

Stability of Gieseker stable sheaves on K3 surfaces in the sense of Bridgeland and some applications

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Abstract We show that some Gieseker stable sheaves on a projective K3 surface X are stable with respect to a stability condition of Bridgeland on the derived category of X if the stability condition is in explicit subsets of the space of stability conditions depending on the sheaves. Furthermore we shall give two applications of the result. As a part of these applications, we show that the fine moduli space of Gieseker stable torsion-free sheaves on a K3 surface with Picard number one is the moduli space of μ -stable locally free sheaves if the rank of the sheaves is not a square number.

1. Introduction

In [1], Bridgeland introduced the notion of stability conditions on arbitrary triangulated categories \mathcal{D} . Let $\text{Stab}(\mathcal{D})$ be the space of stability conditions on \mathcal{D} . If $\text{Stab}(\mathcal{D})$ is not empty, then $\text{Stab}(\mathcal{D})$ is a complex manifold by [1]. Furthermore we can define the notion of σ -stability for objects $E \in \mathcal{D}$.

Suppose that \mathcal{D} is the bounded derived category $D(X)$ of coherent sheaves on a smooth projective variety X over \mathbb{C} . In this paper we study the case where X is a projective K3 surface. Then as is well known, the space $\text{Stab}(X)$ of stability conditions on $D(X)$ is not empty by virtue of [2]. Then for coherent sheaves on X we have the notion of σ -stability in addition to Gieseker stability and μ -stability. It is natural to ask how these three different notions of stability are related for sheaves. The aim of this paper is to give some answers (see Theorems 4.4, 4.9) with applications (see Theorems 5.4, 6.7).

In this article we have two goals. The first goal is to show the σ -stability of Gieseker stable (or μ -stable) sheaves on X if σ is in explicit subsets of $\text{Stab}(X)$ depending on the sheaves. This result will be proved in Theorems 4.4 and 4.9. The second goal is to give two applications of these two theorems.

Now we explain the background of our theorem. We first recall that the space $\text{Stab}(X)$ has the subset $U(X)$. The set $U(X)$ is very roughly the set of stability conditions σ such that for all $x \in X$, the skyscraper sheaf \mathcal{O}_x is σ -stable. This subset $U(X)$ is also a trivial $\tilde{\text{GL}}^+(2, \mathbb{R})$ bundle over a set $V(X)$, where $\tilde{\text{GL}}^+(2, \mathbb{R})$

is the universal cover of $\mathrm{GL}^+(2, \mathbb{R})$ (see also Section 3). In addition $V(X)$ is roughly parameterized by \mathbb{R} -divisors β and \mathbb{R} -ample divisors ω . Hence we can write $\sigma \in U(X)$ as $\sigma = \sigma_{(\beta, \omega)} \cdot \tilde{g}$ where $\sigma_{(\beta, \omega)} \in V(X)$ and $\tilde{g} \in \tilde{\mathrm{GL}}^+(2, \mathbb{R})$.

In [2], it is written that for a fixed \mathbb{R} -divisor β , if we take a limit $\omega \rightarrow \infty$ (so-called large volume limits), the notion of $\sigma_{(\beta, \omega)}$ -stability is expected to coincide with the notion of twisted stability.* Furthermore it is shown in [2] that this conjecture holds for some objects. In some sense we strengthen this result in Theorems 4.4 and 4.9. As a consequence of this, we show that in large-volume limits, we can distinguish the notion of μ -stability of coherent sheaves on X with Picard rank one and the notion of Gieseker stability (see Proposition 5.2).[†] What does this sentence mean?

For a given sheaf E and $\beta \in \mathrm{NS}(X)_{\mathbb{R}}$ we have $\mu_{\omega}(E) > \beta\omega$ or $\mu_{\omega}(E) \leq \beta\omega$. If E is Gieseker stable and $\mu_{\omega}(E) > \beta\omega$, then E is $\sigma_{(\beta, \omega)}$ -stable in the large-volume limit by [2]. In Proposition 5.2, we show that if a Gieseker stable sheaf E with $\mu_{\omega}(E) \leq \beta\omega$ is $\sigma_{(\beta, \omega)}$ -stable in large volume limit, then E is μ -stable.

We also apply Proposition 5.2 to the classification of the fine moduli space of Gieseker stable sheaves on X , which is the first application in Theorem 5.4. More precisely we show that the fine moduli space of Gieseker stable, torsion-free sheaves is the fine moduli space of μ -stable locally free sheaves if the rank of the sheaves is not a square number. We also show that if the rank is a square number, then the fine moduli space is either the moduli space of μ -stable locally free sheaves or the moduli space of properly Gieseker stable, torsion-free sheaves[‡]. Furthermore we show that if the latter case occurs, then the moduli space is isomorphic to X itself. Hence these results show us that any nontrivial Fourier–Mukai partners of X with Picard rank one are isomorphic to the fine moduli space of μ -stable locally free sheaves (see Remark 5.5). The key idea of the proof of Theorem 5.4 is to compare two Jordan–Hölder filtrations of a Gieseker stable sheaf with respect to μ -stability and σ -stability for some $\sigma \in \mathrm{Stab}(X)$. This comparison is enabled by Proposition 5.2. We have to remark that a similar result was proved by [11, Lemma 1.2].

The second application is Theorem 6.7, which is a generalization of [7, Theorem 1.1]. Let $\Phi : D(Y) \rightarrow D(X)$ be an equivalence where X and Y are projective K3 surfaces, and let $\Phi_* : \mathrm{Stab}(Y) \rightarrow \mathrm{Stab}(X)$ be a natural map induced by Φ . We show that, if Φ satisfies the condition $\Phi_*U(Y) = U(X)$, then the equivalence Φ is given by $M \otimes f_*(-)[n]$ where M is a line bundle on X , f is an isomorphism from Y to X , and $n \in \mathbb{Z}$. This is a complete generalization of [7, Theorem 1.1], which shows the same statement under the additional condition that the Picard number is 1.

*Namely, for any sufficiently large $\lambda \gg 0$, if $E \in D(X)$ is $\sigma_{(\beta, \lambda\omega)} \cdot \tilde{g}$ -stable, then E is a (β, ω) -twisted stable sheaf, and vice versa.

[†]Note that if the Picard number is one, then twisted stability is just Gieseker stability.

[‡]Namely, the sheaf is neither μ -stable nor locally free.

We proceed as follows. For an equivalence $\Phi : D(Y) \rightarrow D(X)$ satisfying the assumption $\Phi_*U(Y) = U(X)$, one can see that it is enough to prove that $\Phi(\mathcal{O}_y) = \mathcal{O}_x[n]$ where $x \in X$ and $n \in \mathbb{Z}$. In [7], this was proved by using [7, Theorem 6.6]. Hence the crucial part of the proof of [7, Theorem 1.1] is [7, Theorem 6.6]. A necessary generalization of this result of [7] will be done in Corollary 6.6 by applying Theorem 4.5.

2. Review of classical stability for sheaves

In this section we recall the μ -stability, Gieseker stability, and twisted stability for coherent sheaves on a projective K3 surface.

We first introduce some notation. Throughout this section X is a projective K3 surface over \mathbb{C} . Let A and B be in $D(X)$. If the i th cohomology $H^i(A)$ is concentrated only at degree $i = 0$, we call A a *sheaf*. We put $\text{Hom}_X^n(A, B) := \text{Hom}_{D(X)}(A, B[n])$ and $\text{hom}_X^n(A, B) := \dim_{\mathbb{C}} \text{Hom}_X^n(A, B)$ where $[n]$ means $n \in \mathbb{Z}$ times shifts. We remark that

$$\text{Hom}_X^n(A, B) = \text{Hom}_X^{2-n}(B, A)^*$$

by the Serre duality. Then the Euler pairing $\chi(E, F) = \sum_i (-1)^i \text{hom}_X^i(E, F)$ is a \mathbb{Z} -bilinear symmetric form on the Grothendieck group $K(X)$ of $D(X)$.

Let $\mathcal{N}(X)$ be the quotient of $K(X)$ by numerical equivalence with respect to the Euler pairing χ . Then $\mathcal{N}(X)$ is isomorphic to

$$H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z})$$

where $\text{NS}(X)$ is the Néron–Severi lattice of X . For $E \in D(X)$, we define the Mukai vector $v(E)$ of E by $ch(E)\sqrt{td_X}$. Then $v(E) = r_E \oplus \delta_E \oplus s_E$ is in $H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z})$, and we have $r_E = \text{rank } E$, $\delta_E = c_1(E)$ and $s_E = \chi(\mathcal{O}_X, E) - r_E$.

Let $\langle -, - \rangle$ be the Mukai pairing on $\mathcal{N}(X)$:

$$\langle r \oplus \delta \oplus s, r' \oplus \delta' \oplus s' \rangle = \delta\delta' - rs' - r's,$$

where $r \oplus \delta \oplus s, r' \oplus \delta' \oplus s' \in \mathcal{N}(X)$. Then, by the Riemann–Roch formula, we see

$$\chi(E, F) = -\langle v(E), v(F) \rangle.$$

We secondly recall the notion of μ -stability. For a torsion-free sheaf F and an ample divisor ω , the *slope* $\mu_\omega(F)$ is defined by $(c_1(F) \cdot \omega) / \text{rank } F$. If the inequality $\mu_\omega(A) \leq \mu_\omega(F)$ holds for any nontrivial subsheaf A of F , then F is said to be μ_ω -*semistable*. Moreover if the strict inequality $\mu_\omega(A) < \mu_\omega(F)$ holds for any nontrivial subsheaf A with $\text{rank } A < \text{rank } F$, then F is said to be μ_ω -*stable*. If $\text{NS}(X) = \mathbb{Z}L$, we write μ -(semi)stable instead of μ_L -(semi)stable. The notion of the μ_ω -stability admits the Harder–Narasimhan filtration of F (details in [5]). We define $\mu_\omega^+(F)$ by the maximal slope of semistable factors of F , and we define $\mu_\omega^-(F)$ by the minimal slope of semistable factors of F .

Let β be an \mathbb{R} -divisor, and let ω be an \mathbb{R} -ample divisor on X .^{*} For a pair (β, ω) we recall the notion of (β, ω) -twisted stability introduced in [8]. For a torsion-free sheaf E with $v(E) = r_E \oplus \delta_E \oplus s_E$, we define a polynomial $p_{(\beta, \omega)}(E)$ by

$$p_{(\beta, \omega)}(E) := \frac{\omega^2}{2} \cdot n^2 + \left(\frac{\delta_E}{r_E} - \beta \right) \omega \cdot n + \frac{s_E}{r_E} - \frac{\delta_E \beta}{r_E} + \frac{\beta^2}{2} + 1 \in \mathbb{R}[n].$$

Suppose that ω is an integral class, and put $\omega = \mathcal{O}_X(1)$. Then $p_{(\beta, \omega)}(n)$ is simply given by

$$p_{(\beta, \omega)}(E) = - \frac{\langle v(\mathcal{O}_X(-n)), \exp(-\beta)v(E) \rangle}{r_E}.$$

DEFINITION 2.1

Let E be a torsion-free sheaf on a projective K3 surface X ; E is said to be (β, ω) -twisted stable (resp., semistable) if $p_{(\beta, \omega)}(F) < p_{(\beta, \omega)}(E)$ (resp., $p_{(\beta, \omega)}(F) < p_{(\beta, \omega)}(E)$) for any nontrivial subsheaf F of E .

Moreover if $\beta = 0$, then E is said to be Gieseker (semi)stable with respect to ω . For a torsion-free sheaf E , we write $p_\omega(E)$ instead of $p_{(0, \omega)}(E)$.

REMARK 2.2

For a torsion-free sheaf E , we can easily check the following relation between μ_ω -stability and (β, ω) -twisted stability:

$$\mu_\omega\text{-stable} \Rightarrow (\beta, \omega)\text{-twisted stable} \Rightarrow (\beta, \omega)\text{-twisted semistable} \Rightarrow \mu_\omega\text{-semistable}.$$

We also see the following relation between the μ_ω -stability and the Gieseker stability:

$$\mu_\omega\text{-stable} \Rightarrow \text{Gieseker stable} \Rightarrow \text{Gieseker semistable} \Rightarrow \mu_\omega\text{-semistable}.$$

Finally we cite the following lemma, which plays an important role when we study the space of stability conditions on abelian or K3 surfaces. A prototype of Lemma 2.3 was first proved by Mukai and Bridgeland. Finally [6] refined it.

LEMMA 2.3 ([6, LEMMA 2.7])

Let X be an abelian surface or a K3 surface. Suppose that $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle in $D(X)$. If $\text{hom}_X^i(A, C) = 0$ for any $i \leq 0$ and $\text{hom}_X^j(C, C) = 0$ for any $j < 0$, then we have the following inequality:

$$0 \leq \text{hom}_X^1(A, A) + \text{hom}_X^1(C, C) \leq \text{hom}_X^1(B, B).$$

^{*}Originally the notion of twisted stability is defined on projective surfaces. To avoid the complexity we add the assumption that X is a projective K3 surface.

3. Review of Bridgeland’s work

In this section we briefly recall the theory of stability conditions. The details are in the original articles [1] and [2]. For a projective K3 surface X we put $\text{NS}(X)_{\mathbb{R}} = \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\text{Amp}(X)$ by the set of \mathbb{R} -ample divisors.

Let \mathcal{A} be the heart of a bounded t -structure on the derived category $D(X)$ of X , and let Z be a group homomorphism from $K(X)$ to \mathbb{C} . Notice that $K(X)$ is isomorphic to the Grothendieck group of the heart \mathcal{A} . The morphism Z is called a *stability function* on \mathcal{A} if Z satisfies the following:

$$0 \neq E \in \mathcal{A} \Rightarrow Z(E) = m \exp(\sqrt{-1}\pi\phi_E),$$

where $m \in \mathbb{R}_{>0}$ and ϕ_E is in the interval $(0, 1]$. Then we put $\arg Z(E) = \phi_E$ and call the pair (\mathcal{A}, Z) a *stability pair* on $D(X)$. If we take a stability pair (\mathcal{A}, Z) , we can define the notion of Z -stability for objects in \mathcal{A} as follows.

DEFINITION 3.1

Let (\mathcal{A}, Z) be a stability pair on $D(X)$, and let E be in \mathcal{A} . The object E is said to be Z -(*semi*)*stable* if E satisfies $\arg Z(F) < (<=) \arg Z(E)$ for any nontrivial subobject.

By using the notion of Z -stability, we define a stability condition on $D(X)$ as follows.

DEFINITION 3.2

A stability pair (\mathcal{A}, Z) is said to be a *stability condition* on $D(X)$ if any $E \in \mathcal{A}$ has the filtration $0 = E_0 \subset E_1 \subset \dots \subset E_{n-1} \subset E_n = E$ such that $A_i := E_i/E_{i-1}$ ($i = 1, 2, \dots, n$) is Z -semistable with $\arg Z(A_1) > \dots > \arg Z(A_n)$. We call such a filtration the *Harder–Narasimhan filtration* of E . Moreover if Z factors through $\mathcal{N}(X)$, σ is said to be *numerical*.

Let $\sigma = (\mathcal{A}, Z)$ be a stability condition on $D(X)$. Then we can define the notion of σ -stability for any object in $D(X)$.* An object $E \in D(X)$ is said to be σ -(*semi*)*stable* if there is an integer $n \in \mathbb{Z}$ such that $E[n]$ is in \mathcal{A} and $E[n]$ is Z -(*semi*)*stable*. We define $\arg Z(E)$ by $\arg Z(E[n]) - n$ and call it the *phase* of E .

We put $\mathcal{P}(\phi) = \{E \in \mathcal{D}(X) \mid E \text{ is } Z\text{-semistable with phase } \phi\} \cup \{0\}$. Then $\mathcal{P}(\phi)$ is an abelian category. For an interval $I \subset \mathbb{R}$, we define $\mathcal{P}(I)$ by the extension-closed full subcategory generated by $\mathcal{P}(\phi)$ for all $\phi \in I$. If for any $\phi \in \mathbb{R}$ there is a positive number ϵ such that $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is Artinian and Noetherian, then the stability condition $\sigma = (\mathcal{A}, Z)$ is said to be *locally finite*.

*For a stability pair (\mathcal{A}, Z) , we can logically define the notion of σ -stability for objects in $D(X)$. However in the original article [1], the notion of stability of arbitrary objects in $D(X)$ is defined by a stability condition. Thus we follow the original style.

In general we cannot define the argument of $Z(E)$ for $E \in D(X)$. However if E is in \mathcal{A} (or $\mathcal{A}[-1]$), then we can define the argument of $Z(E)$ uniquely since the argument $\arg Z(E)$ is in $(0, 1]$ (resp., in $(-1, 0]$).

Take a stability condition $\sigma = (\mathcal{A}, Z)$ on $D(X)$. Then we can easily check that there exists the following sequence of distinguished triangles for an arbitrary object $E \in D(X)$:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\
 & & \swarrow & & \swarrow & & & & \swarrow & & \swarrow \\
 & & A_1 & & A_2 & & & & A_n & &
 \end{array}$$

$\begin{matrix} \text{[1]} \\ \text{[1]} \\ \text{[1]} \end{matrix}$

where each A_i is σ -semistable with $\arg Z(A_1) > \cdots > \arg Z(A_n)$. One can easily check that the above sequence is unique up to isomorphism. We also call this sequence the *Harder–Narasimhan filtration* (for short HN filtration). If E is in \mathcal{A} , then the above filtration is nothing but the filtration defined in Definition 3.2. In addition assume that σ is locally finite. Then for a σ -semistable object F with phase ϕ we have the following sequence of distinguished triangles:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & \cdots & \longrightarrow & F_{m-1} & \longrightarrow & F_m = F \\
 & & \swarrow & & \swarrow & & & & \swarrow & & \swarrow \\
 & & S_1 & & S_2 & & & & S_m & &
 \end{array}$$

$\begin{matrix} \text{[1]} \\ \text{[1]} \\ \text{[1]} \end{matrix}$

where each S_j is σ -stable with $\arg Z(S_j) = \phi$. We call this filtration a *Jordan–Hölder filtration* (for short JH filtration). We remark that a JH filtration of F is not unique but the direct sum $\bigoplus_{i=1}^m S_i$ of all stable factors of F is unique up to isomorphism.

Now we put

$$\text{Stab}(X) = \{ \sigma \mid \sigma \text{ is a numerical locally finite stability condition on } D(X) \}.$$

Bridgeland [2] describes a subset $U(X)$ of $\text{Stab}(X)$. We shall recall its definition. We put

$$\Delta^+(X) := \{ v = r \oplus \delta \oplus s \in \mathcal{N}(X) \mid v^2 = -2, r > 0 \}$$

and define a subset $V(X)$ of $\text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X)$ by

$$\begin{aligned}
 V(X) := & \{ (\beta, \omega) \in \text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X) \mid \\
 & \langle \exp(\beta + \sqrt{-1}\omega), v \rangle \notin \mathbb{R}_{\leq 0} (\forall v \in \Delta^+(X)) \}.
 \end{aligned}$$

Let $(\beta, \omega) \in V(X)$. Then (β, ω) gives a numerical locally finite stability condition $\sigma_{(\beta, \omega)} = (\mathcal{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$ in the following way. We put $\mathcal{A}_{(\beta, \omega)}$ by

$$\mathcal{A}_{(\beta, \omega)} := \left\{ E^\bullet \in D(X) \mid H^i(E^\bullet) \begin{cases} \in \mathcal{T}_{(\beta, \omega)} & (i = 0), \\ \in \mathcal{F}_{(\beta, \omega)} & (i = -1), \\ = 0 & (i \neq 0, -1) \end{cases} \right\},$$

where

$$\mathcal{T}_{(\beta,\omega)} := \{E \in \text{Coh}(X) \mid E \text{ is a torsion sheaf or } \mu_{\omega}^-(E/\text{torsion}) > \beta\omega\}$$

and

$$\mathcal{F}_{(\beta,\omega)} := \{E \in \text{Coh}(X) \mid E \text{ is torsion-free and } \mu_{\omega}^+(E) \leq \beta\omega\}.$$

We define a stability function $Z_{(\beta,\omega)}$ by $Z_{(\beta,\omega)}(E) := \langle \exp(\beta + \sqrt{-1}\omega), v(E) \rangle$. Then the pair $\sigma_{(\beta,\omega)} = (\mathcal{A}_{(\beta,\omega)}, Z_{(\beta,\omega)})$ gives a numerical locally finite stability condition by [2].

Then we put $V(X) := \{\sigma_{(\beta,\omega)} \mid (\beta,\omega) \in \mathcal{V}(X)\}$. If σ is in $V(X)$, then for any closed point $x \in X$, \mathcal{O}_x is σ -stable with phase 1 by [2, Lemma 6.3]. Let $\tilde{\text{GL}}^+(2, \mathbb{R})$ be the universal cover of $\text{GL}^+(2, \mathbb{R})$. Then $\text{Stab}(X)$ has the right group action of $\tilde{\text{GL}}^+(2, \mathbb{R})$ by [1, Lemma 8.2]. We put $U(X) := V(X) \cdot \tilde{\text{GL}}^+(2, \mathbb{R})$. We remark that $U(X)$ is isomorphic to $V(X) \times \tilde{\text{GL}}^+(2, \mathbb{R})$.

Let σ be in $\text{Stab}(X)$. Since σ is numerical and the Euler pairing is nondegenerate on $\mathcal{N}(X)$, we have a natural map

$$\pi : \text{Stab}(X) \rightarrow \mathcal{N}(X), \quad \pi(\mathcal{A}, Z) \rightarrow Z^{\vee},$$

where $Z(E) = \langle Z^{\vee}, v(E) \rangle$. The map π gives a complex structure on $\text{Stab}(X)$. In particular each connected component of $\text{Stab}(X)$ is a complex manifold by [1]. If $\pi(\sigma)$ spans a positive real 2-plane and is orthogonal to all (-2) vectors in $\mathcal{N}(X)$ then σ is said to be *good*.

PROPOSITION 3.3 ([2, PROPOSITION 10.3])

The special locus $U(X)$ is written by

$$U(X) = \{\sigma \in \text{Stab}(X) \mid \mathcal{O}_x \text{ is } \sigma\text{-stable with a common phase and } \sigma \text{ is good}\}.$$

Let us consider the boundary $\partial U(X) := \overline{U(X)} \setminus U(X)$ where $\overline{U(X)}$ is the closure of $U(X)$. Then $\partial U(X)$ consists of a locally finite union of real codimension 1 submanifolds by [2, Proposition 9.2]. If $\sigma \in \partial U(X)$ lies on only one of these submanifolds, then σ is said to be *general*.

THEOREM 3.4 ([2, THEOREM 12.1])

Let $\sigma \in \partial U(X)$ be general. Then exactly one of the following holds.

(A^+) *There is a spherical locally free sheaf A such that both A and $T_A(\mathcal{O}_x)$ are stable factors of \mathcal{O}_x for any $x \in X$, where T_A is the spherical twist by A . Moreover a JH filtration of \mathcal{O}_x is given by*

$$A^{\oplus \text{rank } A} \longrightarrow \mathcal{O}_x \longrightarrow T_A(\mathcal{O}_x) \longrightarrow A^{\oplus \text{rank } A}[1].$$

In particular \mathcal{O}_x is properly σ -semistable for all $x \in X$, and A does not depend on $x \in X$.*

*Namely, \mathcal{O}_x is not σ -stable but is σ -semistable.

(A⁻) There is a spherical locally free sheaf A such that both A and $T_A^{-1}(\mathcal{O}_x)$ are stable factors of \mathcal{O}_x for any $x \in X$, where T_A is the spherical twist by A . Moreover a JH filtration of \mathcal{O}_x is given by

$$T_A^{-1}(\mathcal{O}_x) \longrightarrow \mathcal{O}_x \longrightarrow A^{\oplus \text{rank } A}[2] \longrightarrow T_A^{-1}(\mathcal{O}_x)[1].$$

In particular \mathcal{O}_x is properly σ -semistable for all $x \in X$, and A does not depend on $x \in X$.

(C_k) There are a (-2) -curve C and an integer k such that \mathcal{O}_x is σ -stable if $x \notin C$ and \mathcal{O}_x is properly σ -semistable if $x \in C$. Moreover a JH filtration of \mathcal{O}_x for $x \in C$ is given by

$$\mathcal{O}_C(k+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_C(k)[1] \longrightarrow \mathcal{O}_C(k+1)[1].$$

We recall the map $\Phi_* : \text{Stab}(Y) \rightarrow \text{Stab}(X)$ induced by an equivalence $\Phi : D(Y) \rightarrow D(X)$. Let X and Y be projective K3 surfaces, and let $\Phi : D(Y) \rightarrow D(X)$ be an equivalence. Then Φ induces a natural morphism $\Phi_* : \text{Stab}(Y) \rightarrow \text{Stab}(X)$ as follows:

$$\Phi_* : \text{Stab}(Y) \rightarrow \text{Stab}(X), \quad \Phi_*((\mathcal{A}_Y, Z_Y)) = (\mathcal{A}_X, Z_X),$$

$$\text{where } Z_X(E) = Z_Y(\Phi^{-1}(E)) \text{ and } \mathcal{A}_X = \Phi(\mathcal{A}_Y).$$

Then the following proposition is almost obvious.

PROPOSITION 3.5 ([7, PROPOSITION 6.1])

Let X and Y be projective K3 surfaces, and let $\Phi : D(Y) \rightarrow D(X)$ be an equivalence. For $\sigma \in U(X)$, σ is in $\Phi_*(U(Y))$ if and only if $\Phi(\mathcal{O}_y)$ is σ -stable with the same phase for all closed points $y \in Y$.

4. Stability of classically stable sheaves

The goal of this section is to show the σ -stability of Gieseker stable (or μ -stable) sheaves on a projective K3 surface X for some $\sigma \in \text{Stab}(X)$.

We first prepare a function (4.1), which plays an important role in this section. Let L_0 be an ample line bundle on X with $L_0^2 = 2d$. We define a subset $V(X)_{L_0}$ of $V(X)$ by

$$V(X)_{L_0} := \{ \sigma_{(\beta, \omega)} \in V(X) \mid (\beta, \omega) = (xL_0, yL_0) \text{ where } (x, y) \in \mathbb{R} \times \mathbb{R}_{>0} \}.$$

Take an element $\sigma_{(\beta, \omega)} \in V(X)_{L_0}$. For an arbitrary object $F \in D(X)$ we put the Mukai vector $v(F)$ by $v(F) = r_F \oplus \delta_F \oplus s_F$. We have the orthogonal decomposition of δ_F with respect to L_0 in $\text{NS}(X)_{\mathbb{R}}$:

$$\delta_F = n_F L_0 + \nu_F,$$

where ν_F is in $\text{NS}(X)_{\mathbb{R}}$ with $\nu_F L_0 = 0$. Then we have

$$Z(F) = \frac{v(F)^2}{2r_F} + \frac{r_F}{2} \left(\omega + \sqrt{-1} \left(\frac{\delta_F}{r_F} - \beta \right) \right)^2$$

$$= \frac{v(F)^2}{2r_F} + \frac{r_F}{2} \left(\omega + \sqrt{-1} \left(\frac{n_F L_0}{r_F} - \beta \right) \right)^2 - \frac{\nu_F^2}{2r_F}.$$

We see that the imaginary part $\Im Z(F)$ of $Z(F)$ is given by $2\sqrt{-1}y d\lambda_F$ where $\lambda_F = n_F - r_F x$. Put

$$Z^{L_0}(F) := Z(F) + \frac{\nu_F^2}{2r_F}.$$

We define a function $N_{A,E}(x, y)$ for objects A and $E \in D(X)$ and for $\sigma_{(\beta, \omega)} \in V(X)_{L_0}$ by

$$(4.1) \quad N_{A,E}(x, y) := \lambda_E \Re Z^{L_0}(A) - \lambda_A \Re Z^{L_0}(E),$$

where \Re means taking the real part of a complex number.

Now suppose that E is a μ_ω -semistable torsion-free sheaf where $\omega \in \text{Amp}(X)$. For a stability condition $\sigma_{(\beta, \omega)} = (\mathcal{A}, Z) \in V(X)$ we see the following:

- $\mu_\omega(E) > \beta\omega \iff E \in \mathcal{A}$,
- $\mu_\omega(E) \leq \beta\omega \iff E \in \mathcal{A}[-1]$.

We consider the following three cases: $\mu_\omega(E) > \beta\omega$, $\mu_\omega(E) = \beta\omega$, and $\mu_\omega(E) < \beta\omega$. We first treat the case when $\mu_\omega(E) > \beta\omega$.

LEMMA 4.1

Let X be a projective K3 surface, and let $\sigma_{(\beta, \omega)} = (\mathcal{A}, Z) \in V(X)$. Assume that $A \rightarrow E \rightarrow F$ is a nontrivial distinguished triangle in \mathcal{A} . Namely, A , E , and F are in \mathcal{A} . (This means that the triangle gives a short exact sequence in \mathcal{A} .)

- (1) *If E is a torsion-free sheaf, then A is also a torsion-free sheaf.*
- (2) *In addition to (1), if E is Gieseker stable with respect to ω , then we have $p_\omega(A) < p_\omega(E)$.*
- (3) *Let L_0 be an ample line bundle. In addition to (2), assume $\sigma_{(\beta, \omega)} \in V(X)_{L_0}$ and $\mu_\omega(A) = \mu_\omega(E)$. Then we have $\arg Z(A) < \arg Z(E)$.*

Proof

Let $H^i(F)$ be the i th cohomology of F . Since F is in \mathcal{A} , $H^i(F)$ is zero unless i is zero or -1 . Then one can easily check the first assertion by taking cohomologies to the given distinguished triangle. Hence we start the proof of the second assertion (2).

We have the following exact sequence of sheaves:

$$0 \longrightarrow H^{-1}(F) \longrightarrow A \xrightarrow{f} E \longrightarrow H^0(F) \longrightarrow 0.$$

Suppose that $H^{-1}(F)$ is not zero. One can easily see that

$$\mu_\omega(H^{-1}(F)) \leq \mu_\omega^+(H^{-1}(F)) \leq \beta\omega < \mu_\omega^-(A) \leq \mu_\omega(A).$$

Thus we have $\mu_\omega(H^{-1}(F)) < \mu_\omega(A) < \mu_\omega(\text{Im } f)$ where $\text{Im } f$ is the image of the morphism $f : A \rightarrow E$. Thus we have $p_\omega(A) < p_\omega(\text{Im } f) \leq p_\omega(E)$ since E is Gieseker stable with respect to ω .

Suppose that $H^{-1}(F) = 0$. Then A is a subsheaf of E . Since E is Gieseker stable, the assertion is obvious.

Let us prove the third assertion. We put $L_0^2 = 2d$. For E and A in $D(X)$ we put $v(E) = r_E \oplus \delta_E \oplus s_E$ and $v(A) = r_A \oplus \delta_A \oplus s_A$. We decompose δ_E and δ_A by

$$\delta_E = n_E L_0 + \nu_E \quad \text{and} \quad \delta_A = n_A L_0 + \nu_A,$$

where ν_E and ν_A are in $\text{NS}(X)_{\mathbb{R}}$ with $\nu_E L_0 = \nu_A L_0 = 0$. We remark that both ν_E^2 and ν_A^2 are less than or equal to zero and that the number $m_A = \delta_A L_0$ (resp., $m_E = \delta_E L_0$) is an integer. Then we have $n_A = m_A/2d$ (resp., $n_E = m_E/2d$) and

$$\frac{Z(A)}{r_A} = \frac{v(A)^2}{2r_A^2} + \frac{1}{2} \left(\omega + \sqrt{-1} \left(\frac{n_A L_0}{r_A} - \beta \right) \right)^2 - \frac{\nu_A^2}{2r_A^2}.$$

Now we put $J(A) = (1/2r_A^2)(v(A)^2 - \nu_A^2)$ and $J(E) = (1/2r_E^2)(v(E)^2 - \nu_E^2)$. Then we see

$$J(A) = \frac{1}{2} \left(\frac{n_A^2 L_0^2}{r_A^2} - \frac{s_A}{r_A} \right) \quad \text{and} \quad J(E) = \frac{1}{2} \left(\frac{n_E^2 L_0^2}{r_E^2} - \frac{s_E}{r_E} \right).$$

Since $\mu_{\omega}(E) = \mu_{\omega}(A)$ we see $(Z(E)/r_E) - J(E) = (Z(A)/r_A) - J(A)$. Thus we see that $\arg Z(A) < \arg Z(E)$ if and only if $J(E) < J(A)$. Since $p_{\omega}(A) < p_{\omega}(E)$ and $\mu_{\omega}(A) = \mu_{\omega}(E)$, we see that

$$(4.2) \quad \frac{n_A}{r_A} = \frac{n_E}{r_E} \quad \text{and} \quad \frac{s_A}{r_A} < \frac{s_E}{r_E}.$$

Then the inequality $J(E) < J(A)$ follows from the inequality (4.2). Thus we have finished the proof. □

LEMMA 4.2

Let X be a projective K3 surface, let L_0 be an ample line bundle on X , and let $\sigma_{(\beta, \omega)} = (A, Z) \in V(X)_{L_0}$. Assume that $A \rightarrow E \rightarrow F$ is a distinguished triangle in \mathcal{A} with $\text{hom}_X^0(A, A) = 1$ and that E is a torsion-free sheaf with $\delta_E = n_E L_0$ for an integer n_E where we put $v(E) = r_E \oplus \delta_E \oplus s_E$.

(1) If $v(E)^2 = -2$, $\mu_{\omega}(A) < \mu_{\omega}(E)$, and $(\delta_E L_0 - r_E \beta L_0) \leq \omega^2/2$, then $\arg Z(A) < \arg Z(E)$.

(2) If $v(E)^2 \geq 0$, $\mu_{\omega}(A) < \mu_{\omega}(E)$, and

$$(\delta_E L_0 - r_E \beta L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2},$$

then $\arg Z(A) < \arg Z(E)$.

Proof

We first note that A is a torsion-free sheaf by Lemma 4.1. Since there is no $\sigma_{(\beta, \omega)}$ -stable torsion-free sheaf with phase 1 (see [7, Remark 3.5(1)] or [2, Lemma 10.1]), we see $\mu_{\omega}(A) > \beta\omega$. For the Mukai vector $v(A) = r_A \oplus \delta_A \oplus s_A$ of A we put

$$\delta_A = n_A L_0 + \nu_A,$$

where ν_A is in $\text{NS}(X)_{\mathbb{R}}$ with $\nu_A L_0 = 0$. Then we have

$$\begin{aligned} Z(A) &= \frac{v(A)^2}{2r_A} + \frac{r_A}{2} \left(\omega + \sqrt{-1} \left(\frac{n_A}{r_A} L_0 - r_A \beta \right) \right)^2 - \frac{\nu_A^2}{2r_A} \\ &= Z^{L_0}(A) - \frac{\nu_A^2}{2r_A}. \end{aligned}$$

We note that both $\lambda_A = n_A - r_A x$ and $\lambda_E = n_E - r_E x$ are positive. Since F is in \mathcal{A} , we have $\Im m Z(F) \geq 0$. Thus we see that $\lambda_A \leq \lambda_E$. Since $\nu_A^2 \leq 0$, we see $\arg Z(A) \leq \arg Z^{L_0}(A)$. Thus to prove (1), it is enough to show that $\arg Z^{L_0}(A) < \arg Z(E)$. Since $\Im m Z(A) = \Im m Z^{L_0}(A) = 2yd\lambda_A > 0$ and $\Im m Z(E) = 2yd\lambda_E > 0$, we see

$$\arg Z^{L_0}(A) < \arg Z(E) \iff 0 < N_{A,E}(x, y).$$

Note that $Z^{L_0}(E) = Z(E)$.

Now we have

$$\begin{aligned} N_{A,E}(x, y) &= \lambda_E \Re c Z^{L_0}(A) - \lambda_A \Re c Z(E) \\ &= \lambda_E \left(\frac{v(A)^2}{2r_A} + dr_A y^2 - \frac{d\lambda_A^2}{r_A} \right) - \lambda_A \left(\frac{v(E)^2}{2r_E} + dr_E y^2 - \frac{d\lambda_E^2}{r_E} \right) \\ &= dy^2(r_A n_E - r_E n_A) + \lambda_E \frac{v(A)^2}{2r_A} - \lambda_A \frac{v(E)^2}{2r_E} + d\lambda_A \lambda_E \left(\frac{n_E}{r_E} - \frac{n_A}{r_A} \right). \end{aligned}$$

Since $\mu_\omega(A) < \mu_\omega(E)$ we have $(n_E/r_E) - (n_A/r_A) > 0$ and $r_A n_E - r_E n_A > 0$. Since the last term $d\lambda_A \lambda_E ((n_E/r_E) - (n_A/r_A))$ is positive, we have

$$N_{A,E}(x, y) > N'_{A,E}(x, y) := dy^2(r_A n_E - r_E n_A) + \lambda_E \frac{v(A)^2}{2r_A} - \lambda_A \frac{v(E)^2}{2r_E}.$$

Since $\text{hom}_X^0(A, A) = 1$ we have $v(A)^2 \geq -2$. Thus we see

$$N'_{A,E}(x, y) \geq N''_{A,E}(x, y) := dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} - \lambda_A \frac{v(E)^2}{2r_E}.$$

Hence it is enough to prove that $N''_{A,E}(x, y) \geq 0$.

Let us prove the first assertion (1). Since $v(E)^2 = -2$ we have

$$\begin{aligned} N''_{A,E}(x, y) &= dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} + \frac{\lambda_A}{r_E} \\ &> dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A}. \end{aligned}$$

We shall show

$$dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} > 0.$$

Since both $2dn_A$ and $2dn_E$ are integers, using the condition in (1), we see

$$\begin{aligned} \frac{\omega^2}{2} &= dy^2 > (\delta_E L_0 - r_E \beta L_0) = 2d\lambda_E \\ &\geq \frac{2d\lambda_E}{r_A \cdot 2d(r_A n_E - r_E n_A)} = \frac{\lambda_E}{r_A(r_A n_E - r_E n_A)}. \end{aligned}$$

Hence we have

$$dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} \geq 0$$

by $r_A n_E - r_E n_A > 0$. Thus we have proved the assertion.

Let us prove the second assertion. Essentially the proof is the same as the one of the first assertion. Assume that $v(E)^2 \geq 0$. It is enough to show that $N''_{A,E}(x, y) \geq 0$. Since $0 < \lambda_A \leq \lambda_E$ we have

$$\begin{aligned} (4.3) \quad N''_{A,E}(x, y) &= dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} - \mu_A \frac{v(E)^2}{2r_E} \\ &\geq dy^2(r_A n_E - r_E n_A) - \lambda_E - \mu_E \frac{v(E)^2}{2r_E}. \end{aligned}$$

Hence it is enough to show that $dy^2(r_A n_E - r_E n_A) - \lambda_E - \mu_E \frac{v(E)^2}{2r_E} \geq 0$. Similarly to the first assertion, one can easily prove this inequality by using the assumption

$$\frac{\omega^2}{2} \geq (\delta_E L_0 - r_E \beta L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right).$$

Thus we have proved the second assertion. □

COROLLARY 4.3

Notation and assumptions are as in Lemma 4.2. Furthermore we assume that $\text{NS}(X) = \mathbb{Z}L_0$.

(1) *If $v(E)^2 = -2$, $\mu_\omega(A) < \mu_\omega(E)$, and $(1/L_0^2)(\delta_E L_0 - r_E \beta L_0) \leq \omega^2/2$, then $\arg Z(A) < \arg Z(E)$.*

(2) *If $v(E)^2 \geq 0$, $\mu_\omega(A) < \mu_\omega(E)$, and*

$$\frac{1}{L_0^2}(\delta_E L_0 - r_E \beta L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2},$$

then $\arg Z(A) < \arg Z(E)$.

Proof

We use the same notation as in the proof of Lemma 4.2.

Let us prove the first assertion. Suppose that $v(E)^2 = -2$. By using the same argument in the proof of Lemma 4.2, one can see that it is enough to show that

$$(4.4) \quad 0 \leq dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A}.$$

Since $\text{NS}(X) = \mathbb{Z}L_0$, we see $n_A \in \mathbb{Z}$. Thus the inequality (4.4) follows from the assumption $(1/L_0^2)(\delta_E L_0 - r_E \beta L_0) \leq \omega^2/2$.

One can easily prove the second assertion since the proof is essentially the same as the first assertion. In fact one can easily see that it is enough to show

$$(4.5) \quad 0 \leq dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} \left(\frac{v(E)^2}{2r_E} + 1 \right)$$

instead of (4.4) as above. This inequality follows from the assumption $(1/L_0^2) \times (\delta_E L_0 - r_E \beta L_0)((v(E)^2/2r_E) + 1) \leq \omega^2/2$. \square

THEOREM 4.4

Let X be a projective K3 surface, let L_0 be an ample line bundle, and let $\sigma_{(\beta,\omega)} = (\mathcal{A}, Z) \in V(X)_{L_0}$. We assume that E is a Gieseker stable, torsion-free sheaf with respect to L_0 with $\mu_\omega(E) > \beta\omega$ and that the Mukai vector $v(E)$ is $r_E \oplus \delta_E \oplus s_E$ with $\delta_E = n_E L_0$ for some $n_E \in \mathbb{Z}$.

- (1) Assume that $v(E)^2 = -2$. If $\delta_E L_0 - r_E \beta L_0 \leq \omega^2/2$, then E is $\sigma_{(\beta,\omega)}$ -stable.
- (2) Assume that $v(E)^2 \geq 0$. If $(\delta_E L_0 - r_E \beta L_0)((v(E)^2/2r_E) + 1) \leq \omega^2/2$, then E is $\sigma_{(\beta,\omega)}$ -stable.
- (3) Assume that $\text{NS}(X) = \mathbb{Z}L_0$ and that $v(E)^2 = -2$. If $(1/L_0^2)(\delta_E L_0 - r_E \beta L_0) \leq \omega^2/2$, then E is $\sigma_{(\beta,\omega)}$ -stable.
- (4) Assume that $\text{NS}(X) = \mathbb{Z}L_0$ and that $v(E)^2 \geq 0$. If $(1/L_0^2)(\delta_E L_0 - r_E \beta L_0)((v(E)^2/2r_E) + 1) \leq \omega^2/2$, then E is $\sigma_{(\beta,\omega)}$ -stable.

Proof

Suppose to the contrary that E is not $\sigma_{(\beta,\omega)}$ -stable. Then there is a $\sigma_{(\beta,\omega)}$ -stable subobject A of E in \mathcal{A} with $\arg Z(A) \geq \arg Z(E)$ and we have the following distinguished triangle in \mathcal{A} :

$$A \longrightarrow E \longrightarrow F \longrightarrow A[1].$$

Since E is Gieseker stable with respect to $\omega = yL_0$, we see that A is a torsion-free sheaf with $p_\omega(A) < p_\omega(E)$ by Lemma 4.1. Since $p_\omega(A) < p_\omega(E)$ we see $\mu_\omega(A) \leq \mu_\omega(E)$. If $\mu_\omega(A) = \mu_\omega(E)$, then $\arg Z(A) < \arg Z(E)$ by Lemma 4.1. Thus $\mu_\omega(A)$ should be strictly smaller than $\mu_\omega(E)$. Then whether $v(E)^2 = -2$ or $v(E)^2 \geq 0$, we see $\arg Z(A) < \arg Z(E)$ by Lemma 4.2. Hence E is $\sigma_{(\beta,\omega)}$ -stable.

The proofs of the third and fourth assertions are essentially the same as the proof of Theorem 4.4. The difference is to use Corollary 4.3 instead of Lemma 4.2. \square

Next we consider the case $\mu_\omega(E) = \beta\omega$.

PROPOSITION 4.5

Let X be a projective K3 surface, and let $\sigma_{(\beta,\omega)} = (\mathcal{A}, Z) \in V(X)$. Assume that the Mukai vector of an object $E \in D(X)$ is $r_E \oplus \delta_E \oplus s_E$ with $r_E \neq 0$ and $\delta_E \omega / r_E = \beta\omega$.

- (1) If E is a μ_ω -semistable torsion-free sheaf, then E is $\sigma_{(\beta,\omega)}$ -semistable with phase zero.
- (2) The object E is a μ_ω -stable locally free sheaf if and only if E is $\sigma_{(\beta,\omega)}$ -stable with phase zero.

Proof

Let us prove the first assertion. Since E is μ_ω -semistable, E is in $\mathcal{A}[-1]$. Since $\mu_\omega(E) = \beta\omega$, the imaginary part $\Im Z(E)$ of $Z(E)$ is zero. Thus the argument of $Z(E)$ is zero.

Assume that E is not $\sigma_{(\beta,\omega)}$ -semistable. Then there is a $\sigma_{(\beta,\omega)}$ -semistable object $A \in \mathcal{A}[-1]$ such that

$$A \subset E \quad \text{in } \mathcal{A}[-1] \text{ with } \arg Z(A) > \arg Z(E) = 0.$$

This contradicts the fact that A is in $\mathcal{A}[-1]$. Hence E is $\sigma_{(\beta,\omega)}$ -semistable.

Let us prove the second assertion. We assume that E is a μ_ω -stable locally free sheaf. Then E is minimal in $\mathcal{A}[-1]^*$ by [4, Theorem 0.2]. Thus E is $\sigma_{(\beta,\omega)}$ -stable with phase zero.

Conversely we assume that E is $\sigma_{(\beta,\omega)}$ -stable with phase zero. Since the rank of E is not zero, E is a locally free sheaf by [2, Lemma 10.1(b)]. Since E is in $\mathcal{A}[-1]$, we see $E \in \mathcal{F}_{(\beta,\omega)}$. Thus we have

$$\mu_\omega(E) \leq \mu_\omega^+(E) \leq \beta\omega.$$

Thus equalities should hold. Hence E is μ_ω -semistable.

Suppose that E is not μ_ω -stable. Then there is a μ_ω -stable subsheaf A of E such that $\mu_\omega(A) = \mu_\omega(E)$. If necessary, by taking a saturation, we may assume that the quotient E/A is a torsion-free sheaf. We remark that E/A is μ_ω -semistable. Then A is locally free since E is locally free and $\dim X = 2$. Since A is a μ_ω -stable locally free sheaf, A is $\sigma_{(\beta,\omega)}$ -stable with phase zero. Thus the short exact sequence $A \rightarrow E \rightarrow E/A$ defines a distinguished triangle in $\mathcal{A}[-1]$. In particular A is a subobject of E in $\mathcal{A}[-1]$ with phase zero. This contradicts the fact that E is $\sigma_{(\beta,\omega)}$ -stable. \square

Finally we treat the case $\mu_\omega(E) < \beta\omega$.

LEMMA 4.6

Let X be a projective K3 surface, and let $\sigma_{(\beta,\omega)} = (\mathcal{A}, Z) \in V(X)$. Assume that $F \rightarrow E \rightarrow A$ is a distinguished triangle in $\mathcal{A}[-1]$.

- (1) If E is a torsion-free sheaf, then A is a torsion-free sheaf.
- (2) If E is a μ_ω -stable locally free sheaf, then $\mu_\omega(E) < \mu_\omega(A)$.

We remark that the proof of [7, Lemma 4.4] completely works.

Proof

One can easily prove the first assertion by taking cohomologies to the triangle $F \rightarrow E \rightarrow A$. Thus let us prove the second assertion. Since F , E , and A are in $\mathcal{A}[-1]$, we have an exact sequence of sheaves

*Namely, E has no nontrivial subobject in $\mathcal{A}[-1]$.

$$0 \longrightarrow H^0(F) \longrightarrow E \xrightarrow{f} A \longrightarrow H^1(F) \longrightarrow 0,$$

where $H^i(F)$ is the i th cohomology of F .

Assume that $H^0(F) \neq 0$. Since $H^0(F)$ is torsion-free, $\text{rank Im } f < \text{rank } E$, where $\text{Im } f$ is the image of the morphism $f : E \rightarrow A$. Thus $\mu_\omega(E) < \mu_\omega(\text{Im } f)$. By using the fact that $H^1(F) \in \mathcal{T}_{(\beta, \omega)}$, one can prove that $\mu_\omega(\text{Im } f) \leq \mu_\omega(A)$. Thus we have $\mu_\omega(E) < \mu_\omega(A)$.

Assume that $H^0(F) = 0$. Thus $F = H^1(F)$. Then E is a subsheaf of A . If $\text{rank } F$ is not zero, then we have $\mu_\omega(A) \leq \beta\omega < \mu_\omega(F)$. Thus we have $\mu_\omega(E) < \mu_\omega(A)$. Suppose that $\text{rank } F = 0$. If the dimension of the support of F is 1, then $c_1(F)\omega > 0$. Hence we see that $\mu_\omega(E) < \mu_\omega(A)$. Thus suppose that F is a torsion sheaf with $\dim \text{Supp}(F) = 0$. Take a closed point $x \in \text{Supp}(F)$. By taking the right derived functor $\mathbb{R}\text{Hom}_X(\mathcal{O}_x, -)$ to the triangle $E \rightarrow A \rightarrow F$, we have the following exact sequence of \mathbb{C} -vector spaces:

$$\text{Hom}_X^0(\mathcal{O}_x, E) \rightarrow \text{Hom}_X^0(\mathcal{O}_x, A) \rightarrow \text{Hom}_X^0(\mathcal{O}_x, F) \rightarrow \text{Hom}_X^1(\mathcal{O}_x, E).$$

Since E is locally free we see $\text{Hom}_X^0(\mathcal{O}_x, E) = \text{Hom}_X^1(\mathcal{O}_x, E) = 0$. Since x is in the support of F , $\text{Hom}_X^0(\mathcal{O}_x, F)$ should not be zero. This contradicts the torsion freeness of A . Hence we have proved the assertion. \square

LEMMA 4.7

Let X be a projective K3 surface, let L_0 be an ample line bundle, let $\sigma_{(\beta, \omega)} = (A, Z) \in V(X)_{L_0}$, and let $F \rightarrow E \rightarrow A$ be a distinguished triangle in $\mathcal{A}[-1]$. We put $v(E) = r_E \oplus \delta_E \oplus s_E$. Assume that $\text{hom}_X^0(A, A) = 1$, both $\text{rank } E$ and $\text{rank } A$ are positive, and $\delta_E = n_E L_0$ for some integer n_E .

(1) If $v(E)^2 = -2$, $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$ and $r_E \beta L_0 - \delta_E L_0 \leq \omega^2/2$, then $\arg Z(E) < \arg Z(A)$.

(2) If $v(E)^2 \geq 0$, $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$, and

$$(r_E \beta L_0 - \delta_E L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2},$$

then $\arg Z(E) < \arg Z(A)$.

Proof

The proof is essentially the same as that of Lemma 4.2. We put $L_0^2 = 2d$ and $v(A) = r_A \oplus \delta_A \oplus s_A$ with $\delta_A = n_A L_0 + \nu_A$, where $\nu_A \in \text{NS}(X)_{\mathbb{R}}$ with $\nu_A L_0 = 0$. If we put $m_A = \delta_A L_0 \in \mathbb{Z}$, then we have $n_A = m_A/2d$.

Now we have

$$\begin{aligned} Z(A) &= \frac{v(A)^2}{2r_A} + \frac{r_A}{2} \left(\omega + \sqrt{-1} \left(\frac{n_A L_0}{r_A} - \beta \right) \right)^2 - \frac{\nu_A^2}{2r_A} \\ &= Z^{L_0}(A) - \frac{\nu_A^2}{2r_A}. \end{aligned}$$

Since $\nu_A^2 \leq 0$ we have $\arg Z^{L_0}(A) \leq \arg Z(A)$. Thus it is enough to show that $\arg Z(E) < \arg Z^{L_0}(A)$. We put $\lambda_E = n_E - r_E x$ and $\lambda_A = n_A - r_A x$. We remark

that both λ_E and λ_A are negative and $\lambda_E \leq \lambda_A < 0$ by the fact $F \in \mathcal{A}[-1]$. Hence we see

$$\arg Z(E) < \arg Z^{L_0}(A) \iff N_{A,E}(x, y) < 0.$$

Now we have

$$N_{A,E}(x, y) = dy^2(r_A n_E - r_E n_A) + d\lambda_A \lambda_E \left(\frac{n_E}{r_E} - \frac{n_A}{r_A} \right) + \frac{v(A)^2}{2r_A} \lambda_E - \frac{v(E)^2}{2r_E} \lambda_A.$$

Since $\mu_\omega(E) < \mu_\omega(A)$ we see $r_A n_E - r_E n_A < 0$. Thus we have

$$N_{A,E}(x, y) < N'_{A,E}(x, y) := dy^2(r_A n_E - r_E n_A) + \frac{v(A)^2}{2r_A} \lambda_E - \frac{v(E)^2}{2r_E} \lambda_A.$$

Since $\text{hom}_X^0(A, A) = 1$ we have $v(A)^2 \geq -2$. Thus we see

$$N'_{A,E}(x, y) \leq N''_{A,E}(x, y) := dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} - \frac{v(E)^2}{2r_E} \lambda_A.$$

Hence it is enough to show $N''_{A,E}(x, y) \leq 0$.

Let us show (1). Assume that $v(E)^2 = -2$, then

$$\begin{aligned} N''_{A,E}(x, y) &= dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} + \frac{\lambda_A}{r_E} \\ &\leq dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A}. \end{aligned}$$

Hence it is enough to show that

$$(4.6) \quad dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} \leq 0.$$

Recall that $n_A = m_A/2d$ for some integer m_A and $d \in \mathbb{Z}$. Then the inequality (4.6) follows from the assumption

$$r_E \beta L_0 - \delta_E L_0 \leq \frac{\omega^2}{2}.$$

Thus we have finished the proof.

Next we shall show (2). Assume that $v(E)^2 \geq 0$. Then we have

$$\begin{aligned} N''_{A,E}(x, y) &= dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} - \frac{v(E)^2}{2r_E} \lambda_A \\ &\leq dy^2(r_A n_E - r_E n_A) - \lambda_E - \frac{v(E)^2}{2r_E} \lambda_E. \end{aligned}$$

Hence it is enough to show that

$$(4.7) \quad dy^2(r_A n_E - r_E n_A) - \lambda_E - \frac{v(E)^2}{2r_E} \lambda_E \leq 0.$$

The inequality (4.7) is equivalent to the inequality

$$(4.8) \quad \frac{-\lambda_E}{r_A(r_E n_A - r_A n_E)} \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq dy^2.$$

The last inequality (4.8) follows from the assumption

$$(r_E\beta L_0 - \delta_E L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2}.$$

Thus we have proved the assertion. \square

Similarly to the case of Corollary 4.3, we have the following corollary. We omit the proof since the proof is as the same as the proof of Lemma 4.7.

COROLLARY 4.8

Notation and assumptions are as in Lemma 4.7. Furthermore we assume that $\text{NS}(X) = \mathbb{Z}L_0$.

(1) *Assume that $v(E)^2 = -2$, $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$, and $(1/L_0^2)(r_E\beta L_0 - \delta_E L_0) \leq \omega^2/2$. Then we have $\arg Z(E) < \arg Z(A)$.*

(2) *Assume that $v(E)^2 \geq 0$, $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$, and*

$$\frac{1}{L_0^2}(r_E\beta L_0 - \delta_E L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2}.$$

Then we have $\arg Z(E) < \arg Z(A)$.

THEOREM 4.9

Let X be a projective K3 surface, let L_0 be an ample line bundle, and let $\sigma_{(\beta,\omega)} = (\mathcal{A}, Z) \in V(X)_{L_0}$. Assume that E is a μ_{L_0} -stable locally free sheaf. We put $v(E) = r_E \oplus \delta_E \oplus s_E$. Assume that $\delta_E = n_E L_0$ and $\mu_\omega(E) < \beta\omega$ where $n_E \in \mathbb{Z}$.

(1) *Assume that $v(E)^2 = -2$. If $(r_E\beta L_0 - \delta_E L_0) < \omega^2/2$, then E is $\sigma_{(\beta,\omega)}$ -stable.*

(2) *Assume that $v(E)^2 \geq 0$. If $(r_E\beta L_0 - \delta_E L_0)((v(E)^2/2r_E) + 1) < \omega^2/2$, then E is $\sigma_{(\beta,\omega)}$ -stable.*

(3) *Assume that $\text{NS}(X) = \mathbb{Z}L_0$, and assume that $v(E)^2 = -2$. If*

$$\frac{1}{L_0^2}(r_E\beta L_0 - \delta_E L_0) < \frac{\omega^2}{2},$$

then E is $\sigma_{(\beta,\omega)}$ -stable.

(4) *Assume that $\text{NS}(X) = \mathbb{Z}L_0$, and assume that $v(E)^2 \geq 0$. If*

$$\frac{1}{L_0^2}(r_E\beta L_0 - \delta_E L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) < \frac{\omega^2}{2},$$

then E is $\sigma_{(\beta,\omega)}$ -stable.

Proof

Since $\mu_\omega(E) < \beta\omega$, E is in $\mathcal{A}[-1]$ and $\arg Z(E) < 0$. Suppose to the contrary that E is not $\sigma_{(\beta,\omega)}$ -stable. Then there is a $\sigma_{(\beta,\omega)}$ -stable object A such that A is a quotient of E in $\mathcal{A}[-1]$ with $\arg Z(A) \leq \arg Z(E)$. Thus we have a distinguished

triangle in $\mathcal{A}[-1]$:

$$F \longrightarrow E \longrightarrow A \longrightarrow F[1].$$

By Lemma 4.6, A is a torsion-free sheaf with $\mu_\omega(E) < \mu_\omega(A)$. Since A is in $\mathcal{A}[-1]$, we see that $\mu_\omega(A) \leq \beta\omega$. If $\mu_\omega(A) = \beta\omega$, then the imaginary part of $Z(A)$ is zero. Thus A is $\sigma_{(\beta,\omega)}$ -stable with phase zero. This contradicts $\arg Z(A) < \arg Z(E) < 0$. Hence $\mu_\omega(A) < \beta\omega$. Then we see that $\arg Z(E) < \arg Z(A)$ by Lemma 4.7 whether $v(E)^2 = -2$ or $v(E)^2 \geq 0$. This is a contradiction. Thus E is $\sigma_{(\beta,\omega)}$ -stable.

The proofs of (3) and (4) are essentially the same as that of Theorem 4.9. One can easily show these assertions by using Corollary 4.8 instead of Lemma 4.7. \square

5. First application

The goal of this section is to prove Theorem 5.4 as an application of Corollaries 4.4 and 4.9. We shall give a classification of fine moduli spaces of Gieseker stable, torsion-free sheaves on a projective K3 surface with Picard number one. In this section the pair (X, L) is called a *generic K3* if X is a projective K3 surface and $\text{NS}(X)$ is generated by an ample line bundle.

We shall start this section with an easy observation. Suppose that E is a Gieseker stable, torsion-free sheaf on a generic K3 (X, L) . Since E is Gieseker stable we have $v(E)^2 \geq -2$. Assume that $v(E)^2 = -2$. Then $\text{hom}_X^1(E, E) = 0$. Thus E is a spherical sheaf. It is known that E is a μ -stable locally free sheaf (for instance, see [7, Proposition 5.2]). Thus the notion of μ -stability is equivalent to the notion of Gieseker stability if $v(E)^2 = -2$.

Next we consider the case $v(E)^2 \geq 0$. The following proposition plays a key role in this section.

PROPOSITION 5.1

Let X be a projective K3 surface, and let L be an ample line bundle. Assume that E is a Gieseker stable, torsion-free sheaf with respect to L with $v(E)^2 = 0$.

(1) *Assume that $\text{rank } E > 1$. If E is μ -stable with respect to L , then E is locally free.*

(2) *Assume that $\text{NS}(X) = \mathbb{Z}L$. If E is locally free, then E is μ -stable with respect to L .*

In particular if $\text{NS}(X) = \mathbb{Z}L$ and $\text{rank } E > 1$, then the following holds: If E is not μ -stable locally free, then E is neither μ -stable nor locally free.

Proof

The first assertion was proved in step (vii) in the proof of [4, Proposition 4.1].

Hence, let us prove the second assertion. For any $F \in D(X)$ we put $v(F) = r_F \oplus \delta_F \oplus s_F$. Assume that E is not μ -stable. Then there is a μ -stable subsheaf A of E such that $\mu_L(A) = \mu_L(E)$ and the quotient E/A is torsion-free. Since E is locally free, A is also locally free. We remark that $p_L(A) < p_L(E)$ since E is

Gieseker stable. We remark that $s_A/r_A < s_E/r_E$ by $\mu_L(A) = \mu_L(E)$. Hence we have

$$0 = \frac{v(E)^2}{r_E^2} = \frac{\delta_E^2}{r_E^2} - 2\frac{s_E}{r_E} < \frac{\delta_A^2}{r_A^2} - 2\frac{s_A}{r_A} = \frac{v(A)^2}{r_A^2}.$$

Thus $v(A)^2 > 0$.

We choose $\sigma_{(\beta,\omega)} \in V(X)$ such that $\mu_\omega(E) = \mu_\omega(A) = \beta\omega$. Then E is $\sigma_{(\beta,\omega)}$ -semistable with phase zero, and A is $\sigma_{(\beta,\omega)}$ -stable with phase zero by Proposition 4.5. Since $\sigma_{(\beta,\omega)}$ is locally finite we have a distinguished triangle

$$A' \longrightarrow E \longrightarrow E/A',$$

where all stable factors of A' are A and $\text{hom}_X^0(A', E/A') = 0$. Then by Lemma 2.3, we see $\text{hom}_X^1(A', A') \leq 2$. Thus $v(A')^2 \leq 0$. However, since A' is an extension of A , we have $v(A') = \ell v(A)$ for some $\ell \in \mathbb{N}$. Thus $v(A')^2 = \ell^2 v(A)^2 > 0$. This is a contradiction. Hence E is μ -stable. \square

Suppose that (X, L) is a generic K3, and take an element $v = r \oplus \delta \oplus s \in \mathcal{N}(X)$ with $r > 0$ and $v^2 \geq -2$. We define subsets of $V(X)$ depending on v .

Case 1: $v^2 = -2$. We have

$$\begin{aligned} V_v^+ &:= \left\{ \sigma_{(\beta,\omega)} \in V(X) \mid \beta\omega < \frac{\delta\omega}{r}, \frac{1}{L^2}(\delta L - r\beta L) \leq \frac{\omega^2}{2} \right\}, \\ V_v^0 &:= \left\{ \sigma_{(\beta,\omega)} \in V(X) \mid \beta\omega = \frac{\delta\omega}{r} \right\}, \\ V_v^- &:= \left\{ \sigma_{(\beta,\omega)} \in V(X) \mid \beta\omega > \frac{\delta\omega}{r}, \frac{-1}{L^2}(\delta L - r\beta L) \leq \frac{\omega^2}{2} \right\}. \end{aligned}$$

Case 2: $v^2 \geq 0$. We have

$$\begin{aligned} V_v^+ &:= \left\{ \sigma_{(\beta,\omega)} \in V(X) \mid \beta\omega < \frac{\delta\omega}{r}, \frac{1}{L^2}(\delta L - r\beta L) \left(\frac{v^2}{2r} + 1 \right) \leq \frac{\omega^2}{2} \right\}, \\ V_v^0 &:= \left\{ \sigma_{(\beta,\omega)} \in V(X) \mid \beta\omega = \frac{\delta\omega}{r} \right\}, \\ V_v^- &:= \left\{ \sigma_{(\beta,\omega)} \in V(X) \mid \beta\omega > \frac{\delta\omega}{r}, \frac{-1}{L^2}(\delta L - r\beta L) \left(\frac{v^2}{2r} + 1 \right) \leq \frac{\omega^2}{2} \right\}. \end{aligned}$$

For instance, take a Gieseker stable, torsion-free sheaf E on (X, L) with $v(E)^2 = 0$, and put $v = v(E) = r \oplus \delta \oplus s$. Then the picture of the sets V_v^+ , V_v^0 , and V_v^- is given by Figure 1. In Proposition 5.2 (below), we show that the set V_v^0 is a wall if and only if E is not μ -stable locally free but Gieseker stable torsion-free.

PROPOSITION 5.2

Let (X, L) be a generic K3, and let E be a Gieseker stable, torsion-free sheaf with $v(E)^2 \geq 0$.

- (1) If the sheaf E is not locally free, then E is not σ -semistable for any $\sigma \in V_{v(E)}^-$.

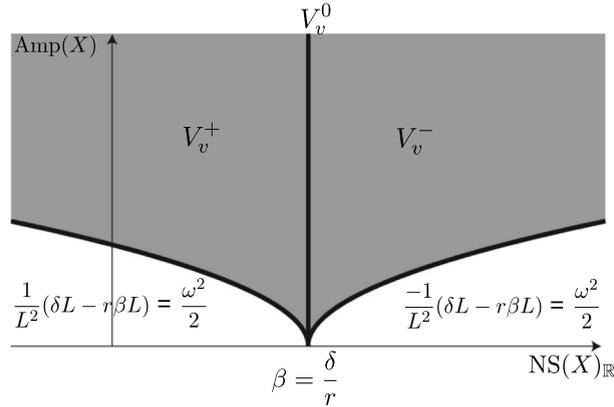


Figure 1

(2) If the sheaf E is not μ -stable, then E is not σ -semistable for any $\sigma \in V_{v(E)}^-$.

(3) Take an arbitrary $\sigma \in V_{v(E)}^-$. For the sheaf E , the following three conditions are equivalent:

- (i) E is σ -stable,
- (ii) E is σ -semistable, and
- (iii) E is μ -stable and locally free.

Proof

For an object $F \in D(X)$ we put $v(F) = r_F \oplus \delta_F \oplus s_F$. Take an arbitrary element $\sigma_0 = (\mathcal{A}, Z) \in V_{v(E)}^-$.

Let us prove assertion (1). Suppose to the contrary that E is σ_0 -semistable. Since E is not locally free, we have the following distinguished triangle by taking double dual of E :

$$S[-1] \longrightarrow E \longrightarrow E^{\vee\vee},$$

where $S = E^{\vee\vee}/E$. Note that S is a torsion sheaf with $\dim \text{Supp}(S) = 0$. Hence $S[-1]$ is σ_0 -semistable with phase zero. Since $\sigma_0 \in V_{v(E)}^-$ we see that $\text{Im} Z(E) < 0$. Hence E is σ -semistable with phase $\phi \in (-1, 0)$. Thus $\arg Z(E) < \arg Z(S[-1])$, and $\text{Hom}_X(S[-1], E)$ should be zero. This contradicts the above triangle. Hence E is not σ_0 -semistable.

Let us prove assertion (2). Suppose to the contrary that E is σ_0 -semistable. Since E is not μ -stable, there is a torsion-free quotient A of E such that A is μ -stable with $\mu_L(A) = \mu_L(E)$. Since E is Gieseker stable we have $p_L(E) < p_L(A)$. Thus we see that $s_E/r_E < s_A/r_A$. Moreover we can assume that A is locally free. In fact, if necessary it is enough to take the double dual of A . Then we see that $\mu_L(A^{\vee\vee}) = \mu_L(E)$, $s_E/r_E < s_{A^{\vee\vee}}/r_{A^{\vee\vee}}$, and $A^{\vee\vee}$ is μ -stable. Thus we can assume that A is a μ -stable locally free sheaf. Note that $\text{Hom}_X^0(E, A) \neq 0$.

We show $V_{v(E)}^- \subset V_{v(A)}^-$. Note that

$$\begin{aligned}
 (5.1) \quad r_A \beta L - \delta_A L &= r_A \left(\beta L - \frac{\delta_A}{r_A} L \right) = r_A \left(\beta L - \frac{\delta_E}{r_E} L \right) \\
 &< r_E \left(\beta L - \frac{\delta_E}{r_E} L \right) = r_E \beta L - \delta_E L.
 \end{aligned}$$

Here by the fact that $\text{NS}(X) = \mathbb{Z}L$, we have $\beta L - (\delta_E/r_E)L > 0$, from which the inequality above holds.

Since A is μ -stable we have $v(A)^2 \geq -2$. By the definition of $V_{v(A)}^-$, we have to consider two cases. We first assume that $v(A)^2 = -2$. Since $v(E)^2 \geq 0$, we have

$$1 \leq \frac{v(E)^2}{2r_E} + 1.$$

Then we see

$$r_A \beta L - \delta_A L < (r_E \beta L - \delta_E L) \left(\frac{v(E)^2}{2r_E} + 1 \right).$$

Hence we see $V_{v(E)}^- \subset V_{v(A)}^-$ by the definition of $V_{v(E)}^-$.

Next suppose that $v(A)^2 \geq 0$. Then by using the fact that $\text{NS}(X) = \mathbb{Z}L$ we have

$$\begin{aligned}
 (5.2) \quad \frac{v(A)^2}{r_A} &= \left(\frac{\delta_A^2}{r_A^2} - 2 \frac{s_A}{r_A} \right) r_A \\
 &< \left(\frac{\delta_E^2}{r_E^2} - 2 \frac{s_E}{r_E} \right) r_A \\
 &< \left(\frac{\delta_E^2}{r_E^2} - 2 \frac{s_E}{r_E} \right) r_E = \frac{v(E)^2}{r_E}.
 \end{aligned}$$

By two inequalities (5.1) and (5.2) we see

$$(r_A \beta L - \delta_A L) \left(\frac{v(A)^2}{2r_A} + 1 \right) < (r_E \beta L - \delta_E L) \left(\frac{v(E)^2}{2r_E} + 1 \right).$$

Thus we have proved $V_{v(E)}^- \subset V_{v(A)}^-$.

Recall that A is a μ -stable locally free sheaf. Since the stability condition σ_0 is in $V_{v(A)}^-$, A is σ_0 -stable by Theorem 4.9. Now we have

$$\begin{aligned}
 \frac{Z(A)}{r_A} &= \frac{v(A)^2}{2r_A^2} + \frac{1}{2} \left(\omega + \sqrt{-1} \left(\frac{\delta_A}{r_A} - \beta \right) \right)^2 \\
 &= \frac{v(A)^2}{2r_A^2} - \frac{v(E)^2}{2r_E^2} + \frac{v(E)^2}{2r_E^2} + \frac{1}{2} \left(\omega + \sqrt{-1} \left(\frac{\delta_E}{r_E} - \beta \right) \right)^2 \\
 &= \frac{Z(E)}{r_E} + \frac{v(A)^2}{2r_A^2} - \frac{v(E)^2}{2r_E^2}.
 \end{aligned}$$

Here we used the fact $\text{NS}(X) = \mathbb{Z}L$ in the second equality. Since $\mu_L(A) = \mu_L(E)$, $s_E/r_E < s_A/r_A$, and $\text{NS}(X) = \mathbb{Z}L$, we see that $(v(A)^2/2r_A^2) - (v(E)^2/2r_E^2)$ is a

negative number. Hence we see that

$$\arg \frac{Z(A)}{r_A} < \arg \frac{Z(E)}{r_E}.$$

This contradicts $\text{Hom}_X^0(E, A) \neq 0$ since both A and E are σ_0 -semistable. Thus E is not σ_0 -semistable.

Let us prove the third assertion. We claim that (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a). The first claim (a) \Rightarrow (b) is trivial. The second claim (b) \Rightarrow (c) follows from the contrapositions of Propositions 5.2(1) and 5.2(2). The third claim (c) \Rightarrow (a) is nothing but Theorem 4.9. Thus we have finished the proof. \square

Take a stability condition $\sigma_{(\beta, \omega)} \in V(X)$ and a μ -semistable torsion-free sheaf E with $\mu_\omega(E) = \beta\omega$. By Proposition 4.5, if E is not a μ -stable locally free sheaf, then E is properly σ -semistable. Hence it makes sense to consider a Jordan–Hölder filtration of E with respect to $\sigma_{(\beta, \omega)}$.

LEMMA 5.3

Let X be a projective K3 surface. Take a $\sigma_{(\beta, \omega)} \in V(X)$. Assume that E is a μ_ω -semistable, torsion-free sheaf with $\mu_\omega(E) = \beta\omega$ and that the filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E$$

is a Jordan–Hölder filtration of E with respect to μ_ω -stability. Namely, $A_i = E_i/E_{i-1}$ ($i = 1, 2, \dots, n$) is a μ_ω -stable, torsion-free sheaf with $\mu_\omega(A_i) = \mu_\omega(E)$. Then σ -stable factors of E consist of all $A_i^{\vee\vee}$ and σ -stable factors of $A_i^{\vee\vee}/A_i[-1]$ ($i = 1, 2, \dots, n$).

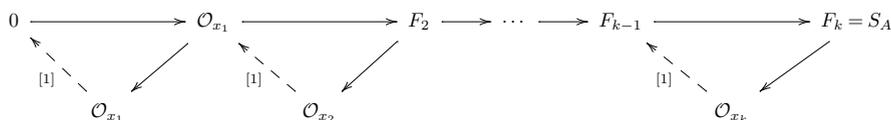
Proof

We put $\sigma = \sigma_{(\beta, \omega)}$. All A_i ($i = 1, 2, \dots, k$) are σ -semistable by Proposition 4.5. If we obtain JH filtrations of A_i , we can construct a JH filtration of E by combining JH filtrations of A_i . Hence it is enough to prove the assertion for μ -stable, torsion-free sheaves.

Suppose that A is a μ_ω -stable, torsion-free sheaf with $\mu_\omega(A) = \beta\omega$, and put $S_A = A^{\vee\vee}/A$. Then we have a distinguished triangle:

$$S_A[-1] \longrightarrow A \longrightarrow A^{\vee\vee} \longrightarrow S_A.$$

Since the dimension of the support of S_A is zero, there are finite closed points $\{x_1, x_2, \dots, x_k\}^*$ such that



*There may be i and j in $\{1, 2, \dots, k\}$, so that $x_i = x_j$.

Since \mathcal{O}_{x_i} ($i = 1, 2, \dots, k$) and $A^{\vee\vee}$ are σ -stable, these are σ -stable factors of A , and the JH filtration of A with respect to σ is given by



Thus we have finished the proof. □

In the next theorem, we give a classification of moduli spaces of Gieseker stable, torsion-free sheaves on a generic K3 (X, L) . Let Y be the fine moduli space of Gieseker-stable, torsion-free sheaves with Mukai vector $v = r \oplus \delta \oplus s$ and with $v^2 = 0$, and let \mathcal{E} be a universal family of the moduli Y . We define an equivalence $\Phi_{\mathcal{E}} : D(Y) \rightarrow D(X)$ by

$$\Phi_{\mathcal{E}}(-) = \mathbb{R}\pi_{X*}(\mathcal{E} \otimes^{\mathbb{L}} \pi_Y^*(-)),$$

where π_X (resp., π_Y) is the projection $X \times Y \rightarrow X$ (resp., $X \times Y \rightarrow Y$). To avoid the complexity in notation, we write V^+ (resp., V^0 and V^-) instead of $V_{v(\Phi(\mathcal{O}_y))}^+$ (resp., $V_{v(\Phi(\mathcal{O}_y))}^0$ and $V_{v(\Phi(\mathcal{O}_y))}^-$) for the given equivalence $\Phi_{\mathcal{E}} : D(Y) \rightarrow D(X)$.

THEOREM 5.4

Let Y be the fine moduli space of Gieseker-stable, torsion-free sheaves on X with Mukai vector $v = r \oplus \delta \oplus s$ and $v^2 = 0$. Suppose that \mathcal{E} is the universal family of Y and that $\Phi_{\mathcal{E}}$ is the equivalence induced by \mathcal{E} .

- (1) *If r is not a square number, then Y is the fine moduli space of a μ -stable locally free sheaf.*
- (2) *Assume that r is a square number. Then one of the following two cases occurs:*
 - (a) *Y is the fine moduli space of μ -stable locally free sheaves.*
 - (b) *Y is the fine moduli space of properly Gieseker stable, torsion-free sheaves and Y is isomorphic to X . Moreover $\Phi_{\mathcal{E}}$ is the spherical twist by a spherical locally free sheaf up to an isomorphism $Y \rightarrow X$.*

Proof

We note that Y is either the fine moduli space of properly Gieseker stable, torsion-free sheaves or the moduli stable of a μ -stable locally free sheaf by Proposition 5.1. Let $\Phi_{\mathcal{E}*}$ be a natural map $\Phi_{\mathcal{E}*} : \text{Stab}(Y) \rightarrow \text{Stab}(X)$ induced by $\Phi_{\mathcal{E}}$. We put $\mathcal{E}_y = \Phi_{\mathcal{E}}(\mathcal{O}_y)$. Then for any $\sigma \in V^+$, \mathcal{E}_y is σ -stable by Theorem 4.4, and the phase of \mathcal{E}_y does not depend on $y \in Y$. Hence we see that $V^+ \subset \Phi_{\mathcal{E}*}U(Y)$.

If $V^0 \cap \Phi_{\mathcal{E}*}U(Y) \neq \emptyset$, by using Proposition 5.2 we see that any \mathcal{E}_y is a μ -stable locally free sheaf as follows. Since $\Phi_{\mathcal{E}*}U(Y)$ is open, there exist $\sigma \in V^0$ and an open neighborhood N_{σ} of σ in V^0 such that $N_{\sigma} \subset \Phi_{\mathcal{E}*}U(Y)$. And then N_{σ} intersects the set V^- . Thus we see that $\Phi_{\mathcal{E}*}U(Y) \cap V^-$ is not empty. Then by Proposition 5.2, we see that \mathcal{E}_y is a μ -stable locally free sheaf for all $y \in Y$.

Suppose to the contrary that $V^0 \cap \Phi_{\mathcal{E}^*}U(Y) = \emptyset$. We first show that V^0 is contained in the boundary $\partial\Phi_{\mathcal{E}^*}U(Y)$ under the assumption $V^0 \cap \Phi_{\mathcal{E}^*}U(Y) = \emptyset$. Since V^0 is in the closure of V^+ , V^0 is also in the closure of $\Phi_{\mathcal{E}^*}U(Y)$. Then we claim that $V^- \cap \Phi_{\mathcal{E}^*}U(Y) = \emptyset$. In fact, if $V^- \cap \Phi_{\mathcal{E}^*}U(Y) \neq \emptyset$, then \mathcal{E}_y is a μ -stable locally free sheaf for all $y \in Y$ by Proposition 5.2. Moreover V^0 is in $\Phi_{\mathcal{E}^*}U(Y)$ by Proposition 4.5. This contradicts $V^0 \cap \Phi_{\mathcal{E}^*}U(Y) = \emptyset$. Hence we see that $V^- \cap \Phi_{\mathcal{E}^*}U(Y) = \emptyset$. Thus V^0 is contained in the boundary $\partial(\Phi_{\mathcal{E}^*}U(Y))$. Moreover any $\sigma \in V^0$ is general in $\partial(\Phi_{\mathcal{E}^*}U(Y))$ since there are no walls in $V(X)$.

Take a stability condition $\sigma_0 \in V^0$. Recall that the Picard number of X is 1. Since Y is a Fourier–Mukai partner of X , the Picard number of Y is also 1. Since there is no (-2) -curve in Y , \mathcal{O}_y is properly $\Phi_{\mathcal{E}^*}^{-1}\sigma_0$ -semistable for all $y \in Y$ by Theorem 3.4. Hence \mathcal{E}_y is not σ_0 -stable but σ_0 -semistable. Moreover we see that \mathcal{E}_y is not a locally free sheaf by Propositions 5.1 and 5.2. Hence we have the following distinguished triangle by taking the double dual of \mathcal{E}_y :

$$S_y[-1] \longrightarrow \mathcal{E}_y \longrightarrow \mathcal{E}_y^{\vee\vee} \longrightarrow S_y,$$

where $S_y = \mathcal{E}_y^{\vee\vee}/\mathcal{E}_y \neq 0$. By Lemma 2.3, we see that

$$\mathrm{hom}_X^1(S_y, S_y) = 2 \quad \text{and} \quad \mathrm{hom}_X^1(\mathcal{E}_y^{\vee\vee}, \mathcal{E}_y^{\vee\vee}) = 0.$$

Thus there is a closed point $x \in X$ such that $S_y = \mathcal{O}_x$. Since σ_0 is in $V(X)$, \mathcal{O}_x is a σ_0 -stable factor of \mathcal{E}_y . By Theorem 3.4, $\mathcal{E}_y^{\vee\vee}$ is a direct sum of a spherical object S . Since $\mathcal{E}_y^{\vee\vee}$ is a locally free sheaf, S is also a locally free sheaf with $\mu_L(S) = \mu_L(\mathcal{E}_y^{\vee\vee})$. Thus we can put $\mathcal{E}_y^{\vee\vee} = S^{\oplus \ell}$.

Since $v(\mathcal{E}_y) = v(\mathcal{E}_y^{\vee\vee}) - 0 \oplus 0 \oplus 1$, we have

$$(5.3) \quad 0 = v(\mathcal{E}_y)^2 = v(\mathcal{E}_y^{\vee\vee})^2 - 2\langle v(\mathcal{E}_y^{\vee\vee}), v(\mathcal{O}_x) \rangle.$$

Furthermore we have $v(\mathcal{E}_y^{\vee\vee})^2 = -2\ell^2$ and

$$\langle v(\mathcal{E}_y^{\vee\vee}), v(\mathcal{O}_x) \rangle = -\mathrm{rank} \mathcal{E}_y^{\vee\vee} = -\mathrm{rank} \mathcal{E}_y = -r.$$

Thus we have

$$2\ell^2 = 2r.$$

Hence if r is not a square number, then we have $V^0 \cap \Phi_{\mathcal{E}^*}U(Y) \neq \emptyset$. Thus \mathcal{E}_y is a μ -stable locally free sheaf for all $y \in Y$ by Proposition 4.5. This gives the proof of assertion (1).

Suppose that $\mathrm{rank} \mathcal{E}_y$ is a square number. Then a JH filtration of \mathcal{E}_y is given by the following triangle:

$$\mathcal{O}_x[-1] \longrightarrow \mathcal{E}_y \longrightarrow S^{\oplus r} \longrightarrow \mathcal{O}_x.$$

Since $\mathcal{O}_x[-1]$ is the unique stable factor of \mathcal{E}_y with an isotropic Mukai vector, one of the following two cases will occur by Theorem 3.4 and by the uniqueness of stable factors up to permutations.

- (i) For any $y \in Y$, there is a closed point $x \in X$ such that $\Phi_{\mathcal{E}} \circ T_B(\mathcal{O}_y) = \mathcal{O}_x[-1]$ where B is a spherical locally free sheaf on Y and T_B is the spherical twist by B .

(ii) For any $y \in Y$, there is a closed point $x \in X$ such that $\Phi_{\mathcal{E}} \circ T_B^{-1}(\mathcal{O}_y) = \mathcal{O}_x[-1]$ where B is a spherical locally free sheaf on Y .

We remark that B does not depend on y by Theorem 3.4.

Assume that case (i) occurs. Then, as is well known, there is a line bundle M on X and an isomorphism $f : Y \rightarrow X$ such that $\Phi_{\mathcal{E}} \circ T_B(-) = M \otimes f_*(-)[-1]$.^{*} Thus we have

$$(5.4) \quad \Phi_{\mathcal{E}}(\mathcal{O}_y) = M \otimes f_*(T_B^{-1}(\mathcal{O}_y))[-1].$$

Then the right-hand side of (5.4) is properly complex and the left-hand side is a sheaf. This is contradiction. Hence case (ii) should occur. Then $\Phi_{\mathcal{E}}(-)$ is given by $M \otimes f_*(T_B(-))[-1]$. This gives the proof of assertion (2). \square

REMARK 5.5

As is well known, any Fourier–Mukai partner of K3 surfaces is isomorphic to fine moduli spaces of Gieseker stable sheaves by [9], and the equivalence between them can be chosen as the Fourier–Mukai transformation by the universal sheaf. Then, by Theorem 5.4, we see that any nontrivial Fourier–Mukai partners of projective K3 surfaces with Picard number one are isomorphic to fine moduli spaces of μ -stable locally free sheaves. This gives another proof of [4, Proposition 4.1]. Remarkably, we do not use the lattice theory except Orlov’s theorem.

EXAMPLE 5.6

Let (X, L) be a generic K3, and let E be a Gieseker stable, torsion-free sheaf with $v(E) = r \oplus nL \oplus s$. Since $\text{NS}(X) = \mathbb{Z}L$, E is μ -stable if $\text{gcd}\{r, n\} = 1$ by [5, Lemma 1.2.14]. Then E is a μ -stable locally free sheaf by Proposition 5.1. Moreover if $\text{gcd}\{r, nL^2, s\} = 1$, then the moduli space containing E is a fine moduli space.

Let (X, L) be a generic K3 with $L^2 = 6$. Take $v \in \mathcal{N}(X)$ as $v = 12 \oplus 10L \oplus 25$. Then by [5, Corollary 4.6.7] the moduli space $M_L(v)$ of Gieseker stable, torsion-free sheaves with Mukai vector v is the fine moduli space since $\text{gcd}\{12, 10L^2, 25\} = 1$. By Theorem 5.4, $M_L(v)$ is the moduli space of μ_L -stable locally free sheaves, although $\text{gcd}\{12, 10\} = 2$.

6. Second application

The goal of this section is to generalize [7, Theorem 1.1] to arbitrary projective K3 surfaces.

In [7] the author describes a picture of $(T_L)_*U(X) \cap V(X)$ by using [7, Theorem 1.2] where T_L is a spherical twist by an ample line bundle L . Instead of the theorem we use Lemma 6.1 (below). Before we state the lemma we prepare the notation. Let X be a projective K3 surface, and take an ample line bundle L . For the line bundle L we define the subset $V_L^{>0}$ of $V(X)$ by

^{*}For instance see [3, Corollary 5.23].

$$V_L^{>0} := \left\{ \sigma_{(\beta, \omega)} \in V(X)_L \mid L^2 - \beta L \leq \frac{\omega^2}{2} \right\}.$$

The following lemma, a special case of Lemma 4.4, is frequently used in this section.

LEMMA 6.1

Notation is as above. The set $V_L^{>0}$ is contained in $T_{L}U(X)$.*

Proof

Recall that $T_L(\mathcal{O}_x) = L \otimes \mathcal{I}_x[1]$ where \mathcal{I}_x is the kernel of the evaluation map $\mathcal{O}_X \rightarrow \mathcal{O}_x$. If σ is in $V_L^{>0}$, then $L \otimes \mathcal{I}_x$ is σ -stable for all $x \in X$ by Theorem 4.4. Furthermore the phase of $L \otimes \mathcal{I}_x$ does not depend on $x \in X$. Thus we have proved the assertion. □

The following lemma is also used in [7]. By using Lemma 6.2, we can see that $\Phi(\mathcal{O}_y)$ is a sheaf up to shifts if an equivalence $\Phi : D(Y) \rightarrow D(X)$ satisfies the condition $\Phi_*U(Y) = U(X)$.

LEMMA 6.2 ([2, PROPOSITION 14.2], [10, PROPOSITION 6.4])

Let X be a projective K3 surface, let E be in $D(X)$, and let $\sigma_{(\beta, \omega)} = (\mathcal{A}, Z) \in V(X)$. We put $v(E) = r_E \oplus \delta_E \oplus s_E$.

(1) *Assume that $r_E > 0$ and $E \in \mathcal{A}$. If there exists a positive real number ℓ_0 such that E is $\sigma_{(\beta, \ell\omega)}$ -stable for all $\ell > \ell_0$, then E is a torsion-free sheaf and is (β, ω) -twisted stable.*

(2) *Assume that $r_E = 0$ and $E \in \mathcal{A}$. If there exists a positive real number ℓ_0 such that E is $\sigma_{(\beta, \ell\omega)}$ -stable for all $\ell > \ell_0$, then E is a pure torsion sheaf.*

In [7] the author proves that some spherical twists send sheaves to complexes in some special cases. In the following lemma we generalize this result to arbitrary projective K3 surfaces.

LEMMA 6.3

Let X be a projective K3 surface, and let E and A be coherent sheaves with positive rank. We assume that $v(E)^2 = 0$ and $v(A)^2 = -2$ and put $v(E) = r_E \oplus \delta_E \oplus s_E$ and $v(A) = r_A \oplus \delta_A \oplus s_A$.

(1) *If $((\delta_E/r_E) - (\delta_A/r_A))^2 \geq 0$, then $\chi(A, E) > 0$.*

(2) *In addition to (1), assume that A is spherical and $\text{hom}_X^0(A, E) = 0$. Then the spherical twist $T_A(E)$ of E by A is a complex $H^0(T_A(E)) \neq 0$ and $H^1(T_A(E)) \neq 0$.*

Proof

We first show the first assertion. Since r_E and r_A are positive, it is enough to show that $(\chi(A, E))/(r_A r_E)$ is positive. We have

$$\begin{aligned} \frac{\chi(A, E)}{r_A r_E} &= -\left\langle 1 \oplus \frac{\delta_A}{r_A} \oplus \frac{s_A}{r_A}, 1 \oplus \frac{\delta_E}{r_E} \oplus \frac{s_E}{r_E} \right\rangle \\ &= \frac{s_A}{r_A} + \frac{s_E}{r_E} - \frac{\delta_A \delta_E}{r_A r_E}. \end{aligned}$$

Since $v(A)^2 = -2$ and $v(E)^2 = 0$ we have

$$\frac{s_A}{r_A} = \frac{1}{2} \frac{\delta_A^2}{r_A^2} + \frac{1}{r_A^2} \quad \text{and} \quad \frac{s_E}{r_E} = \frac{1}{2} \frac{\delta_E^2}{r_E^2}.$$

Thus we have

$$\begin{aligned} \frac{\chi(A, E)}{r_A r_E} &= \frac{1}{2} \frac{\delta_A^2}{r_A^2} + \frac{1}{r_A^2} + \frac{1}{2} \frac{\delta_E^2}{r_E^2} - \frac{\delta_A \delta_E}{r_A r_E} \\ &= \frac{1}{2} \left(\frac{\delta_A}{r_A} - \frac{\delta_E}{r_E} \right)^2 + \frac{1}{r_A^2} > 0. \end{aligned}$$

Thus we have proved the first assertion.

We show the second assertion. By the assumption and Lemma 6.3(1) we have $\chi(A, E) = -\text{hom}_X^1(A, E) + \text{hom}_X^2(A, E) > 0$. Hence $\text{hom}_X^2(A, E)$ is not zero. By the computing of the i th cohomology H^i of $T_A(E)$, we can prove the assertion. In fact we have the following exact sequence of sheaves:

$$\begin{aligned} \text{Hom}_X^0(A, E) \otimes A &\longrightarrow E \longrightarrow H^0 \\ \longrightarrow \text{Hom}_X^1(A, E) \otimes A &\longrightarrow 0 \longrightarrow H^1 \\ \longrightarrow \text{Hom}_X^2(A, E) \otimes A &\longrightarrow 0. \end{aligned}$$

Since $\text{hom}_X^2(A, E)$ is not zero, we see that $H^1 \neq 0$. Since $\text{hom}_X^0(A, E)$ is zero, the sheaf H^0 contains E . Thus H^0 is not zero. □

For an equivalence Φ satisfying the condition $\Phi_{\mathcal{E}*}U(Y) = U(X)$ and for a closed point $y \in Y$, it is enough to prove $\Phi(\mathcal{O}_y) = \mathcal{O}_x[n]$ for some $x \in X$ and $n \in \mathbb{Z}$. By Lemma 6.2, if $\Phi_*U(Y) = U(X)$, then $\Phi(\mathcal{O}_y)$ should be a sheaf up to shifts. Thus we have to exclude the case in which $\Phi(\mathcal{O}_y)$ is a torsion-free sheaf F or pure torsion sheaf T with $\dim \text{Supp}(T) = 1$ (up to shifts). If the Picard number of X is one, then it is not necessary to consider the case $\Phi(\mathcal{O}_y) = T$ with $\dim \text{Supp}(T) = 1$ since $v(\Phi(\mathcal{O}_y))^2 = 0$. We need the following lemma to exclude the case $\Phi(\mathcal{O}_y) = T$ with $\dim \text{Supp}(T) = 1$.

LEMMA 6.4

Let X be a projective K3 surface, let E be a pure torsion sheaf with $\dim \text{Supp}(E) = 1$, and let L be a line bundle on X . If $\chi(L, E) < 0$, then the spherical twist $T_L(E)$ of E is a sheaf containing a torsion sheaf or is properly complex. In particular $T_L(E)$ is not a torsion-free sheaf.

Proof

We have $\text{hom}_X^1(L, E) \neq 0$ by $\chi(L, E) < 0$. We can compute the i th cohomology H^i of $T_L(E)$ in the following way:

Now we put $E' = T_L(E)[1]$ and $v(E') = r' \oplus \delta' \oplus s'$. Since $r' = \chi(L, E) - r_E$, r' is positive. We choose a divisor β so that

$$\beta = bL \quad (b \in \mathbb{R}) \quad \text{and} \quad \beta\omega_0 < \min\left\{L\omega_0, \frac{\delta'\omega_0}{r'}\right\}.$$

We consider the following family of stability conditions:

$$\left\{ \sigma_y := \sigma_{(\beta, yL_0)} \in V_L^{>0} \mid L_0^2 - \beta L_0 \leq \frac{(yL_0)^2}{2} \right\}.$$

We put $\sigma_y = (Z_y, \mathcal{P}_y)$. By Lemma 6.1, a stability condition σ_y is in $(T_L)_*U(X)$. Since E is τ -stable for all $\tau \in U(X)$, the object E' is $(T_L)_*\tau$ -stable. Thus E' is σ_y -stable since σ_y is in $(T_L)_*U(X)$. By the choice of β we have $\Im Z_y(E') > 0$. Hence E' should be a torsion-free sheaf up to shifts by Lemma 6.2(1). This contradicts the fact that two cohomologies of E' survive.

Let us prove assertion (2). We choose an arbitrary stability condition $\sigma_{(\beta_0, \omega_0)} = (\mathcal{A}_0, Z_0) \in V(X)$ and fix it. Since E is $\sigma_{(\beta_0, \omega_0)}$ -stable we can assume that E is in \mathcal{A}_0 by shifts if necessary. By taking a limit $\omega_0 \rightarrow \infty$ we see that E is a pure torsion sheaf by Lemma 6.2(2).

We shall show $\delta_E = 0$. Suppose to the contrary that $\delta_E \neq 0$. Then $\delta_E L$ is positive for any ample line bundle L . Thus there is a sufficiently ample line bundle L_0 such that $\chi(L_0, E) < 0$. Here we put $v(T_{L_0}(E)) = r \oplus \delta \oplus s$. Since $r = -\chi(L_0, E)$, we see $r > 0$. Similarly to (1) we consider the following family of stability conditions:

$$\left\{ \sigma_y := \sigma_{(0, yL_0)} = (\mathcal{A}_y, Z_y) \mid L_0^2 \leq \frac{(yL_0)^2}{2} \right\}.$$

Since $\mu_{L_0}(L_0) = L_0^2 > 0$, σ_y is in $(T_{L_0})_*U(X)$ by Lemma 6.1. Moreover we have

$$\frac{\delta L_0}{r} = \frac{\delta_E - \chi(L_0, E)L_0}{r} L_0 > 0.$$

Thus $\Im Z_y(T_{L_0}(E)) > 0$. Hence we can assume that $T_{L_0}(E)$ is in \mathcal{A}_y up to even shifts. By Lemma 6.2(1), $T_{L_0}(E)$ should be a torsion-free sheaf. This contradicts Lemma 6.4. Thus we have $\delta_E = 0$.

Since $\delta_E = 0$, E is a pure torsion sheaf with $\dim \text{Supp}(E) = 0$. Since E is σ -stable we have $\text{hom}_X^0(E, E) = 1$. Thus E is a length 1 torsion sheaf up to shifts. We have proved the assertions. □

COROLLARY 6.6

Let X be a projective K3 surface, and let E be in $D(X)$ with $v(E)^2 = 0$. If E is σ -stable for all $\sigma \in V(X)$, then E is $\mathcal{O}_x[n]$ for some $x \in X$ and $n \in \mathbb{Z}$.

Proof

We put $v(E) = r_E \oplus \delta_E \oplus s_E$. If $r_E \neq 0$, then this contradicts Proposition 6.5(1). Hence $r_E = 0$. The assertion follows from Proposition 6.5(2). □

THEOREM 6.7

Let X and Y be projective K3 surfaces, and let $\Phi : D(Y) \rightarrow D(X)$ be an equivalence. If $\Phi_*U(Y) = U(X)$, then Φ can be written as

$$\Phi(-) = L \otimes f_*(-)[n]$$

where L is a line bundle on X , f is an isomorphism $f : Y \rightarrow X$, and $n \in \mathbb{Z}$.

Proof

Take an element $\sigma \in \text{Stab}(X)$. By the definition of $\tilde{\text{GL}}^+(2, \mathbb{R})$ -action we see that an object E is σ -stable if and only if E is $\sigma\tilde{g}$ -stable for all $\tilde{g} \in \tilde{\text{GL}}^+(2, \mathbb{R})$. Hence if $\Phi_*U(Y) = U(X)$, then $\Phi(\mathcal{O}_y)$ is written by $\mathcal{O}_x[n]$ for some $x \in X$ and $n \in \mathbb{Z}$ by Corollary 6.6. Then the assertion follows from [3, Corollary 5.23]. \square

Then we immediately obtain the following corollary.

COROLLARY 6.8

We put

$$\text{Aut}(D(X), U(X)) := \{\Phi \in \text{Aut}(D(X)) \mid \Phi_*U(X) = U(X)\}.$$

Then $\text{Aut}(D(X), U(X)) = (\text{Aut}(X) \times \text{Pic}(X)) \times \mathbb{Z}[1]$.

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