad s_{-1} = s_{-2} = 3.$$

Proof

By (6.1), we have $f_{E_6/U}(t) = \frac{(1-t^{10})(1-t^{12})(1-t^{16})(1-t^{18})(1-t^{24})}{(1-t^2)^4(1-t^6)}$. Then the proof is completed by Lemma 2.2 and Theorem 6.4. □

6.2. KO-theory of E_6/T

Let $\rho_i \in \mathbb{Z}/2[t_1, \ldots, t_6, \gamma_3]$ be as in (6.2), (6.3), and (6.4). The mod 2 cohomology of E_6/T is calculated in [KI2] as

$$H^*(E_6/T; \mathbb{Z}/2) = \mathbb{Z}/2[t_1, \ldots, t_6, \gamma_3]/(\rho_2, \rho_3, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{12}),$$

where $\mathrm{Sq}^2\gamma_3 = c_4$. For the projection $\pi : E_6/T \to E_6/U$, we have

$$\begin{aligned} &\pi^*(t_i) = t_i \quad (i = 1, 2, 3), \qquad \pi^*(\hat{c}_1) = t_4 + t_5 + t_6, \\ &\pi^*(\hat{c}_2) = t_4 t_5 + t_5 t_6 + t_6 t_4, \qquad \pi^*(\hat{c}_3) = t_4 t_5 t_6. \end{aligned} \tag{6.5}$$

Define a map $\lambda : (E_6/T)_{(2)} \to BT^6_{(2)}$ by $\lambda^*(t_i = t_i)$ for $i = 1, \ldots, 6$. Then there is a lift $\tilde{\lambda} : (E_6/T)_{(2)} \to B\widetilde{T}^6_{(2)}$ satisfying

$$\tilde{\lambda}^*(t_i) = t_i \quad (i = 1, \ldots, 6), \qquad \tilde{\lambda}^*(\gamma_3) = \gamma_3, \tag{6.6}$$

where the second equality is shown in [KI1].

PROPOSITION 6.6

The Sq^2-cohomology of E_6/T is given as

$$H^*(E_6/T; \mathrm{Sq}^2) = \Lambda(x_3, x_7, x_{11}, x_{15}), \quad |x_i| = 2i,$$

where $\tilde{\lambda}^*(x_3) = x_3$, $\pi^*(x_7) = x_7$, $\pi^*(x_{11}) = x_{11}$, and $\pi^*(x_{15}) = x_{15}$.

Proof

Define a differential graded algebra A as $A = \mathbb{Z}/2[t_1, \ldots, t_6]$ with $|t_i| = 2$ and $dt_i = t_i^2$. Then we calculate the cohomology of a differential graded algebra $A \otimes$

$\mathbb{Z}/2[\gamma_3]/(\rho_2,\rho_3,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{12})$, where $d\gamma_3 = c_4$. This is done quite similarly to the proof of Proposition 6.3. The second assertion follows from (6.5) and (6.6). $\square$

THEOREM 6.7

The spectral sequence $E_r(E_6/T)$ collapses at the E_3-term. In particular, we have

$$g_{E_6/T}(t) = (1+t^6)(1+t^{14})(1+t^{22})(1+t^{30}).$$

Proof

By Theorem 3.7 and Proposition 6.6, $\iota^{-1}(x_3)$ in the 2-localized spectral sequence $E_3^{6,-1}(E_6/T)_{(2)}$ is a permanent cycle, implying that $\iota^{-1}(x_3)$ in the integral spectral sequence $E_3^{6,-1}(E_6/T)$ is also a permanent cycle since the 2-localization $E_3^{p,q}(E_6/T) \to E_3^{p,q}(E_6/T)$ is injective. By Theorem 6.4 and Proposition 6.6, $\iota^{-1}(x_i) \in E_3^{*,-1}(E_6/T)$ is also a permanent cycle for $i = 7, 11, 15$. Thus the result follows from Lemma 2.1. $\square$

Proof of Theorem 1.1 for E_6

The result follows from (2.5), Lemma 2.2, and Corollary 6.7. $\square$

REMARK 6.8

We cannot apply the same calculation method to E_7/T and E_8/T for which there is no control on elements γ_5, γ_9 in their mod 2 cohomology (see [KI2]).

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