The Heisenberg ultrahyperbolic equation: K-finite and polynomial solutions

Anthony C. Kable

Abstract A family of partial differential operators on the Heisenberg group is introduced and studied. These operators may be regarded as analogues of the ultrahyperbolic operator on Euclidean space. Each of them is conformally invariant under the special linear group. The main focus is on the space of smooth solutions that extend to smooth sections of a suitable line bundle over a generalized flag manifold that contains the Heisenberg group as a dense open subset. The space of polynomial solutions is also considered from the point of view of conformal invariance.

Contents

1. Introduction

Let M be a manifold, let $\mathfrak{g}$ be a Lie algebra of first-order differential operators on M, let $\mathcal{V} \to M$ be a vector bundle equipped with a $\mathfrak{g}$-action, and let D be a differential operator that acts on sections of $\mathcal{V}$. We say that D is conformally invariant under $\mathfrak{g}$ if there is a map C from $\mathfrak{g}$ to $C^\infty(M)$ such that $[X, D] = C(X)D$ for all $X \in \mathfrak{g}$. If the action of $\mathfrak{g}$ derives from the action of a group G on M then the preceding condition is the infinitesimal version of the condition $gDg^{-1} = c(g)D$ for all $g \in G$; this latter condition defines conformal invariance under G. There are similar definitions for a system of operators.

Many of the most intensively studied differential operators admit large conformal symmetry groups. A familiar example is the Laplacian Δ on $\mathbb{R}^n$. This visibly admits the group $\mathrm{O}(n) \ltimes \mathbb{R}^n$ as a group of symmetries. It is well known that this group may be enlarged to the group $\mathrm{O}(n+1,1)$ of conformal symmetries

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of Δ by adjoining the Kelvin transform $\mathbb{K}$ that is defined by

$$(\mathbb{K}f)(x) = \|x\|^{2-n} f\Big(\frac{x}{\|x\|^2}\Big)$$

and satisfies the conformal identity

$$\mathbb{K} \circ \Delta \circ \mathbb{K} = \|x\|^4 \Delta.$$

A number of authors, notably Ehrenpreis [3] and Kostant [15], have considered conformally invariant operators from a general perspective. In [2], some of their insights were generalized to conformally invariant systems. That work is part of a project whose aims are to construct and classify conformally invariant systems admitting specified groups G and to study the properties of these systems with particular reference to their conformal invariance. The meaning of this latter clause is spelled out in greater detail in the introduction to [8].

Let G be either $\mathrm{SL}(d+2,\mathbb{R})$ or $\mathrm{SU}(p,q)$ with $p \geq q$ and $p+q=d+2$. Each of these groups has a real parabolic subgroup Q such that the real flag manifold G/Q contains the Heisenberg group H_d as a dense open subset. By applying the general theory of conformally invariant systems, one finds that there is a one-parameter family of operators $\square_z$ (with $z \in \mathbb{C}$), each of which is conformally invariant under G. In the case where $G = \mathrm{SU}(d+1,1)$, $\square_z$ is the Heisenberg Laplacian operator, which has been the subject of a great deal of work. Here we focus on the group $G = \mathrm{SL}(d+2,\mathbb{R})$ instead. With this choice, $\square_z$ is an operator that stands in the same relationship to the Euclidean ultrahyperbolic operator

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j \, \partial y_j}$$

on $\mathbb{R}^d \oplus \mathbb{R}^d$ as the Heisenberg Laplacian does to the Euclidean Laplacian. We refer to $\square_z$ as the Heisenberg ultrahyperbolic operator. It may be viewed either as an operator on suitable functions on the Heisenberg group or as an operator on suitable sections of a line bundle over the generalized flag manifold G/Q.

We are now ready to describe the contents of the present work. Sections 2 and 3 are preliminary in nature. The former is devoted to establishing the basic framework and notation that are used throughout, while the latter introduces the Heisenberg ultrahyperbolic operator, establishes its conformal invariance, and derives a number of facts from the general theory of such operators.

James Clerk Maxwell famously proved that all harmonic polynomials on Euclidean space may be derived from the radial solution $\|x\|^{2-n}$ to Laplace's equation by taking repeated directional derivatives and then applying the Kelvin transform to the resulting function. (We assume that $n \geq 3$ for simplicity. In addition, we do not recall the more precise aspects of Maxwell's result.) From the point of view of conformal invariance, Maxwell's result may be reexpressed by saying that the constant polynomial 1 is a cyclic vector in a certain module that may be associated to the Laplace operator.

Korányi [14] proved an analogue of Maxwell's result for the Heisenberg Laplacian. He showed that, provided the parameter z does not lie in one of two arithmetic progressions, the constant polynomial 1 is a cyclic vector in the module associated with the Heisenberg Laplacian. Thus all Heisenberg harmonic polynomials may be generated by a procedure similar to Maxwell's. Korányi's proof of this theorem is quite different from the classical proofs of Maxwell's theorem, and turns on the fact that the Heisenberg Laplacian is hypoelliptic when z does not belong to the aforementioned arithmetic progressions. In Section 4, a slightly more precise version of Korányi's result is re-proved as a by-product of analyzing the structure of the module in question. (Note that, as far as polynomial solutions are concerned, there is no essential difference between the Heisenberg Laplacian and the Heisenberg ultrahyperbolic operator, since there is a change of variables that takes one to the other and preserves polynomials. This is, of course, far from true for other types of solutions.) Instead of showing that 1 is a cyclic vector for the module of polynomial solutions, it is shown that the module is simple. It follows, of course, that *every* vector is a cyclic vector for the module. It is also shown that the restriction on z is precise, and the situation when z does fall into one of the arithmetic progressions is determined. Our method of proof is different from Korányi's (as it must be, since the Heisenberg ultrahyperbolic operator is never hypoelliptic; see [9]). One reason for seeking a different approach to Korányi's result is that there are a number of other families of conformally invariant operators on generalized flag manifolds that are quite similar to the family presently being discussed. However, in most of these other examples, none of the available operators is hypoelliptic, and so Korányi's approach cannot be adapted, whereas it is expected that our approach can be. The reader can find the structure of the relevant module in Theorem 4.5, the cyclic vector version of Korányi's result in Theorem 4.9, and the simple module version in Corollary 4.10.

As we remarked above, the Heisenberg ultrahyperbolic operator acts on sections of certain line bundles $\mathcal{L} \to G/Q$ over the generalized flag manifold G/Q. The main object of the present work is the study of the space $\Gamma(\mathcal{L})^{\square_z}$ of global smooth solutions to the Heisenberg ultrahyperbolic equation. The conformal invariance of the Heisenberg ultrahyperbolic operator implies that this space affords a representation of G. This representation is smooth and admissible and we may consider the underlying Harish-Chandra module $\mathrm{HC}(\Gamma(\mathcal{L})^{\square_z})$ of K-finite solutions, where K is a maximal compact subgroup of G. A basic framework for studying such spaces of K-finite solutions was established in [7]. By applying the results of that work, the problem of determining the K-finite solutions to the Heisenberg ultrahyperbolic equation is reduced to a purely algebraic problem. It should be helpful to describe this problem in more detail. We have a certain element Υ_z in the universal enveloping algebra $\mathcal{U}(\mathfrak{k})$ of $\mathfrak{k} = \mathfrak{so}(d+2)$. For each irreducible representation V of K that contains nonzero vectors annihilated by $\mathfrak{so}(d)$, we must determine the null space $\mathbb{M}(V)$ of $\Upsilon_{\bar{z}}$ in the subspace $V^{\mathfrak{so}(d)}$. There is a finite abelian subgroup F of $\mathrm{SO}(d+2)$ that acts on $\mathbb{M}(V)$ and we may decompose $\mathbb{M}(V)$ into the direct sum of the common eigenspaces $\mathbb{M}_\varepsilon(V)$ under the action of F, where ε runs over the characters of F. There is then an isomorphism

$$\mathrm{Hom}_K\big(V, \Gamma(\mathcal{L}_\varepsilon)^{\Box_z}\big) \cong \overline{\mathbb{M}_\varepsilon(V)},$$

where the bar denotes the complex conjugate vector space and $\mathcal{L}_\varepsilon$ is one of the line bundles on which $\Box_z$ acts. Moreover, if a vector in $\mathbb{M}_\varepsilon(V)$ is known explicitly then we obtain a corresponding explicit solution.

In principle, the facts rehearsed in the previous paragraph solve the problem of finding all K-finite solutions to the Heisenberg ultrahyperbolic equation. By using them, we are able to determine the precise decomposition of the space

$$\bigoplus_\varepsilon \mathrm{HC}\big(\Gamma(\mathcal{L}_\varepsilon)^{\Box_z}\big)$$

as a representation of K. The results may be found in Theorems 5.12 and 5.13, which come at the end of Section 5. The bulk of the work in this section is devoted to developing the ideas necessary to implement the general solution in this specific case. We now outline the method by which this is done. If V is an irreducible representation of K such that $V^{\mathfrak{so}(d)} \neq \{0\}$ then certainly $V^{\mathfrak{so}(d)}$ is a representation of $\mathfrak{so}(2)$. It is a remarkable fact that $V^{\mathfrak{so}(d)}$ is, in fact, an irreducible representation of $\mathfrak{su}(2)$ in such a way that the $\mathfrak{so}(2)$-action is obtained by restriction. Unfortunately, the known proofs of this fact do not seem to allow the $\mathfrak{su}(2)$ action on $V^{\mathfrak{so}(d)}$ to be written explicitly. The author at first attempted to resolve this problem, but instead uncovered a striking and unexpected phenomenon that proved to be an adequate replacement. Namely, it was found that $V^{\mathfrak{so}(d)}$ is naturally a module for the associative algebra $\mathcal{H}$ generated by three elements Z, R_+, and R_- subject to the relations $[Z, R_+] = 2iR_+$, $[Z, R_-] = -2iR_-$, and

$$[R_+, R_-] = -4iZ(h + 2Z^2),$$

where h is a natural number depending on V. Of course, these relations are strongly reminiscent of the commutator relations satisfied by one of the standard sets of generators for $\mathfrak{su}(2)$ and so $\mathcal{H}$ may be thought of as resembling $\mathcal{U}(\mathfrak{su}(2))$. However, there are substantial differences between the two algebras; for example, there are only a finite number of isomorphism classes of finite-dimensional simple Hermitian $\mathcal{H}$-modules, in contrast to $\mathcal{U}(\mathfrak{su}(2))$, for which this number is infinite. Theorem 5.6 gives the complete classification of finite-dimensional Hermitian $\mathcal{H}$-modules. There are some exotic modules appearing on the list that look nothing like modules for $\mathfrak{su}(2)$, but fortunately we are able to show (in Corollary 5.8) that these exotic modules do not appear in the $\mathcal{H}$-module $V^{\mathfrak{so}(d)}$ for any V. Once this fact is in hand, one has enough information to write the matrix for Υ_z acting on $V^{\mathfrak{so}(d)}$ in a judiciously chosen basis. There is one further technical problem to be solved before one arrives at the main theorems, namely, the evaluation of the determinant of a certain tridiagonal matrix. Unfortunately, this determinant does not appear to fall into the classes of tridiagonal determinants treated systematically by Askey [1] and so it has to be treated by an *ad hoc* argument (Proposition 5.11).

In the last section, Section 6, we continue the study of the space $\Gamma(\mathcal{L})^{\square_z}$ by proving an analogue of the celebrated range theorem due to Fritz John [6] and its generalizations. John's original result is that the solution space of the Euclidean ultrahyperbolic equation on $\mathbb{R}^4$ in suitable function spaces coincides with the image of a certain integral transform (the so-called X-ray transform) up to an elementary factor. When $d \geq 3$ and z does not lie in $(d/2)+\mathbb{Z}$, we show that $\Gamma(\mathcal{L})^{\square_z}$ is either identically zero or coincides precisely with the image of a certain integral transform (Theorem 6.6). A similar statement can also be obtained for $d=1$ and $d=2$, but the details are more complicated and so those cases are not considered here. The author hopes to return to them elsewhere. The integral transform that is involved in this result has to be defined for some values of z by analytic continuation if its domain is to be large enough to make the claim true. On more restricted domains, the integral transform converges for all z and the analytic continuation procedure is superfluous. In representation-theoretic terms the integral transform is an intertwining operator from a degenerate principal series representation. It can be constructed in this way because the real parabolic subgroup Q of G is not maximal amongst all real parabolic subgroups. There is no analogous construction for the Heisenberg Laplacian, for example, because in that case Q is maximal amongst real parabolic subgroups.

In a series of papers [11]–[13], Kobayashi and Ørsted studied the minimal representation of $\mathrm{O}(p,q)$ in great depth. They gave several different realizations of this representation, including one on a suitable space of solutions to a Euclidean ultrahyperbolic equation. This construction is analogous to what is done here, but there are some salient differences. Two technical differences are that the parabolic subgroup that plays the role of Q in Kobayashi and Ørsted's construction has abelian, rather than Heisenberg, unipotent radical and that the degenerate principal series representation that contains the solution space is K-multiplicity-free. Wang [19] considered the analogue of Kobayashi and Ørsted's construction for the operator with conformal group $G=\mathrm{SU}(p,q)$ that was mentioned above. Thus Wang's work is more directly comparable to what is done here. However, Wang obtains relatively little information about his representations in [19]. Much remains to be done beyond that reference in order to reach a degree of completeness comparable to that which Kobayashi and Ørsted attained in their work. There is also a significant difference of emphasis between the work of Kobayashi and Ørsted and of Wang and what is done here. Briefly, whereas Kobayashi, Ørsted, and Wang foreground the representation, our aim is to foreground the equation. The representation associated to the Heisenberg ultrahyperbolic equation is much more familiar than the representations constructed by Kobayashi and Ørsted, and by Wang, and so the identification of the solution space with this representation should be viewed as throwing light on the equation rather than on the representation.

The direct algebraic study of the Harish–Chandra modules underlying degenerate principal series representations has been undertaken by a number of authors, including notably Howe and his collaborators ([5] is one example of this program). This approach has advanced farthest in the case of K-multiplicity-free representations, but some other cases have also been investigated. In [16], Miyazaki studied the principal series representations of $\mathrm{SL}(3,\mathbb{R})$, which are precisely those induced from the Heisenberg parabolic subgroup in this case by direct algebraic methods. As well as providing a useful entry point to the theory, Miyazaki's work gives some sense of how challenging this approach is. In particular, it does not appear that it at present provides an alternative route to the identification of the kernel of the Heisenberg ultrahyperbolic operators.

We close this introduction with a confession of some of our sins of omission. The integral transform considered in Section 6 can also be thought of as a geometric transform somewhat akin to the X-ray transform (or, more precisely, as the Mellin transform in one variable of a truly geometric integral transform in the remaining variables). Because of length considerations, this aspect of the integral transform is not discussed here beyond this statement. For the same reason, further investigation of its analytic behavior is not undertaken. In particular, we have not attempted to derive an inversion formula for the transform, nor to elucidate its behavior on other function spaces. Also, although $\Gamma(\mathcal{L})^{\square_z}$ may be analyzed along similar lines when $z \in (d/2)+\mathbb{Z}$, the precise statements seem to become rather complicated, and so we have not attempted to describe this exceptional case further.

2. Framework and review

In this section, we describe the setting in which our work takes place, introduce notation and notational conventions, and review the essential background.

Let $G=\mathrm{SL}(m,\mathbb{R})$ with $m \geq 3$, let Q (respectively, $\bar{Q}$) be the standard block upper-triangular (respectively, lower-triangular) subgroup of G with blocks of size $(1,d,1)$ with $d=m-2$, let $L=Q\cap\bar{Q}$, and let N (respectively, $\bar{N}$) be the unipotent radical of Q (respectively, $\bar{Q}$). Let ν_1 and ν_m be the characters of L given by $\nu_j(\mathrm{diag}(a_1,l,a_m))=a_j$. Let $\eta=\nu_1\nu_m^{-1}$, and define $\gamma:\mathfrak{l}\to\mathbb{C}$ to be $\gamma=d\eta$. Note that the restriction of γ to the standard Cartan subalgebra of $\mathfrak{g}$ coincides with the highest root for the standard positive system. For $z\in\mathbb{C}$ and $u\in\mathbb{R}^\times$, let $|u|_+^z=|u|^z$ and $|u|_-^z=\mathrm{sgn}(u)|u|^z$. For $z\in\mathbb{C}$ and $\varepsilon_1,\varepsilon_2\in\{\pm\}$, we define an analytic character $\chi(z,\varepsilon_1,\varepsilon_2):L\to\mathbb{C}^\times$ by

$$\chi(z,\varepsilon_1,\varepsilon_2)\big(\mathrm{diag}(a_1,h,a_m)\big)=|a_1|_{\varepsilon_1}^{z+z_0}|a_m|_{\varepsilon_2}^{z-z_0}$$

with $z_0=d/2$. We shall usually denote this character simply by χ, with z, ε_1, and ε_2 understood from context, and also regard it as a character of $\bar{Q}$ by extending it trivially to $\bar{N}$. There is a homogeneous line bundle $\mathcal{L}_\chi\to G/\bar{Q}$ associated to χ. The space $\Gamma(U,\mathcal{L}_\chi)$ of smooth sections of $\mathcal{L}_\chi$ over an open set $U\subset G/\bar{Q}$ may be identified with the space of smooth functions $\varphi:W\to\mathbb{C}$ that satisfy $\varphi(g\bar{q})=\chi(\bar{q})\varphi(g)$ for $g\in G$ and $\bar{q}\in\bar{Q}$, where $W\subset G$ is the preimage of U under the projection $G \to G/\bar{Q}$. The space $\Gamma(\mathcal{L}_\chi)$ of smooth global sections of $\mathcal{L}_\chi$ with the left-translation action of G is a model of the smooth induced representation $\operatorname{Ind}(G, \bar{Q}, \chi^{-1})$. Let π_χ denote this representation of G. We consistently employ the convention that the Lie algebra of a real Lie group is denoted by the corresponding fraktur letter with subscript zero, while the complexification of this Lie algebra is denoted by removing the subscript. In particular, $\mathfrak{g}_0 = \mathfrak{sl}(m, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{C})$. From π_χ we obtain a derived representation Π_χ of $\mathfrak{g}$. This realizes $\mathfrak{g}$ as an algebra of first-order differential operators on $\mathcal{L}_\chi \to G/\bar{Q}$ and hence extends to $\Gamma(U, \mathcal{L}_\chi)$ for any open set $U \subset G/\bar{Q}$. Note that Π_χ is independent of ε_1 and ε_2 and, for this reason, we may write it as Π_z.

It follows from the Bruhat decomposition that $N\bar{Q} = NL\bar{N}$ is a dense open subset of G. Since $N \cap \bar{Q} = \{e\}$ and $L \cap \bar{N} = \{e\}$, if $g \in N\bar{Q}$ then g has a unique factorization in the form

$$g = \zeta(g) a(g) \bar{\zeta}(g) \tag{2.1}$$

with $\zeta(g) \in N$, $a(g) \in L$, and $\bar{\zeta}(g) \in \bar{N}$. Suppose that $g_1, g_2 \in G$ and $n \in N$ are such that $g_2 n \in N\bar{Q}$ and $g_1 g_2 n \in N\bar{Q}$. Then it follows that $g_1 \zeta(g_2 n) \in N\bar{Q}$, that

$$\zeta(g_1 g_2 n) = \zeta\big(g_1 \zeta(g_2 n)\big), \tag{2.2}$$

and that

$$a(g_1 g_2 n) = a\big(g_1 \zeta(g_2 n)\big) a(g_2 n). \tag{2.3}$$

If $g \in G$, $n \in N$, and $gn \in N\bar{Q}$ then we may define $g * n = \zeta(gn)$. It is well known that $n \mapsto g * n$ is a rational map of N, its domain being the open dense set $\{n \in N \mid gn \in N\bar{Q}\}$. The identity (2.2) implies that $(g, n) \mapsto g * n$ defines a rational action of G on N. That is, if both $g_1 g_2 * n$ and $g_2 * n$ are defined then $g_1 * (g_2 * n)$ is defined and $g_1 g_2 * n = g_1 * (g_2 * n)$. With this notation, identity (2.3) may be reexpressed as

$$a(g_1 g_2 n) = a\big(g_1 (g_2 * n)\big) a(g_2 n). \tag{2.4}$$

A useful consequence of this identity is

$$a(g^{-1} n) = a\big(g(g^{-1} * n)\big)^{-1}, \tag{2.5}$$

which follows on taking $g_1 = g$ and $g_2 = g^{-1}$ in (2.4). If $n' \in N$ then $n' * n = n'n$ and $a(n'n) = e$ for all $n \in N$. If $h \in L$ then $h * n = hnh^{-1}$ and $a(hn) = h$ for all $n \in N$.

The restriction map $\Gamma(\mathcal{L}_\chi) \to C^\infty(N)$ is injective and the image is dense if $C^\infty(N)$ is given the smooth topology. If $g \in G$ and $\varphi \in \Gamma(\mathcal{L}_\chi)$, then we have

$$\big(\pi_\chi(g)\varphi\big)(n) = \chi\big(a(g^{-1} n)\big) \varphi(g^{-1} * n) \tag{2.6}$$

for n in the domain of $n \mapsto g^{-1} * n$. Note that the assumption that $\varphi \in \Gamma(\mathcal{L}_\chi)$ implies that the right-hand side of (2.6) extends from the domain of $n \mapsto g^{-1} * n$ to a smooth function on all of N. In fact, this extension property for all $g \in G$ (or, equivalently, for all g in a complete set of (Q, Q)-double coset representatives) characterizes the image of $\Gamma(\mathcal{L}_\chi)$ in $C^\infty(N)$. Even when $\varphi \in C^\infty(N)$ does not lie in the image of $\Gamma(\mathcal{L}_\chi)$, the right-hand side of (2.6) defines a smooth function on the domain of $n \mapsto g^{-1} * n$. By abuse of notation, we continue to denote this function by $\pi_\chi(g)\varphi$.

It will be convenient to have coordinates available on N. Let $\mathrm{Mat}(i,j)$ denote the space of i-by-j matrices (with the scalar field determined by context). For $x \in \mathrm{Mat}(1,d)$, $y \in \mathrm{Mat}(d,1)$, and $t \in \mathbb{R}$ let

$$n(x,y,t) = \begin{pmatrix} 1 & x & t \\ 0 & I_d & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $N = \{n(x,y,t) \mid x \in \mathrm{Mat}(1,d), y \in \mathrm{Mat}(d,1), t \in \mathbb{R}\}$. This makes it apparent that N is isomorphic to the Heisenberg group of dimension $2d+1$. Let

$$w_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_d & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

LEMMA 2.1

Let $x \in \mathrm{Mat}(1,d)$, let $y \in \mathrm{Mat}(d,1)$, and let $t \in \mathbb{R}$. Then, on the set where $t(t-xy) \neq 0$, we have

$$w_0 * n(x,y,t) = n\Big(-\frac{x}{t-xy}, -\frac{y}{t}, -\frac{1}{t}\Big)$$

and

$$a\big(w_0 n(x,y,t)\big) = \mathrm{diag}\Big(\frac{-1}{t-xy}, I_d - t^{-1}yx, -t\Big).$$

Proof

We have

$$w_0 n(x,y,t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_d & y \\ -1 & -x & -t \end{pmatrix}$$

and the required factorization is

$$w_0 n(x,y,t) = n\Big(-\frac{x}{t-xy}, -\frac{y}{t}, -\frac{1}{t}\Big)\,\mathrm{diag}\Big(\frac{-1}{t-xy}, I_d - t^{-1}yx, -t\Big)\bar{n}$$

with

$$\bar{n} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{y}{t-xy} & I_d & 0 \\ \frac{1}{t} & \frac{x}{t} & 1 \end{pmatrix}.$$

□

By using (2.6) and Lemma 2.1, we obtain

$$\big(\pi_\chi(w_0^{-1})\varphi\big)\big(n(x,y,t)\big) = \varepsilon_1\varepsilon_2|t-xy|_{\varepsilon_1}^{-(z+z_0)}|t|_{\varepsilon_2}^{z-z_0}\varphi\Big(n\Big(-\frac{x}{t-xy}, -\frac{y}{t}, -\frac{1}{t}\Big)\Big).$$

The inverse transformation is

$$\big(\pi_\chi(w_0)\varphi\big)\big(n(x,y,t)\big) = |t-xy|_{\varepsilon_1}^{-(z+z_0)}|t|_{\varepsilon_2}^{z-z_0}\varphi\bigg(n\Big(\frac{x}{t-xy},\frac{y}{t},-\frac{1}{t}\Big)\bigg).$$

Let $E_{i,j}\in \mathrm{Mat}(m,m)$ denote the matrix whose only nonzero entry is a 1 in the (i,j)-place. For $1\le j\le d$, define $X_j = E_{1,j+1}$, define $Y_j = E_{j+1,m}$, and let $T = E_{1,m}$. Then

$$X_1,\dots,X_d,Y_1,\dots,Y_d,T$$

is a basis for $\mathfrak{n}_0$, T is a basis for $\mathfrak{z}(\mathfrak{n}_0)$, and the bracket on $\mathfrak{n}_0$ satisfies $[X_i,Y_j] = \delta_{ij}T$ for $1\le i,j\le d$. Similarly, for $1\le j\le d$, define $\bar{X}_j = E_{m,j+1}$, define $\bar{Y}_j = E_{j+1,1}$, and let $\bar{T} = E_{m,1}$. Then

$$\bar{X}_1,\dots,\bar{X}_d,\bar{Y}_1,\dots,\bar{Y}_d,\bar{T}$$

is a basis for $\bar{\mathfrak{n}}_0$, $\bar{T}$ is a basis for $\mathfrak{z}(\bar{\mathfrak{n}}_0)$, and the bracket on $\bar{\mathfrak{n}}_0$ satisfies $[\bar{X}_i,\bar{Y}_j] = \delta_{ij}\bar{T}$ for $1\le i,j\le d$. We have the relations $[X_i,\bar{X}_j] = [Y_i,\bar{Y}_j] = 0$, $[X_i,\bar{Y}_j] = \delta_{ij}E_{1,1} - E_{j+1,i+1}$, $[Y_i,\bar{X}_j] = E_{i+1,j+1} - \delta_{ij}E_{m,m}$, $[X_i,\bar{T}] = -\bar{X}_i$, $[Y_i,\bar{T}] = \bar{Y}_i$, $[T,\bar{X}_i] = X_i$, $[T,\bar{Y}_i] = -Y_i$, and $[T,\bar{T}] = E_{1,1} - E_{m,m}$.

Let $H_0 = E_{1,1} - E_{m,m}$. Then $\gamma(H_0) = 2$ and we have $\mathfrak{l}_0 = \mathbb{R}H_0 \oplus \mathfrak{l}_0^\gamma$, where $\mathfrak{l}_0^\gamma$ denotes the kernel of γ in $\mathfrak{l}_0$. The algebra $\mathfrak{l}_0^\gamma$ is isomorphic to $\mathfrak{gl}(d,\mathbb{R})$ and it will be convenient to choose notation to render this isomorphism transparent. Thus we define

$$F_{i,j} = E_{i+1,j+1} - \frac{1}{m}\delta_{ij}I_m$$

for $1\le i,j\le d$. Then $\{F_{i,j}\}$ is a basis for $\mathfrak{l}_0^\gamma$ and the assignment $F_{i,j}\mapsto E_{i,j}$ extends linearly to a Lie algebra isomorphism from $\mathfrak{l}_0^\gamma$ to $\mathfrak{gl}(d,\mathbb{R})$. The center of $\mathfrak{l}_0^\gamma$ is spanned by the element

$$W_0 = F_{1,1}+\cdots+F_{d,d}.$$

Note that $\mathfrak{l}_0^\gamma = \mathbb{R}W_0\oplus\mathfrak{m}_0$, where $\mathfrak{m}_0$ is the preimage of $\mathfrak{sl}(d,\mathbb{R})$ under the isomorphism from $\mathfrak{l}_0^\gamma$ to $\mathfrak{gl}(d,\mathbb{R})$. We have the bracket relations $[F_{i,j},X_k] = -\delta_{ik}X_j$ and $[F_{i,j},Y_k] = \delta_{jk}Y_i$.

If $n_0\in N$ then $\pi_\chi(n_0)\varphi(n) = \varphi(n_0^{-1}n)$ and it follows that

$$\Pi_z(X_i) = -\frac{\partial}{\partial x_i} - y_i\frac{\partial}{\partial t},$$

$$\Pi_z(Y_i) = -\frac{\partial}{\partial y_i},$$

$$\Pi_z(T) = -\frac{\partial}{\partial t}.$$

If $h\in L$ then $\pi_\chi(h)\varphi(n) = \chi(h)^{-1}\varphi(h^{-1}nh)$. From this, we obtain

$$\Pi_z(H_0) = -\mathbb{E}_x - \mathbb{E}_y - 2t\frac{\partial}{\partial t} - 2z_0,$$

$$\Pi_z(F_{i,j}) = x_i\frac{\partial}{\partial x_j} - y_j\frac{\partial}{\partial y_i} + \frac{2z}{m}\delta_{ij},$$

where

$$\mathbb{E}_x = \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i}$$

and

$$\mathbb{E}_y = \sum_{i=1}^{d} y_i \frac{\partial}{\partial y_i}$$

are the Euler operators with respect to the x and y variables, respectively.

LEMMA 2.2
We have

$$\Pi_z(\bar{X}_i) = -(z - z_0)y_i + y_i\mathbb{E}_y + (t - xy)\frac{\partial}{\partial x_i} + y_i t\frac{\partial}{\partial t},$$

$$\Pi_z(\bar{Y}_i) = (z + z_0)x_i + x_i\mathbb{E}_x - t\frac{\partial}{\partial y_i},$$

$$\Pi_z(\bar{T}) = (z + z_0)(t - xy) - (z - z_0)t + (t - xy)\mathbb{E}_x + t\mathbb{E}_y + t^2\frac{\partial}{\partial t}$$

for $1 \le i \le d$.

Proof
Let $n = n(x,y,t)$, and write $n' = n(x',y',t') = w_0 * n(x,y,t)$. Let u be one of the coordinates (x,y,t), and define rational functions α, β, ρ^{x_j}, ρ^{y_j}, and ρ^t on N by $\alpha(n) = (t - xy)^{-1}\,\partial(t - xy)/\partial u$, $\beta(n) = t^{-1}\,\partial t/\partial u$, $\rho^{x_j}(n) = \partial x_j'/\partial u$, $\rho^{y_j}(n) = \partial y_j'/\partial u$, and $\rho^t(n) = \partial t'/\partial u$. Let $\chi = \chi(z,+,+)$. By the chain rule and the formula for $\pi_\chi(w_0^{-1})$ given above, we have

$$\begin{aligned}
&\frac{\partial}{\partial u}\pi_\chi(w_0^{-1})\varphi(n)\\
&\quad = |t - xy|^{-(z+z_0)}|t|^{z-z_0}\Big(\big(-(z + z_0)\alpha(n) + (z - z_0)\beta(n)\big)\varphi(n')\\
&\qquad + \sum_{j=1}^{d}\rho^{x_j}(n)\frac{\partial\varphi}{\partial x_j}(n') + \sum_{j=1}^{d}\rho^{y_j}(n)\frac{\partial\varphi}{\partial y_j}(n') + \rho^t(n)\frac{\partial\varphi}{\partial t}(n')\Big)
\end{aligned}$$

and it follows that

$$\begin{aligned}
\pi_\chi(w_0)\frac{\partial}{\partial u}\pi_\chi(w_0^{-1}) &= -(z + z_0)\alpha(n'') + (z - z_0)\beta(n'')\\
&\quad + \sum_{j=1}^{d}\rho^{x_j}(n'')\frac{\partial}{\partial x_j} + \sum_{j=1}^{d}\rho^{y_j}(n'')\frac{\partial}{\partial y_j} + \rho^t(n'')\frac{\partial}{\partial t},
\end{aligned}$$

where $n'' = w_0^{-1} * n$. We have $\mathrm{Ad}(w_0)X_i = -\bar{X}_i$, $\mathrm{Ad}(w_0)Y_i = \bar{Y}_i$, and $\mathrm{Ad}(w_0)T = -\bar{T}$. Thus, for example, $\Pi_z(\bar{T}) = -\pi_\chi(w_0)\Pi_z(T)\pi_\chi(w_0^{-1})$, and this and the fact that $\Pi_z(T) = -\partial/\partial t$ make it routine to compute $\Pi_z(\bar{T})$. The other computations are similar. $\square$

3. The ultrahyperbolic operator

In this section we introduce the Heisenberg ultrahyperbolic operator $\square_z$ and derive some of its properties from the general theory presented in [2] and [7].

Each element $X \in \mathfrak{n}_0$ defines a differential operator $R(X)$ on N by

$$\big(R(X) \bullet \varphi\big)(n) = \frac{d}{d\tau}\Big|_{\tau=0} \varphi\big(n \exp(\tau X)\big).$$

Since left and right translation commute with one another, we have $[\Pi_z(X), R(Y)] = 0$ for all $X, Y \in \mathfrak{n}_0$. In terms of the coordinates (x, y, t) on N, we find that

$$\begin{aligned} R(X_i) &= \frac{\partial}{\partial x_i}, \\ R(Y_i) &= \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial t}, \\ R(T) &= \frac{\partial}{\partial t}. \end{aligned}$$

The map R from $\mathfrak{n}_0$ into the algebra $\mathbb{D}_0[N]$ of smooth differential operators with real coefficients on N is an $\mathbb{R}$-linear Lie algebra homomorphism. We may extend it to a $\mathbb{C}$-linear Lie algebra homomorphism from $\mathfrak{n}$ into $\mathbb{D}[N]$, the complexification of $\mathbb{D}_0[N]$, and then to an algebra homomorphism from the universal enveloping algebra $\mathcal{U}(\mathfrak{n})$ into $\mathbb{D}[N]$. We also denote these extensions by R. Let

$$\omega_0 = \sum_{i=1}^{d} Y_i X_i$$

in $\mathcal{U}(\mathfrak{n})$. Note that the group L acts adjointly on the algebra $\mathcal{U}(\mathfrak{n})$.

LEMMA 3.1

If $h \in L$ then $\mathrm{Ad}(h)\omega_0 = \eta(h)\omega_0$ and $\mathrm{Ad}(h)T = \eta(h)T$.

Proof

Every element of L may be uniquely expressed as a product of an element of the form $h_1 = \mathrm{diag}(a, a^{-1}, I_d)$ with $a \in \mathbb{R}^\times$, an element of the form $h_2 = \mathrm{diag}(1, f, 1)$ with $f \in \mathrm{SL}(d, \mathbb{R})$, and an element of the form $h_3 = \mathrm{diag}(b, I_d, b^{-1})$ with $b \in \mathbb{R}^\times$. It suffices to verify the claimed identity for each of these elements. We have

$$\mathrm{Ad}(h_1)X_i = \begin{cases} a^2 X_1 & \text{if } i = 1, \\ a X_i & \text{if } 2 \le i \le d, \end{cases}$$

$$\mathrm{Ad}(h_1)Y_i = \begin{cases} a^{-1} Y_1 & \text{if } i = 1, \\ Y_i & \text{if } 2 \le i \le d, \end{cases}$$

and $\mathrm{Ad}(h_1)T = aT$. It follows from this that $\mathrm{Ad}(h_1)\omega_0 = a\omega_0 = \eta(h_1)\omega_0$ and $\mathrm{Ad}(h_1)T = \eta(h_1)T$. We have $\mathrm{Ad}(h_2)X_i = \sum_{j=1}^{d} \bar{f}_{ij} X_j$ where $\bar{f}_{ij}$ is the (i,j)-entry in f^{-1}, $\mathrm{Ad}(h_2)Y_i = \sum_{j=1}^{d} f_{ji} Y_j$, and $\mathrm{Ad}(h_2)T = T$. A brief calculation using these evaluations shows that $\mathrm{Ad}(h_2)\omega_0 = \omega_0 = \eta(h_2)\omega_0$. Finally, $\mathrm{Ad}(h_3)X_i = bX_i$, $\mathrm{Ad}(h_3)Y_i = bY_i$, and $\mathrm{Ad}(h_3)T = b^2T$. Thus $\mathrm{Ad}(h_3)\omega_0 = b^2\omega_0 = \eta(h_3)\omega_0$ and $\mathrm{Ad}(h_3)T = \eta(h_3)T$. $\square$

The Heisenberg ultrahyperbolic operator is defined by

$$\square_z = R\big(\omega_0 + (z+z_0)T\big) \tag{3.1}$$

for $z \in \mathbb{C}$. In the coordinates (x, y, t) on N, $\square_z$ takes the form

$$\square_z = \Delta + \big(\mathbb{E}_x + (z+z_0)\big)\frac{\partial}{\partial t},$$

where

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i\, \partial y_i}$$

is the Euclidean ultrahyperbolic operator.

Let $\chi = \chi(z, \varepsilon_1, \varepsilon_2)$, and denote by $\mathbb{C}_{d\chi}$ the one-dimensional $\bar{\mathfrak{q}}$-module on which $\bar{\mathfrak{q}}$ acts via $d\chi$. The map $\mathcal{U}(\mathfrak{n}) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\chi}$ given by $u \mapsto u \otimes 1$ is a vector space isomorphism from $\mathcal{U}(\mathfrak{n})$ onto the generalized Verma module $\mathcal{M}(d\chi) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\chi}$.

LEMMA 3.2

The subspace of $\mathcal{M}(d\chi)$ spanned by $(\omega_0 + (z+z_0)T) \otimes 1$ is a $\bar{\mathfrak{q}}$-submodule of $\mathcal{M}(d\chi)$.

Proof

Let $u = (\omega_0 + (z+z_0)T) \otimes 1$. It follows from Lemma 3.1 that $\mathfrak{l}u \subset \mathbb{C}u$. Thus it suffices to show that $\bar{\mathfrak{n}}u = \{0\}$. On calculation, we find that

$$\begin{aligned} \bar{X}_i(\omega_0 \otimes 1) &= (z+z_0)X_i \otimes 1, \\ \bar{Y}_i(\omega_0 \otimes 1) &= -(z+z_0)Y_i \otimes 1, \\ \bar{T}(\omega_0 \otimes 1) &= 2z_0(z+z_0)1 \otimes 1, \end{aligned}$$

and

$$\begin{aligned} \bar{X}_i(T \otimes 1) &= -X_i \otimes 1, \\ \bar{Y}_i(T \otimes 1) &= Y_i \otimes 1, \\ \bar{T}(T \otimes 1) &= -2z_0 1 \otimes 1. \end{aligned}$$

It follows from these evaluations that $\bar{\mathfrak{n}}u = \{0\}$, as required. $\square$

Lemma 3.2 verifies the hypotheses of [2, Theorem 15] and thus allows us to conclude that the operator $\square_z$ is conformally invariant on the bundle $\mathcal{L}_\chi$. This means that there is a structure operator $C : \mathfrak{g} \to C^\infty(N)$ such that

$$[\Pi_z(X), \square_z] = C(X)\square_z \tag{3.2}$$

for all $X \in \mathfrak{g}$. Note that, at this point, C might depend on z, but we shall shortly see that it does not. For the terminology of structure operators and the concept of conformal invariance that is being used here the reader may consult the [2, Introduction, Section 2]. In fact, since this conformally invariant system consists of a single operator, the relevant concept of conformal invariance is the one employed by Kostant in [15].

In order to be able to apply the results of [7] to the operator $\square_z$, we must first note a few more of its properties. Since $\square_z$ lies in the image of R, we have $C(X) = 0$ for all $X \in \mathfrak{n}$. In the terminology of [2, Section 4], this means that the system $\square_z$ is straight. Lemma 3.1 implies that we have $\mathrm{Ad}(h)(\omega_0 + (z + z_0)T) = \eta(h)(\omega_0 + (z + z_0)T)$ for all $h \in L$. That is, in the terminology of [2, Section 6], the system $\square_z$ is L-stable. Finally, by an easy computation, we have $[\Pi_z(H_0), \square_z] = 2\square_z$. This implies, again in the terminology of [2, Section 6], that the system $\square_z$ is homogeneous. These are the additional hypotheses (beyond conformal invariance) that are required in order to apply the results of [7]. In particular, when applied to the present situation, [7, Proposition 2.3] and Lemma 3.1 imply the following result when $z \in \mathbb{R}$. The restriction that z be real arises because it was assumed in [7] that the character χ is real-valued. However, both sides of the identity in Theorem 3.3 are holomorphic functions of z and so we may deduce the truth of the identity for all $z \in \mathbb{C}$ from its truth for all $z \in \mathbb{R}$.

THEOREM 3.3

For all $g \in G$ we have

$$\pi_\chi(g) \circ \square_z \circ \pi_\chi(g^{-1}) = P_g \square_z$$

with $P_g(n) = \eta(a(g^{-1}n))^{-1}$, the identity being valid on the dense open subset of N on which both sides are defined. In particular, the space of all $\varphi \in \Gamma(\mathcal{L}_\chi)$ such that $\square_z \bullet \varphi = 0$ is invariant under G.

The operators $\pi_\chi(w_0)$ are particularly interesting, since they are analogues of the Kelvin transform associated with the Laplacian on $\mathbb{R}^n$. For this reason, we record in a more explicit form the special case of Theorem 3.3 with $g = w_0$.

COROLLARY 3.4

We have

$$\pi_\chi(w_0) \circ \square_z \circ \pi_\chi(w_0^{-1}) = P_{w_0} \square_z$$

with $P_{w_0}(n(x, y, t)) = t(t - xy)$.

Proof

We may evaluate $P_{w_0}(n(x, y, t))$ by combining (2.5), Lemma 2.1, and the fact that

$$w_0^{-1} * n(x, y, t) = n\Big(\frac{x}{t - xy}, \frac{y}{t}, -\frac{1}{t}\Big).$$

The claim then follows from Theorem 3.3. □

It is convenient to have an evaluation of the structure operator C appearing in (3.2). To express this evaluation, note that there is a direct sum decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{l} \oplus \bar{\mathfrak{n}}$ and associated projection operators onto each summand. We denote the projection onto the middle summand by $\mathrm{pr}_{\mathfrak{l}} : \mathfrak{g} \to \mathfrak{l}$.

PROPOSITION 3.5

For $X \in \mathfrak{g}$ and $n \in N$ we have

$$C(X)(n) = \gamma\big(\mathrm{pr}_{\mathfrak{l}}(\mathrm{Ad}(n^{-1})X)\big).$$

Proof

This evaluation is a consequence of a more general evaluation of the structure operator (for which see [2, Theorem 15]). However, having obtained Theorem 3.3 from the general theory, it is easy to evaluate C directly. Indeed, it follows from Theorem 3.3 that

$$C(X)(n) = \frac{d}{d\tau}\Big|_{\tau=0} \eta\big(a(e^{-\tau X}n)\big)^{-1}. \tag{3.3}$$

We have

$$n^{-1}e^{-\tau X}n = \big(n^{-1}\zeta(e^{-\tau X}n)\big)a(e^{-\tau X}n)\bar{\zeta}(e^{-\tau X}n)$$

from which we obtain

$$\frac{d}{d\tau}\Big|_{\tau=0} a(e^{-\tau X}n) = -\,\mathrm{pr}_{\mathfrak{l}}\big(\mathrm{Ad}(n^{-1})X\big). \tag{3.4}$$

The required evaluation results from combining (3.3) and (3.4). □

For later use, we observe that $\Box_z$ may be expressed in terms of operators in the image of Π_z.

LEMMA 3.6

We have

$$\Box_z = -\Pi_z(T)\big(\Pi_z(W_0) + d\chi(W_0)\big) + \sum_{i=1}^{d} \Pi_z(Y_i)\Pi_z(X_i) - (z - z_0)\Pi_z(T).$$

Proof

This result comes from a calculation based upon the explicit evaluations of the various operators that were given in Section 2. □

Note that the operator $\Pi_z(W_0) + d\chi(W_0)$ is independent of z; in fact,

$$\Pi_z(W_0) + d\chi(W_0) = \mathbb{E}_x - \mathbb{E}_y.$$

Lemma 3.6 allows us to understand the interaction of the Heisenberg ultrahyperbolic operator and the map on $C^\infty(N)$ induced by inversion on N. If $n = n(x, y, t)$

then $n^{-1} = n(-x, -y, xy - t)$ and this leads us to define an operator $\mathbb{I}$ on functions on N by

$$\mathbb{I}\varphi\big(n(x,y,t)\big) = \varphi\big(n(-x,-y,xy-t)\big).$$

PROPOSITION 3.7

We have

$$\mathbb{I} \circ \Box_z \circ \mathbb{I} = -R(T)\big(\Pi_z(W_0) + d\chi(W_0)\big) + \Box_{-z}.$$

In particular, if $(\Pi_z(W_0) + d\chi(W_0)) \bullet \varphi = \lambda\varphi$ *and* $\Box_z \bullet \varphi = 0$ *then* $\Box_{-(z+\lambda)} \bullet \mathbb{I}\varphi = 0$.

Proof

It follows immediately from the definitions that if $X \in \mathfrak{n}$ then $\mathbb{I} \circ \Pi_z(X) \circ \mathbb{I} = R(X)$ and if $Z \in \mathfrak{l}$ then $\mathbb{I} \circ \Pi_z(Z) \circ \mathbb{I} = \Pi_z(Z)$. The first claim follows from this observation and Lemma 3.6. It also follows that $\mathbb{I}$ leaves the λ-eigenspace of $\Pi_z(W_0) + d\chi(W_0)$ stable. The second claim then follows on applying the displayed identity to $\mathbb{I}\varphi$ under the given assumptions. $\Box$

4. The structure of a $\mathcal{U}(\mathfrak{g})$-module

In this section, we analyze the structure of a $\mathcal{U}(\mathfrak{g})$-module that is associated with the Heisenberg ultrahyperbolic operator. As a by-product, we give a slightly more precise version of Korányi's analogue of Maxwell's theorem [14], with a different method of proof.

Let $\mathcal{U}_k(\mathfrak{n})$ denote the kth step in the standard filtration of the universal enveloping algebra $\mathcal{U}(\mathfrak{n})$. The graded algebra associated to this filtration is isomorphic to the symmetric algebra $S(\mathfrak{n})$ and we identify the two. The symmetric algebra inherits its standard grading from this identification, and we let $S_k(\mathfrak{n})$ denote the space of homogeneous elements with respect to this grading. Let $\theta : \mathcal{U}(\mathfrak{n}) \to S(\mathfrak{n})$ be the canonical projection. Let $\psi : S(\mathfrak{n}) \to \mathcal{U}(\mathfrak{n})$ be the symmetrization map that is defined by

$$\psi(U_1 \cdots U_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} U_{\sigma(1)} \cdots U_{\sigma(k)}$$

on $S_k(\mathfrak{n})$. Since $\mathcal{U}(\mathfrak{n})/\mathcal{U}(\mathfrak{n})T$ is commutative, if $u \in \mathcal{U}_k(\mathfrak{n})$ then

$$\psi\big(\theta(u)\big) = u + Tv \tag{4.1}$$

for some $v \in \mathcal{U}_{k-2}(\mathfrak{n})$. For the same reason, if $p_1 \in S_k(\mathfrak{n})$ and $p_2 \in S_l(\mathfrak{n})$ then

$$\psi(p_1 p_2) = \psi(p_1)\psi(p_2) + Tv \tag{4.2}$$

for some $v \in \mathcal{U}_{k+l-2}(\mathfrak{n})$.

In the discussion that follows, the case where $m = 3$ is sometimes anomalous and has to be dealt with separately. All the conclusions that we draw are valid in the case $m = 3$, but the discussion may sometimes require proper interpretation to cover the anomalous case.

Let W_1 denote the subspace of $\mathfrak{n}$ that is spanned by $X_1, \ldots, X_d$, and let W_2 denote the subspace of $\mathfrak{n}$ that is spanned by $Y_1, \ldots, Y_d$. The group L acts via the adjoint action on $\mathfrak{n}$, and $\mathfrak{n} = W_1 \oplus W_2 \oplus \mathfrak{z}(\mathfrak{n})$ is the decomposition of $\mathfrak{n}$ into irreducible summands under this action. The subgroup L^η of L is isomorphic to $\{\pm 1\} \times \mathrm{GL}^+(d, \mathbb{R})$. Under the action of this subgroup, W_1 and W_2 are duals of one another. Let $W = W_1 \oplus W_2$. The structure of the symmetric algebra $S(W)$ as a module for L^η is well known (see [4, Theorem 2.5.4], for example) and we recall it here. Define

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial X_i \, \partial Y_i},$$

define

$$P = \sum_{i=1}^{d} X_i Y_i,$$

and let

$$\mathcal{H}(W) = \{ u \in S(W) \mid \Delta \bullet u = 0 \}$$

be the harmonic subspace of $S(W)$. It is known that $S(W) \cong \mathbb{C}[P] \otimes_{\mathbb{C}} \mathcal{H}(W)$ via the inverse of the map that sends the tensor $P^i \otimes h$ to hP^i. The space $\mathcal{H}(W)$ is invariant under L^η and decomposes under this group as a direct sum of irreducible L^η-modules. This decomposition has multiplicity one. If $m = 3$, then the summands in this decomposition are $\mathbb{C}1$, $\mathbb{C}X_1^a$, and $\mathbb{C}Y_1^a$ with $a \geq 1$. If $m \geq 4$, then the summands are generated by the highest weight vectors $Y_1^a X_d^b$ for $a, b \geq 0$.

Let $\chi = \chi(z, \varepsilon_1, \varepsilon_2)$, and let

$$\mathcal{J}_\chi = \mathcal{U}(\mathfrak{n})\big(\omega_0 + (z + z_0)T\big)$$

be the indicated left ideal in $\mathcal{U}(\mathfrak{n})$.

LEMMA 4.1
Let $u \in \mathcal{U}(\mathfrak{n})$. Then there is some $p \geq 0$ and $h_0, \ldots, h_p \in \mathcal{H}(W)$ such that

$$u + \mathcal{J}_\chi = \sum_{i=0}^{p} \psi(h_i) T^i + \mathcal{J}_\chi$$

in the module $\mathcal{U}(\mathfrak{n})/\mathcal{J}_\chi$.

Proof
Assume that $u \in \mathcal{U}_k(\mathfrak{n})$. We proceed by induction on k, the cases of $k = 0$ and $k = 1$ being trivial. We have $\theta(u) \in S_k(\mathfrak{n})$ and so we may write $\theta(u) = p_0 + p_1 T$ with $p_0 \in S_k(W)$ and $p_1 \in S_{k-1}(\mathfrak{n})$. By appealing to the structure of $S(W)$ as described above, we may then write $p_0 = h_0 + p_2 P$ with $h_0 \in \mathcal{H}_k(W)$ and $p_2 \in S_{k-2}(W)$. It follows that

$$\psi\big(\theta(u)\big) = \psi(h_0) + \psi(p_2 P) + \psi(p_1 T).$$

In light of (4.1) and (4.2), this identity is equivalent to

$$u = \psi(h_0) + \psi(p_2)\psi(P) + \psi(p_1)T + Tv$$

for some $v \in \mathcal{U}_{k-2}(\mathfrak{n})$. By calculation, $\psi(P) = \omega_0 + z_0 T = [\omega_0 + (z + z_0)T] - zT$, and so we obtain

$$u + \mathcal{J}_\chi = \psi(h_0) + \bigl(-z\psi(p_2) + \psi(p_1) + v\bigr)T + \mathcal{J}_\chi.$$

Now $v' = -z\psi(p_2) + \psi(p_1) + v \in \mathcal{U}_{k-1}(\mathfrak{n})$ and so the induction hypothesis implies that we may write

$$v' + \mathcal{J}_\chi = \sum_{i=1}^{p} \psi(h_i)T^{i-1} + \mathcal{J}_\chi$$

for certain $h_1, \dots, h_p \in \mathcal{H}(W)$. The claim follows on substituting this into the previous expression and noting that $\mathcal{J}_\chi T \subset \mathcal{J}_\chi$. $\square$

LEMMA 4.2

Suppose that $u \in \mathcal{U}(\mathfrak{n})$, and suppose that $uT \in \mathcal{J}_\chi$. Then $u \in \mathcal{J}_\chi$.

Proof

By hypothesis, there is an element $y \in \mathcal{U}(\mathfrak{n})$ such that

$$uT = y\bigl(\omega_0 + (z + z_0)T\bigr). \tag{4.3}$$

Since $[\mathfrak{n}, \mathfrak{n}] = \mathbb{C}T$, we have $\mathcal{U}(\mathfrak{n})/(T) \cong S(W)$. In particular, $\mathcal{U}(\mathfrak{n})/(T)$ is an integral domain. The image of ω_0 in this quotient is nonzero, but $y\omega_0 + (T) = 0$ by (4.3). Thus $y \in (T)$ and so we may write $y = y'T$ for some $y' \in \mathcal{U}(\mathfrak{n})$. The ring $\mathcal{U}(\mathfrak{n})$ has no zero divisors and so the equation

$$T\bigl(u - y'(\omega_0 + (z + z_0)T)\bigr) = 0$$

implies that $u = y'(\omega_0 + (z + z_0)T) \in \mathcal{J}_\chi$. $\square$

PROPOSITION 4.3

The $\mathbb{C}$-linear map

$$\Psi : \mathcal{H}(W) \otimes_{\mathbb{C}} \mathbb{C}[T] \to \mathcal{U}(\mathfrak{n})/\mathcal{J}_\chi$$

such that $\Psi(h \otimes T^i) = \psi(h)T^i + \mathcal{J}_\chi$ is an isomorphism of vector spaces.

Proof

By Lemma 4.1, Ψ is onto. Now suppose that $\Psi\bigl(\sum_{i=0}^{p} h_i \otimes T^i\bigr) = 0$, so that we have $\sum_{i=0}^{p} \psi(h_i)T^i \in \mathcal{J}_\chi$. We have $\mathcal{U}(\mathfrak{n})/((T) + \mathcal{J}_\chi) \cong S(W)/(P) \cong \mathcal{H}(W)$, and the image of $\sum_{i=0}^{p} \psi(h_i)T^i$ in this quotient is h_0. It follows that $h_0 = 0$ and that $T\sum_{i=0}^{p-1} \psi(h_{i+1})T^i \in \mathcal{J}_\chi$. Lemma 4.2 then implies that $\sum_{i=0}^{p-1} \psi(h_{i+1})T^i \in \mathcal{J}_\chi$. We may use this to establish inductively that $h_1 = \cdots = h_p = 0$. Thus Ψ is one-to-one. $\square$

We may identify $\mathcal{U}(\mathfrak{n})$ with the generalized Verma module $\mathcal{M}(d\chi)$ via the map $u \mapsto u \otimes 1$. It follows from Lemma 3.2 that the ideal $\mathcal{J}_\chi$ corresponds under this identification to a submodule of $\mathcal{M}(d\chi)$. We denote this submodule by $\mathcal{N}(d\chi)$. The quotient $\mathcal{U}(\mathfrak{n})/\mathcal{J}_\chi$ corresponds to the quotient $\mathcal{M}(d\chi)/\mathcal{N}(d\chi)$. The quotient $\mathcal{U}(\mathfrak{n})/\mathcal{J}_\chi$ thus acquires the structure of a $\mathcal{U}(\mathfrak{g})$-module. This is the module whose structure we seek to determine.

LEMMA 4.4

Let $Z = E_{m,m} - E_{2,2}$. Then in $\mathcal{U}(\mathfrak{g})$ we have

$$\begin{aligned} ZY_1^a &= -2aY_1^a + Y_1^a Z, \\ \bar{X}_1 Y_1^a &= -a(a-1)Y_1^{a-1} + aY_1^{a-1}Z + Y_1^a \bar{X}_1, \\ \bar{X}_1 X_d^b &= X_d^b \bar{X}_1, \\ ZT^c &= -cT^c + T^c Z, \\ \bar{X}_1 T^c &= -cX_1 T^{c-1} + T^c \bar{X}_1 \end{aligned}$$

for $a, b, c \geq 0$. If, in addition, $m \geq 4$, then $ZX_d^b = X_d^b Z$ for $b \geq 0$.

Let $U = E_{1,1} - E_{m-1,m-1}$. Then in $\mathcal{U}(\mathfrak{g})$ we have

$$\begin{aligned} UX_d^b &= 2bX_d^b + X_d^b U, \\ \bar{Y}_d X_d^b &= -b(b-1)X_d^{b-1} - bX_d^{b-1}U + X_d^b \bar{Y}_d, \\ \bar{Y}_d Y_1^a &= Y_1^a \bar{Y}_d, \\ UT^c &= cT^c + T^c U, \\ \bar{Y}_d T^c &= cY_d T^{c-1} + T^c \bar{Y}_d \end{aligned}$$

for $a, b, c \geq 0$. If, in addition, $m \geq 4$, then $UY_1^a = Y_1^a U$ for $a \geq 0$.

Proof

All the proposed identities may be established by induction on the relevant parameter. □

THEOREM 4.5

If $z \notin ((z_0 + \mathbb{N}) \cup (-z_0 - \mathbb{N}))$ then $\mathcal{M}(d\chi)/\mathcal{N}(d\chi)$ is a simple $\mathcal{U}(\mathfrak{g})$-module. If $z \in ((z_0 + \mathbb{N}) \cup (-z_0 - \mathbb{N}))$, then $\mathcal{M}(d\chi)/\mathcal{N}(d\chi)$ has a unique nonzero proper submodule. If $z = z_0 + (a-1)$ with $a \geq 1$, then this submodule is generated by $Y_1^a \otimes 1 + \mathcal{N}(d\chi)$. If $z = -z_0 - (b-1)$ with $b \geq 1$, then this submodule is generated by $X_d^b \otimes 1 + \mathcal{N}(d\chi)$.

Proof

First note that $z_0 \geq 1/2$, so that $(-z_0 - \mathbb{N}) \cap (z_0 + \mathbb{N}) = \emptyset$. Thus the conditions $z \in z_0 + \mathbb{N}$ and $z \in -z_0 - \mathbb{N}$ are mutually exclusive. Let $\mathcal{Y}$ be a nonzero submodule of $\mathcal{M}(d\chi)/\mathcal{N}(d\chi)$. Then $\mathcal{Y}$ is a sum of finite-dimensional irreducible $\mathcal{U}(\mathfrak{l})$-modules.

Each such module contains a highest weight vector, which is necessarily the image under the map Ψ of a highest weight vector in $\mathcal{H}(W)$ and a weight vector in $\mathbb{C}[T]$. We described the structure of the former above and the latter are simply monomials in T. Let us assume for the moment that $m \geq 4$, since the case $m = 3$ is anomalous. Then $[Y_1, X_d] = 0$ and so $\psi(Y_1^a X_d^b) = Y_1^a X_d^b$ for all $a, b \geq 0$. We conclude that there are $a, b, c \geq 0$ such that

$$Y_1^a X_d^b T^c \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}. \tag{4.4}$$

A computation based upon Lemma 4.4 reveals that

$$\bar{X}_1 Y_1^a X_d^b T^c \otimes 1 = a\big((z - z_0) - (a + c - 1)\big) Y_1^{a-1} X_d^b T^c \otimes 1 - c Y_1^a X_d^b X_1 T^{c-1} \otimes 1 \tag{4.5}$$

in $\mathcal{M}(d\chi)$. The element $E_{2,m-1} \in \mathfrak{l}$ commutes with Y_1, X_d, and T, and satisfies $[E_{2,m-1}, X_1] = -X_d$ and $d\chi(E_{2,m-1}) = 0$. Thus

$$E_{2,m-1} \bar{X}_1 Y_1^a X_d^b T^c \otimes 1 = c Y_1^a X_d^{b+1} T^{c-1} \otimes 1 \tag{4.6}$$

in $\mathcal{M}(d\chi)$. It follows from (4.4) and (4.6) that if $c > 0$, then

$$Y_1^a X_d^{b+1} T^{c-1} \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}.$$

By continuing in this way, we conclude that

$$Y_1^a X_d^{b+c} \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}.$$

Thus we may as well assume from the start that $c = 0$, so that

$$Y_1^a X_d^b \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}. \tag{4.7}$$

By (4.5) we have

$$\bar{X}_1 Y_1^a X_d^b \otimes 1 = a\big((z - z_0) - (a - 1)\big) Y_1^{a-1} X_d^b \otimes 1 \tag{4.8}$$

and a computation based upon Lemma 4.4 shows that

$$\bar{Y}_d Y_1^a X_d^b \otimes 1 = -b\big((z + z_0) + (b - 1)\big) Y_1^a X_d^{b-1} \otimes 1. \tag{4.9}$$

By (4.7), (4.8), (4.9), and the observation made at the start of the proof, we may conclude that either $\mathcal{Y}$ contains $Y_1^a \otimes 1 + \mathcal{N}(d\chi)$ for some $a \geq 0$ or $\mathcal{Y}$ contains $X_d^b \otimes 1 + \mathcal{N}(d\chi)$ for some $b \geq 0$.

The conclusion of the previous paragraph was reached under the assumption that $m \geq 4$. We next show that the same conclusion may be reached when $m = 3$ also. The initial difference between this case and the case $m \geq 4$ is that the highest weight vectors have a simpler form. In light of this, we conclude either that there are $a, c \geq 0$ such that

$$Y_1^a T^c \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y} \tag{4.10}$$

or that there are $b, c \geq 0$ such that

$$X_1^b T^c \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}. \tag{4.11}$$

A calculation similar to those used in the other case shows that

$$\bar{T} Y_1^a T^c \otimes 1 = -c(a + c) Y_1^a T^{c-1} \otimes 1$$

and

$$\bar{T}X_1^bT^c \otimes 1 = -c(b+c)X_1^bT^{c-1} \otimes 1.$$

Thus if (4.10) holds, then $Y_1^a \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}$, and if (4.11) holds, then $X_1^b \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}$. This brings us to the same point in the analysis of the case $m = 3$ that we previously reached in the case $m \geq 4$.

By specializing (4.8) and (4.9), we obtain

$$\bar{X}_1Y_1^a \otimes 1 = a\big((z - z_0) - (a-1)\big)Y_1^{a-1} \otimes 1 \tag{4.12}$$

and

$$\bar{Y}_dX_d^b \otimes 1 = -b\big((z + z_0) + (b-1)\big)X_d^{b-1} \otimes 1. \tag{4.13}$$

Even though (4.8) and (4.9) are only generally valid when $m \geq 4$, these special cases are also valid when $m = 3$. By combining these identities with our previous conclusions, we conclude that if $z \notin ((z_0 + \mathbb{N}) \cup (-z_0 - \mathbb{N}))$, then $1 \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}$ and so $\mathcal{Y} = \mathcal{M}(d\chi)/\mathcal{N}(d\chi)$. This establishes the first claim.

Now suppose that there is some $a_0 \geq 1$ such that $z = z_0 + (a_0 - 1)$. We conclude from (4.13) that if $X_d^b \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}$ for some $b \geq 0$ then $1 \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}$ and so $\mathcal{Y} = \mathcal{M}(d\chi)/\mathcal{N}(d\chi)$. We draw the same conclusion from (4.12) if $Y_1^a \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}$ for some $0 \leq a < a_0$. If, on the other hand, $Y_1^a \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}$ for some $a \geq a_0$, then we conclude that $Y_1^{a_0} \otimes 1 + \mathcal{N}(d\chi) \in \mathcal{Y}$. These observations imply that the only possibility for a nonzero proper submodule of $\mathcal{M}(d\chi)/\mathcal{N}(d\chi)$ is $\mathcal{Y} = \mathcal{U}(\mathfrak{g})(Y_1^{a_0} \otimes 1 + \mathcal{N}(d\chi))$. It remains to establish that this is, indeed, a submodule. For this purpose, it is sufficient to show that $\bar{\mathfrak{n}}(Y_1^{a_0} \otimes 1) = \{0\}$. By direct calculation, we have $\bar{Y}_jY_1^{a_0} \otimes 1 = 0$ for $1 \leq j \leq d$. It follows from (4.12) that $\bar{X}_1Y_1^{a_0} \otimes 1 = 0$. If $2 \leq j \leq d$, then we multiply this equation by $E_{2,j+1}$ to obtain $\bar{X}_jY_1^{a_0} \otimes 1 = 0$. Finally, since $[\bar{X}_1, \bar{Y}_1] = \bar{T}$, we conclude that $\bar{T}Y_1^{a_0} \otimes 1 = 0$, and this gives the required conclusion.

If there is some $b_0 \geq 1$ such that $z = -z_0 - (b_0 - 1)$, then we may analyze the situation in the same way as in the previous paragraph. We conclude that $\mathcal{Y} = \mathcal{U}(\mathfrak{g})(X_d^{b_0} \otimes 1 + \mathcal{N}(d\chi))$ is a submodule of $\mathcal{M}(d\chi)/\mathcal{N}(d\chi)$ and is the only nonzero proper submodule. This completes the proof. □

Let $R = \mathbb{C}[x_1, \ldots, x_d, y_1, \ldots, y_d, t]$. For $k \geq 0$, we let $R\langle k\rangle$ be the span of the monomials $x^ay^bt^c$ such that $|a| + |b| + 2c = k$, where we use vector exponents and write $|a| = a_1 + \cdots + a_d$ for $a \in \mathbb{N}^d$. The $R\langle k\rangle$ are the homogeneous subspaces for a grading of R. The operator $\Box_z$ is homogeneous of degree -2 for this grading. We let $\mathcal{P} \subset R$ be the space of polynomial solutions to the equation $\Box_z \bullet f = 0$. The homogeneity of $\Box_z$ implies that $\mathcal{P} = \bigoplus_{k\geq 0} \mathcal{P}\langle k\rangle$ where $\mathcal{P}\langle k\rangle = \mathcal{P} \cap R\langle k\rangle$.

LEMMA 4.6

The map $\Box_z : R\langle k+2\rangle \to R\langle k\rangle$ is surjective for all $k \geq 0$.

Proof
We have

$$\square_z = \Delta + \mathbb{E}_x \frac{\partial}{\partial t} + (z + z_0)\frac{\partial}{\partial t},$$

where

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j\, \partial y_j}$$

is the Euclidean ultrahyperbolic operator. It is well known that the map

$$\Delta : \mathbb{C}[x_1, \ldots, x_d, y_1, \ldots, y_d]\langle k+2\rangle \to \mathbb{C}[x_1, \ldots, x_d, y_1, \ldots, y_d]\langle k\rangle$$

is surjective. This can be proved by joint induction on d and the degree with respect to the last pair of variables, for example. Given this, we prove by induction on $k+l$ that if $p \in \mathbb{C}[x,y]\langle k-2l\rangle$, then there is some $q \in R\langle k+2\rangle$ of degree at most l in t such that $\square_z \bullet q = t^l p$. The initial case is trivial, so suppose that $k+l \geq 1$, and consider $p \in \mathbb{C}[x,y]\langle k-2l\rangle$. If $l = 0$, then the claim follows from the surjectivity of Δ, so suppose that $l \geq 1$. By hypothesis there is some $q \in R\langle k\rangle$ of degree at most $l-1$ in t such that $\square_z \bullet q = t^{l-1}p$. We have $[\square_z, t] = \mathbb{E}_x + (z+z_0)$ and so $\square_z \bullet (tq) = t^l p + (\mathbb{E}_x + (z+z_0)) \bullet q$. Now $r = (\mathbb{E}_x + (z+z_0)) \bullet q \in R\langle k\rangle$ is a sum of terms homogeneous of degree at most $l-1$ in t. By hypothesis there is some $s \in R\langle k+2\rangle$ of degree at most $l-1$ in t such that $\square_z \bullet s = r$. Then $\square_z \bullet (tq - s) = t^l p$ and $tq - s \in R\langle k+2\rangle$ has degree at most l in t. This completes the inductive step. $\square$

The dimension of $R\langle k\rangle$ is

$$\dim(R\langle k\rangle) = \sum_{l=0}^{\lfloor k/2\rfloor} \binom{2m-5+k-2l}{2m-5}$$

and it follows from this and Lemma 4.6 that

$$\dim(\mathcal{P}\langle k\rangle) = \binom{2m-5+k}{2m-5}.$$

Note that this is equal to $\dim \mathbb{C}[x,y]\langle k\rangle$ for all $k \geq 0$.

By (3.2), if a function φ satisfies $\square_z \bullet \varphi = 0$ on some set and $X \in \mathfrak{g}$ then $\square_z \bullet (\Pi_z(X) \bullet \varphi) = 0$ on the same set. It follows that $\square_z \bullet (\Pi_z(u) \bullet \varphi) = 0$ for all $u \in \mathcal{U}(\mathfrak{g})$. Also, we have seen in Section 2 that $\Pi_z(X)$ is a polynomial differential operator on N for all $X \in \mathfrak{g}$. Thus $\mathcal{P}$ is a $\mathcal{U}(\mathfrak{g})$-module via Π_z. Although it is not usually stated in this form, the basic analogue of Maxwell's theorem for the Heisenberg ultrahyperbolic equation is the statement that $1 \in \mathcal{P}$ is a cyclic vector for this module. We shall see below how this is consistent with the usual statement of Maxwell's theorem.

If $\chi = \chi(z, \varepsilon_1, \varepsilon_2)$ then we let $\chi' = \chi(-z, \varepsilon_1, \varepsilon_2)$.

LEMMA 4.7
We have $d\chi' = -d\chi \circ \mathrm{Ad}(w_0)$ *on* $\mathfrak{l}$.

Proof
Both sides of the proposed identity vanish on the semisimple part of $\mathfrak{l}$, so it suffices to verify that they agree on H_0 and W_0. We have $\mathrm{Ad}(w_0)H_0 = -H_0$ and $\mathrm{Ad}(w_0)W_0 = W_0$, and so the identity amounts to $d\chi'(H_0) = d\chi(H_0)$ and $d\chi'(W_0) = -d\chi(W_0)$. For $\chi = \chi(z, \varepsilon_1, \varepsilon_2)$, a calculation gives $d\chi(H_0) = 2z_0$ and $d\chi(W_0) = -2dz/m$, and the identity follows. $\square$

Since the Heisenberg ultrahyperbolic equation and the infinitesimal representation Π_z are independent of the parities of χ, we henceforth take $\chi = \chi(z, +, +)$ for simplicity. Let

$$\varphi_0 = \pi_\chi(w_0^{-1})1$$

be the basic solution corresponding to this choice. Explicitly, this function is given by

$$\varphi_0(x, y, t) = |t - xy|^{-(z+z_0)}|t|^{z-z_0}$$

on the set $U = \{(x, y, t) \mid t(t - xy) \neq 0\}$. Since $\square_z \bullet 1 = 0$, it follows from Corollary 3.4 that $\square_z \bullet \varphi_0 = 0$ on U. Thus $\square_z \bullet (\Pi_z(u) \bullet \varphi_0) = 0$ for all $u \in \mathcal{U}(\mathfrak{g})$. We let $\mathcal{P}' = \Pi_z(\mathcal{U}(\mathfrak{g})) \bullet \varphi_0$ be the $\mathcal{U}(\mathfrak{g})$-module generated by φ_0. If $u \in \mathcal{U}(\mathfrak{g})$, then

$$\pi_\chi(w_0)\Pi_z(u) \bullet \varphi_0 = \pi_\chi(w_0)\Pi_z(u)\pi_\chi(w_0^{-1})1 = \Pi_z\big(\mathrm{Ad}(w_0)u\big) \bullet 1$$

and so

$$\Pi_z(u) \bullet \varphi_0 = \pi_\chi(w_0^{-1})\Pi_z\big(\mathrm{Ad}(w_0)u\big) \bullet 1. \tag{4.14}$$

In particular, $\pi_\chi(w_0)\mathcal{P}' \subset \mathcal{P}$, with equality if and only if 1 is a $\mathcal{U}(\mathfrak{g})$-cyclic vector for $\mathcal{P}$.

If $\bar{X} \in \bar{\mathfrak{n}}$, then $\mathrm{Ad}(w_0)\bar{X} \in \mathfrak{n}$ and so, by (4.14),

$$\Pi_z(\bar{X}) \bullet \varphi_0 = \pi_\chi(w_0^{-1})\Pi_z\big(\mathrm{Ad}(w_0)\bar{X}\big) \bullet 1 = 0. \tag{4.15}$$

If $H \in \mathfrak{l}$, then by (4.14),

$$\begin{aligned} \Pi_z(H) \bullet \varphi_0 &= \pi_\chi(w_0^{-1})\Pi_z\big(\mathrm{Ad}(w_0)H\big) \bullet 1 \\ &= -d\chi\big(\mathrm{Ad}(w_0)H\big)\pi_\chi(w_0^{-1})1 = d\chi'(H)\varphi_0, \end{aligned} \tag{4.16}$$

where we have used Lemma 4.7 at the last step. It follows from (4.15) and (4.16) that there is $\mathcal{U}(\mathfrak{g})$-module homomorphism Λ from $\mathcal{M}(d\chi')$ onto $\mathcal{P}'$ that sends $1 \otimes 1$ to φ_0. It also follows that $\mathcal{P}' = \Pi_z(\mathcal{U}(\mathfrak{n})) \bullet \varphi_0$, so that $\pi_\chi(w_0)\mathcal{P}' = \pi_\chi(w_0)\Pi_z(\mathcal{U}(\mathfrak{n})) \bullet \varphi_0$. Thus 1 is a $\mathcal{U}(\mathfrak{g})$-cyclic vector for $\mathcal{P}$ if and only if $\mathcal{P} = \pi_\chi(w_0)\Pi_z(\mathcal{U}(\mathfrak{n})) \bullet \varphi_0$. On recalling that $\pi_\chi(w_0)$ is the analogue of the classical Kelvin transform, this formulation makes the connection with the usual version of Maxwell's theorem clearer.

Recall that the submodule $\mathcal{N}(d\chi')$ of $\mathcal{M}(d\chi')$ is generated by the element $u_0 \otimes 1$, where

$$u_0 = \sum_{i=1}^{d} Y_i X_i - (z - z_0)T.$$

Now $\Pi_z(W_0) \bullet \varphi_0 = d\chi'(W_0)\varphi_0 = -d\chi(W_0)\varphi_0$ and so

$$\big(\Pi_z(W_0) + d\chi(W_0)\big) \bullet \varphi_0 = 0.$$

It follows from this relation and Lemma 3.6 that

$$\Lambda(u_0 \otimes 1) = \Pi_z(u_0) \bullet \varphi_0 = \Box_z \bullet \varphi_0 = 0.$$

Thus Λ induces a map, which we continue to denote by the same symbol, from the quotient module $\mathcal{M}(d\chi')/\mathcal{N}(d\chi')$ onto $\mathcal{P}'$.

LEMMA 4.8

For $a, b \geq 0$, we have

$$\Lambda(Y_1^a \otimes 1) = (z + z_0)_a \pi_\chi(w_0^{-1}) x_1^a$$

and

$$\Lambda(X_d^b \otimes 1) = (-1)^b (z_0 - z)_b \pi_\chi(w_0^{-1}) y_d^b.$$

Proof

We proceed by induction. If $a = 0$ or $b = 0$, the statement follows from the definition of Λ. Now $\mathrm{Ad}(w_0)(Y_1) = \bar{Y}_1$ and so

$$\begin{aligned}
\Lambda(Y_1^a \otimes 1) &= \Pi_z(Y_1) \bullet \Lambda(Y_1^{a-1} \otimes 1)\\
&= (z + z_0)_{a-1} \Pi_z(Y_1) \bullet \pi_\chi(w_0^{-1}) x_1^{a-1}\\
&= (z + z_0)_{a-1} \pi_\chi(w_0^{-1}) \Pi_z(\bar{Y}_1) \bullet x_1^{a-1}\\
&= (z + z_0)_{a-1} \pi_\chi(w_0^{-1})[(z + z_0) x_1^a + (a-1) x_1^a]\\
&= (z + z_0)_{a-1} (z + z_0 + a - 1) \pi_\chi(w_0^{-1}) x_1^a\\
&= (z + z_0)_a \pi_\chi(w_0^{-1}) x_1^a,
\end{aligned}$$

where we have used Lemma 2.2 from the third line to the fourth. The other identity is proved similarly, making use of the fact that $\mathrm{Ad}(w_0) X_d = -\bar{X}_d$. $\Box$

We are now ready to state the analogue of Maxwell's theorem for the Heisenberg ultrahyperbolic equation.

THEOREM 4.9

Let $\mathcal{P}$ be the space of polynomial solutions to the Heisenberg ultrahyperbolic equation $\Box_z \bullet f = 0$, regarded as a $\mathcal{U}(\mathfrak{g})$-module via Π_z. Then $1 \in \mathcal{P}$ is a $\mathcal{U}(\mathfrak{g})$-cyclic vector if and only if $z \notin ((z_0 + \mathbb{N}) \cup (-z_0 - \mathbb{N}))$.

Proof

First suppose that $z \notin ((z_0 + \mathbb{N}) \cup (-z_0 - \mathbb{N}))$. Then $-z$ satisfies the same condition and so, by Theorem 4.5, $\mathcal{M}(d\chi')/\mathcal{N}(d\chi')$ is a simple $\mathcal{U}(\mathfrak{g})$-module. Since $\Lambda(1 \otimes 1) \neq 0$, it follows that $\mathcal{P}'$ is isomorphic to $\mathcal{M}(d\chi')/\mathcal{N}(d\chi')$ as a $\mathcal{U}(\mathfrak{g})$-module.

We have already introduced the grading of the polynomial ring R that derives from assigning weight 1 to the variables $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$, and weight 2 to the variable t. In Section 2, we gave an explicit expression for the operator $\Pi_z(H_0)$ and it follows from this expression that $R\langle k\rangle$ is precisely the k-eigenspace of $-\Pi_z(H_0) - 2z_0$ in R. That is, the grading on R derives from the action of this element. The element $H_0 - 2z_0$ acts semisimply on the module $\mathcal{M}(d\chi')$ with eigenvalues in $\mathbb{N}$ and we have a corresponding decomposition

$$\mathcal{M}(d\chi') = \bigoplus_{k\in\mathbb{N}} \mathcal{M}(d\chi')\langle k\rangle.$$

This is perhaps best seen by identifying $\mathcal{M}(d\chi')$ with $\mathcal{U}(\mathfrak{n})$. Under this identification, the submodule $\mathcal{N}(d\chi')$ is identified with the left ideal $\mathcal{J}_{\chi'}$, which is generated by the homogeneous element $u_0 = \omega_0 - (z - z_0)T \in \mathcal{M}(d\chi')\langle 2\rangle$. It follows that $\mathcal{M}(d\chi')/\mathcal{N}(d\chi') \cong \mathcal{U}(\mathfrak{n})/\mathcal{J}_\chi$ is also graded by the $(H_0 - 2z_0)$-eigenvalue. Let $u \in \mathcal{M}(d\chi')/\mathcal{N}(d\chi')\langle k\rangle$, so that $(H_0 - 2z_0)u = ku$. Then $(\Pi_z(H_0) - 2z_0) \bullet \Lambda(u) = k\Lambda(u)$. We may apply $\pi_\chi(w_0)$ to both sides of this equation to conclude that

$$\big(-\Pi_z(H_0) - 2z_0\big) \bullet \pi_\chi(w_0)\Lambda(u) = k\pi_\chi(w_0)\Lambda(u),$$

since $\mathrm{Ad}(w_0)H_0 = -H_0$. That is, under the map $\pi_\chi(w_0) \circ \Lambda$ from $\mathcal{M}(d\chi')/\mathcal{N}(d\chi')$ to $\mathcal{P}$, the two gradings that we have described correspond.

To establish that $1 \in \mathcal{P}$ is a cyclic vector, we have to show that the map $\pi_\chi(w_0) \circ \Lambda$ is onto. We have seen above that the map Λ is one-to-one and the map $\pi_\chi(w_0)$ is certainly so. Thus $\pi_\chi(w_0) \circ \Lambda$ is one-to-one and

$$\big(\pi_\chi(w_0) \circ \Lambda\big)\big(\mathcal{M}(d\chi')/\mathcal{N}(d\chi')\langle k\rangle\big) \subset \mathcal{P}\langle k\rangle$$

for all $k \in \mathbb{N}$. This inclusion reduces us to showing that

$$\dim\big(\mathcal{M}(d\chi')/\mathcal{N}(d\chi')\langle k\rangle\big) = \dim(\mathcal{P}\langle k\rangle)$$

for all $k \in \mathbb{N}$. This can be done by evaluating both sides, but it is unnecessary to do so. Instead, we can compute as follows:

$$\begin{aligned}
\dim\big(\mathcal{M}(d\chi')/\mathcal{N}(d\chi')\langle k\rangle\big) &= \dim\big(\mathcal{U}(\mathfrak{n})/\mathcal{J}_{\chi'}\langle k\rangle\big) \\
&= \dim\big(\mathcal{U}(\mathfrak{n})\langle k\rangle\big) - \dim(\mathcal{J}_{\chi'}\langle k\rangle) \\
&= \dim\big(\mathcal{U}(\mathfrak{n})\langle k\rangle\big) - \dim\big(\mathcal{U}(\mathfrak{n})\langle k-2\rangle\big) \\
&= \dim\big(S(\mathfrak{n})\langle k\rangle\big) - \dim\big(S(\mathfrak{n})\langle k-2\rangle\big) \\
&= \dim(R\langle k\rangle) - \dim(R\langle k-2\rangle) \\
&= \dim(\mathcal{P}\langle k\rangle).
\end{aligned}$$

The main facts that justify this computation are that the projection $\theta : \mathcal{U}(\mathfrak{n}) \to S(\mathfrak{n})$ is a vector space isomorphism, that $S(\mathfrak{n})$ admits an H_0-action for which θ is intertwining, that $S(\mathfrak{n}) \cong R$ via an isomorphism that respects the gradings, and Lemma 4.6, which justifies the last equality. This completes the proof of the reverse implication.

Now suppose that $z \in -z_0 - \mathbb{N}$. Then $-z \in z_0 + \mathbb{N}$ and we may write $-z = z_0 + (a-1)$ with $a \geq 1$. It follows from Theorem 4.5 that $\mathcal{M}(d\chi')/\mathcal{N}(d\chi')$ has a unique nonzero proper submodule $\mathcal{Y}$ and that $\mathcal{Y}$ is generated by $Y_1^a \otimes 1 + \mathcal{N}(d\chi')$. By Lemma 4.8,

$$\Lambda\big(Y_1^a \otimes 1 + \mathcal{N}(d\chi')\big) = (z+z_0)_a \pi_\chi(w_0^{-1}) x_1^a = (1-a)_a \pi_\chi(w_0^{-1}) x_1^a = 0,$$

and we conclude that $\ker(\Lambda) = \mathcal{Y}$ and $\mathcal{P}' \cong \mathcal{M}(d\chi')/\mathcal{Y}$. In this case, we are required to show that the map $\pi_\chi(w_0) \circ \Lambda : \mathcal{M}(d\chi')/\mathcal{Y} \to \mathcal{P}$ is not onto. With respect to the grading of $\mathcal{M}(d\chi')/\mathcal{N}(d\chi')$ induced by $H_0 - 2z_0$, $\mathcal{Y}$ is generated by a homogeneous element of degree a. Note that $Y_1^a \in \mathcal{H}(W)$ is its own symmetrization, and so it follows from Proposition 4.3 that $Y_1^a + \mathcal{J}_{\chi'} \neq \mathcal{J}_{\chi'}$ or, equivalently, that $Y_1^a \otimes 1 + \mathcal{N}(d\chi') \neq \mathcal{N}(d\chi')$. But $Y_1^a \otimes 1 + \mathcal{N}(d\chi') \in \mathcal{Y}$ and so

$$\begin{aligned}\dim\big(\mathcal{M}(d\chi')/\mathcal{Y}\langle a\rangle\big) &< \dim\big(\mathcal{M}(d\chi')/\mathcal{N}(d\chi')\langle a\rangle\big)\\ &= \dim(\mathcal{P}\langle a\rangle).\end{aligned}$$

Since $\pi_\chi(w_0) \circ \Lambda$ respects the gradings, it is not onto.

Finally, suppose that $z \in z_0 + \mathbb{N}$. Then $-z \in -z_0 - \mathbb{N}$ and we may write $-z = -z_0 - (b-1)$ with $b \geq 1$. The $\mathcal{U}(\mathfrak{g})$-module $\mathcal{M}(d\chi')/\mathcal{N}(d\chi')$ has a unique nonzero proper submodule $\mathcal{Y}$, which is generated by $X_d^b \otimes 1 + \mathcal{N}(d\chi')$. By Lemma 4.8,

$$\begin{aligned}\Lambda\big(X_d^b \otimes 1 + \mathcal{N}(d\chi')\big) &= (-1)^b (z_0 - z)_b \pi_\chi(w_0^{-1}) y_d^b\\ &= (-1)^b (1-b)_b \pi_\chi(w_0^{-1}) y_d^b = 0,\end{aligned}$$

and we conclude that $\ker(\Lambda) = \mathcal{Y}$ and $\mathcal{P}' \cong \mathcal{M}(d\chi')/\mathcal{Y}$. From this point, it follows as before that $\pi_\chi(w_0) \circ \Lambda : \mathcal{M}(d\chi')/\mathcal{Y} \to \mathcal{P}$ is not onto. This completes the proof. □

COROLLARY 4.10

With the notation and assumptions as in Theorem 4.9, the cyclic $\mathcal{U}(\mathfrak{g})$-submodule of $\mathcal{P}$ generated by 1 is always simple.

Proof

We saw in the proof of Theorem 4.9 that, up to conjugation by $\pi_\chi(w_0)$ (which does not affect simplicity), the cyclic submodule of $\mathcal{P}$ generated by 1 is isomorphic to $\mathcal{M}(d\chi')/\mathcal{N}(d\chi')$ when $z \notin ((z_0 + \mathbb{N}) \cup (-z_0 - \mathbb{N}))$ and to $\mathcal{M}(d\chi')/\mathcal{Y}$ when $z \in ((z_0 + \mathbb{N}) \cup (-z_0 - \mathbb{N}))$. By Theorem 4.5, this is a simple module in both cases. □

With a little additional work, one may identify the cyclic submodule of $\mathcal{P}$ generated by 1 in the exceptional cases. When $z \in -z_0 - \mathbb{N}$ and we write $z = -z_0 - (b-1)$ with $b \geq 1$, this submodule consists of all solutions of the form

$$\sum_{|c| \leq b-1} p_c(y,t) x^c.$$

When $z \in z_0 + \mathbb{N}$ and we write $z = z_0 + (a-1)$ with $a \geq 1$, it consists of all solutions of the form

$$\sum_{|c| \leq a-1} p_c(x, t - xy) y^c.$$

Here we use vector exponent notation in both cases and each p_c is a polynomial in the indicated variables.

5. K-finite solutions to the ultrahyperbolic equation

The purpose of this section is to study the space of K-finite solutions to the Heisenberg ultrahyperbolic equation.

Let $K = \mathrm{SO}(m)$ be the standard maximal compact subgroup of G. For $1 \leq p < q \leq m$ write $Z_{p,q} = E_{p,q} - E_{q,p}$. The set $\{Z_{p,q} \mid 1 \leq p < q \leq m\}$ is a basis for $\mathfrak{k}_0$. If we extend the notation so that $Z_{p,q} = -Z_{q,p}$ when $p > q$ and $Z_{p,p} = 0$, then we have the bracket relation

$$[Z_{p,q}, Z_{r,s}] = \delta_{p,s} Z_{q,r} + \delta_{q,r} Z_{p,s} - \delta_{p,r} Z_{q,s} - \delta_{q,s} Z_{p,r}.$$

The rank of $\mathfrak{k}$ is $l = \lfloor m/2 \rfloor$ and the set $\{Z_{2p-1,2p} \mid 1 \leq j \leq l\}$ spans an abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{k}_0$ such that $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{k}$. Moreover, $A = \exp(\mathfrak{a}_0)$ is a maximal torus in K. The real subspace $\mathfrak{a}_{\mathbb{R}}$ has $\{iZ_{2p-1,2p}\}$ as a basis and we let $\{\lambda_p\}$ be the dual basis of $\mathfrak{a}_{\mathbb{R}}^*$. We give $\mathfrak{a}_{\mathbb{R}}^*$ the inner product with respect to which $\{\lambda_p\}$ is orthonormal. We use the standard ordering on $\mathfrak{a}_{\mathbb{R}}^*$, so that the sum of the positive roots is

$$2\rho = \sum_{p=1}^{l} (m - 2p) \lambda_p.$$

If $\varpi \in \mathfrak{a}_{\mathbb{R}}^*$ is dominant and algebraically integral, then we let V_ϖ denote the corresponding highest weight representation of $\mathfrak{k}$. This space is also a representation of K when ϖ is, in addition, analytically integral. All the weights that we have to consider below satisfy this additional condition. We require the Casimir element

$$\Omega = -\sum_{1 \leq p < q \leq m} Z_{p,q}^2.$$

With our normalizations, Ω acts on the representation V_ϖ via the scalar

$$c(\varpi) = (\varpi, \varpi + 2\rho).$$

The group $K \cap L$ has four connected components and its component group is a Klein 4-group generated by the elements

$$a_1 = \mathrm{diag}(-1, -1, 1, \ldots, 1)$$

and

$$a_2 = \mathrm{diag}(1, -1, 1, \ldots, 1, -1).$$

These elements are chosen so that if $\chi = \chi(z, \varepsilon_1, \varepsilon_2)$, then $\chi(a_1) = \varepsilon_1$ and $\chi(a_2) = \varepsilon_2$, where we are equivocating between the sets $\{\pm\}$ and $\{\pm 1\}$. The connected component of the identity in $K \cap L$ is isomorphic to $\mathrm{SO}(m-2)$, embedded as the central block in K. The irreducible representations of K that appear in the space $\Gamma(\mathcal{L}_\chi)$ for some $\chi = \chi(z, \varepsilon_1, \varepsilon_2)$ are precisely those that contain a vector fixed under $(K \cap L)^\circ$ and so we must study these representations. Note that a vector is fixed by $(K \cap L)^\circ$ if and only if it is annihilated by $\mathfrak{k} \cap \mathfrak{l}$. If $m = 3$ then $(K \cap L)^\circ$ is the trivial group and so every representation of K consists entirely of vectors fixed by $(K \cap L)^\circ$. The case $m = 4$ is also anomalous. For $m \geq 5$, the description of these representations of K is uniform.

LEMMA 5.1

If $m = 4$ then every irreducible representation of K contains a nonzero vector fixed by $(K \cap L)^\circ$. In the representation $V_{a\lambda_1 + b\lambda_2}$ with $a \geq |b|$, the dimension of the space of $(K \cap L)^\circ$-fixed vectors is $a - |b| + 1$. If $m \geq 5$, then the irreducible representation V_ϖ of K contains nonzero $(K \cap L)^\circ$-fixed vectors if and only if $\varpi = a\lambda_1 + b\lambda_2$ with $a \geq b \geq 0$. In the representation $V_{a\lambda_1+b\lambda_2}$, the dimension of the space of $(K \cap L)^\circ$-fixed vectors is $a - b + 1$. In both cases, the element $Z_{1,m}$ acts on the space $V^{\mathfrak{k}\cap\mathfrak{l}}_{a\lambda_1+b\lambda_2}$ and the spectrum of $Z_{1,m}$ on $V^{\mathfrak{k}\cap\mathfrak{l}}_{a\lambda_1+b\lambda_2}$ is $\{ip \mid |p| \leq a - |b|, p \equiv a - |b| \pmod 2\}$. Moreover, every element of this spectrum has multiplicity one.

Proof

This follows from [10, Theorem 9.77] and its proof. According to the cited theorem, the space $V^{\mathfrak{k}\cap\mathfrak{l}}_{a\lambda_1+b\lambda_2}$ is, in fact, a representation of $\mathrm{U}(2)$ and, as such, has highest weight $(a, |b|)$. The proof of the theorem reveals that the element $Z_{1,m}$ is the standard generator of the standard $\mathfrak{so}(2) \subset \mathfrak{su}(2) \subset \mathfrak{u}(2)$ and this implies the claims about the spectrum. $\square$

In $\mathfrak{U}(\mathfrak{k})$, we introduce the elements

$$\Upsilon = \sum_{p=2}^{m-1} Z_{p,m} Z_{1,p},$$

$$\Xi_1 = \sum_{p=2}^{m-1} Z_{1,p}^2,$$

$$\Xi_2 = \sum_{p=2}^{m-1} Z_{p,m}^2,$$

$$\Omega_0 = -\sum_{2 \leq p < q \leq m-1} Z_{p,q}^2,$$

and set $\Upsilon_z = \Upsilon + (z + z_0) Z_{1,m}$ and $\Xi = \Xi_2 - \Xi_1$. We also define an equivalence relation $\doteqdot$ on $\mathfrak{U}(\mathfrak{k})$ by $u_1 \doteqdot u_2$ if and only if $u_1 - u_2 \in \mathfrak{U}(\mathfrak{k})(\mathfrak{k} \cap \mathfrak{l})$. The significance of this equivalence relation is that if $u_1 \doteqdot u_2$ and $v \in V^{\mathfrak{k}\cap\mathfrak{l}}$ for some $\mathfrak{k}$-module V then $u_1 v = u_2 v$. Although it is not logically necessary, it may be helpful to make an observation about the equivalence relation $\doteqdot$ and the associated quotient. Of course, $\mathcal{U}(\mathfrak{k})/\mathcal{U}(\mathfrak{k})(\mathfrak{k}\cap\mathfrak{l})$ is merely a left $\mathcal{U}(\mathfrak{k})$-module. However, the subspace

$$\mathcal{A} = \big(\mathcal{U}(\mathfrak{k})^{\mathfrak{k}\cap\mathfrak{l}} + \mathcal{U}(\mathfrak{k})(\mathfrak{k}\cap\mathfrak{l})\big)/\mathcal{U}(\mathfrak{k})(\mathfrak{k}\cap\mathfrak{l})$$

is an algebra under the product induced by the product on $\mathcal{U}(\mathfrak{k})^{\mathfrak{k}\cap\mathfrak{l}}$, which is easily seen to be well defined. If V is a $\mathfrak{k}$-module then the subspace $V^{\mathfrak{k}\cap\mathfrak{l}}$ is an $\mathcal{A}$-module. Much of what we do below could be interpreted in terms of this construction.

PROPOSITION 5.2

We have

$$\Omega = \Omega_0 - \Xi_1 - \Xi_2 - Z_{1,m}^2$$

and

$$\Upsilon_0 = \frac{1}{2}\sum_{p=2}^{m-1}(Z_{1,p}Z_{p,m} + Z_{p,m}Z_{1,p}).$$

If $Z \in \mathfrak{k}\cap\mathfrak{l}$ and $u \in \{Z_{1,m}, \Upsilon, \Xi_1, \Xi_2, \Omega_0\}$, then $[Z,u] = 0$. We also have

$$\begin{aligned}[Z_{1,m}, \Xi_1] &= 2\Upsilon_0,\\ [Z_{1,m}, \Xi_2] &= -2\Upsilon_0,\\ [Z_{1,m}, \Upsilon_0] &= \Xi,\end{aligned}$$

and, consequently,

$$\begin{aligned}[Z_{1,m}, \Xi + 2i\Upsilon_0] &= 2i(\Xi + 2i\Upsilon_0),\\ [Z_{1,m}, \Xi - 2i\Upsilon_0] &= -2i(\Xi - 2i\Upsilon_0).\end{aligned}$$

Finally, we have

$$\begin{aligned}[\Xi_1, \Upsilon_0] &\doteq 2Z_{1,m}\Xi_1 - 2\Upsilon_0 - 2z_0(z_0-1)Z_{1,m},\\ [\Xi_2, \Upsilon_0] &\doteq -2Z_{1,m}\Xi_2 - 2\Upsilon_0 + 2z_0(z_0-1)Z_{1,m},\end{aligned}$$

and, consequently,

$$[\Xi, \Upsilon_0] \doteq 2Z_{1,m}\big(\Omega + Z_{1,m}^2 + 2z_0(z_0-1)\big).$$

Proof

The first identity follows directly from the definitions. The second is a consequence of the definitions and the commutator $[Z_{1,p}, Z_{p,m}] = Z_{1,m}$ for $2 \le p \le m-1$. If $Z \in \mathfrak{k}\cap\mathfrak{l}$, then $[Z, \Omega_0] = 0$ because Ω_0 is the Casimir element of $\mathfrak{k}\cap\mathfrak{l}$. It suffices to check the other instances of $[Z,u] = 0$ for $Z = Z_{q,r}$ with $2 \le q < r \le m-1$. For $u = Z_{1,m}$ this is immediate. The other three elements require similar computations, so we present one of them:

$$\begin{aligned}[Z_{q,r}, \Upsilon] &= \sum_{p=2}^{m-1}[Z_{q,r}, Z_{p,m}Z_{1,p}]\\ &= [Z_{q,r}, Z_{q,m}Z_{1,q}] + [Z_{q,r}, Z_{r,m}Z_{1,r}]\end{aligned}$$

$$
\begin{aligned}
&= -Z_{r,m}Z_{1,q} + Z_{q,m}Z_{r,1} + Z_{q,m}Z_{1,r} - Z_{r,m}Z_{q,1} \\
&= -Z_{r,m}Z_{1,q} - Z_{q,m}Z_{1,r} + Z_{q,m}Z_{1,r} + Z_{r,m}Z_{1,q} \\
&= 0.
\end{aligned}
$$

The next three identities are also similar. For example,

$$
\begin{aligned}
[Z_{1,m}, \Xi_1] &= \sum_{p=2}^{m-1} [Z_{1,m}, Z_{1,p}^2] \\
&= \sum_{p=2}^{m-1} ([Z_{1,m}, Z_{1,p}]Z_{1,p} + Z_{1,p}[Z_{1,m}, Z_{1,p}]) \\
&= \sum_{p=2}^{m-1} (-Z_{m,p}Z_{1,p} - Z_{1,p}Z_{m,p}) \\
&= 2\Upsilon_0.
\end{aligned}
$$

The consequences of these identities follow easily.

The final group of equivalences is a little more difficult to obtain. In the following, the indices p and r are understood to range from 2 to $m-1$. To begin with, we have

$$
\begin{aligned}
[Z_{1,r}, \Upsilon_0] &= \frac{1}{2}\sum_p ([Z_{1,r}, Z_{1,p}Z_{p,m}] + [Z_{1,r}, Z_{p,m}Z_{1,p}]) \\
&= \frac{1}{2}\sum_p ([Z_{1,r}, Z_{1,p}]Z_{p,m} + Z_{1,p}[Z_{1,r}, Z_{p,m}] \\
&\qquad + [Z_{1,r}, Z_{p,m}]Z_{1,p} + Z_{p,m}[Z_{1,r}, Z_{1,p}]) \\
&= \frac{1}{2}(Z_{1,r}Z_{1,m} + Z_{1,m}Z_{1,r}) - \frac{1}{2}\sum_{p\neq r}(Z_{r,p}Z_{p,m} + Z_{p,m}Z_{r,p}) \\
&= Z_{1,m}Z_{1,r} + \frac{1}{2}[Z_{1,r}, Z_{1,m}] - \frac{1}{2}\sum_{p\neq r}(2Z_{p,m}Z_{r,p} + [Z_{r,p}, Z_{p,m}]) \\
&= Z_{1,m}Z_{1,r} - \frac{1}{2}Z_{r,m} - \frac{1}{2}\sum_{p\neq r}(2Z_{p,m}Z_{r,p} + Z_{r,m}) \\
&= Z_{1,m}Z_{1,r} - z_0 Z_{r,m} - \sum_{p\neq r} Z_{p,m}Z_{r,p}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&[Z_{1,r}^2, \Upsilon_0] \\
&\quad = Z_{1,r}[Z_{1,r}, \Upsilon_0] + [Z_{1,r}, \Upsilon_0]Z_{1,r} \\
&\quad = (Z_{1,r}Z_{1,m}Z_{1,r} + Z_{1,m}Z_{1,r}^2) - z_0(Z_{1,r}Z_{r,m} + Z_{r,m}Z_{1,r})
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \sum_{p\neq r} (Z_{1,r} Z_{p,m} Z_{r,p} + Z_{p,m} Z_{r,p} Z_{1,r}) \\
&\doteq 2Z_{1,m} Z_{1,r}^2 + [Z_{1,r}, Z_{1,m}] Z_{1,r} - z_0 (Z_{1,r} Z_{r,m} + Z_{r,m} Z_{1,r}) \\
&\quad - \sum_{p\neq r} Z_{p,m} [Z_{r,p}, Z_{1,r}] \\
&\doteq 2Z_{1,m} Z_{1,r}^2 - Z_{r,m} Z_{1,r} - z_0 (Z_{1,r} Z_{r,m} + Z_{r,m} Z_{1,r}) + \sum_{p\neq r} Z_{p,m} Z_{1,p}
\end{aligned}
$$

and so

$$
\begin{aligned}
[\Xi_1, \Upsilon_0] &\doteq 2Z_{1,m}\Xi_1 - \Upsilon - 2z_0 \Upsilon_0 + \sum_r \sum_{p\neq r} Z_{p,m} Z_{1,p} \\
&\doteq 2Z_{1,m}\Xi_1 - \Upsilon - 2z_0 \Upsilon_0 + \sum_p \sum_{r\neq p} Z_{p,m} Z_{1,p} \\
&\doteq 2Z_{1,m}\Xi_1 - \Upsilon - 2z_0 \Upsilon_0 + (d-1) \sum_p Z_{p,m} Z_{1,p} \\
&\doteq 2Z_{1,m}\Xi_1 - \Upsilon - 2z_0 \Upsilon_0 + (d-1)\Upsilon \\
&\doteq 2Z_{1,m}\Xi_1 - 2z_0 \Upsilon_0 + (d-2)\Upsilon \\
&\doteq 2Z_{1,m}\Xi_1 - 2z_0 \Upsilon_0 + 2(z_0-1)(\Upsilon_0 - z_0 Z_{1,m}) \\
&\doteq 2Z_{1,m}\Xi_1 - 2\Upsilon_0 - 2z_0(z_0-1) Z_{1,m},
\end{aligned}
$$

as claimed. The computation for $[\Xi_2, \Upsilon_0]$ is similar, but a little longer. First one finds that

$$
[Z_{r,m}, \Upsilon_0] = -Z_{1,m} Z_{r,m} - z_0 Z_{1,r} - \sum_{p\neq r} Z_{1,p} Z_{r,p}
$$

and hence

$$
\begin{aligned}
&[Z_{r,m}^2, \Upsilon_0] \\
&\quad = [Z_{r,m}, \Upsilon_0] Z_{r,m} + Z_{r,m} [Z_{r,m}, \Upsilon_0] \\
&\quad = -(Z_{1,m} Z_{r,m}^2 + Z_{r,m} Z_{1,m} Z_{r,m}) - z_0 (Z_{1,r} Z_{r,m} + Z_{r,m} Z_{1,r}) \\
&\qquad - \sum_{p\neq r} (Z_{r,m} Z_{1,p} Z_{r,p} + Z_{1,p} Z_{r,p} Z_{r,m}) \\
&\quad \doteq -2Z_{1,m} Z_{r,m}^2 - [Z_{r,m}, Z_{1,m}] Z_{r,m} - z_0 (Z_{1,r} Z_{r,m} + Z_{r,m} Z_{1,r}) \\
&\qquad - \sum_{p\neq r} Z_{1,p} [Z_{r,p}, Z_{r,m}] \\
&\quad \doteq -2Z_{1,m} Z_{r,m}^2 + Z_{r,1} Z_{r,m} - z_0 (Z_{1,r} Z_{r,m} + Z_{r,m} Z_{1,r}) + \sum_{p\neq r} Z_{1,p} Z_{p,m} \\
&\quad \doteq -2Z_{1,m} Z_{r,m}^2 + Z_{r,m} Z_{r,1} + [Z_{r,1}, Z_{r,m}] - z_0 (Z_{1,r} Z_{r,m} + Z_{r,m} Z_{1,r})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{p\neq r}(Z_{p,m}Z_{1,p}+[Z_{1,p},Z_{p,m}])\\
&\doteqdot -2Z_{1,m}Z_{r,m}^2+Z_{r,m}Z_{r,1}-Z_{1,m}-z_0(Z_{1,r}Z_{r,m}+Z_{r,m}Z_{1,r})\\
&+\sum_{p\neq r}(Z_{p,m}Z_{1,p}+Z_{1,m})\\
&\doteqdot -2Z_{1,m}Z_{r,m}^2-Z_{r,m}Z_{1,r}+(d-2)Z_{1,m}-z_0(Z_{1,r}Z_{r,m}+Z_{r,m}Z_{1,r})\\
&+\sum_{p\neq r}Z_{p,m}Z_{1,p}.
\end{aligned}
$$

By summing this equivalence over r, we obtain

$$
\begin{aligned}
[\Xi_2,\Upsilon_0] &\doteqdot -2Z_{1,m}\Xi_2-\Upsilon+d(d-2)Z_{1,m}-2z_0\Upsilon_0+\sum_r\sum_{p\neq r}Z_{p,m}Z_{1,p}\\
&\doteqdot -2Z_{1,m}\Xi_2-\Upsilon+d(d-2)Z_{1,m}-2z_0\Upsilon_0+(d-1)\Upsilon\\
&\doteqdot -2Z_{1,m}\Xi_2+(d-2)\Upsilon+d(d-2)Z_{1,m}-2z_0\Upsilon_0\\
&\doteqdot -2Z_{1,m}\Xi_2+(2z_0-2)(\Upsilon+dZ_{1,m})-2z_0\Upsilon_0\\
&\doteqdot -2Z_{1,m}\Xi_2+2(z_0-1)(\Upsilon_0+z_0Z_{1,m})-2z_0\Upsilon_0\\
&\doteqdot -2Z_{1,m}\Xi_2-2\Upsilon_0+2z_0(z_0-1)Z_{1,m},
\end{aligned}
$$

as required. For the final equivalence, note that

$$
\begin{aligned}
[\Xi,\Upsilon_0] &= [\Xi_2,\Upsilon_0]-[\Xi_1,\Upsilon_0]\\
&\doteqdot -2Z_{1,m}(\Xi_1+\Xi_2)+4z_0(z_0-1)Z_{1,m}\\
&\doteqdot 2Z_{1,m}(\Omega-\Omega_0+Z_{1,m}^2)+4z_0(z_0-1)Z_{1,m}\\
&\doteqdot 2Z_{1,m}\big(\Omega+Z_{1,m}^2+2z_0(z_0-1)\big),
\end{aligned}
$$

since $\Omega_0\doteqdot 0$. This completes the verification of all the stated identities and equivalences. $\square$

If we define R_+ and R_- in $\mathcal{U}(\mathfrak{k})$ by

$$
\begin{aligned}
R_+ &= \Xi+2i\Upsilon_0,\\
R_- &= \Xi-2i\Upsilon_0,
\end{aligned}
$$

then it follows from Proposition 5.2 that

$$
\begin{aligned}
[Z_{1,m},R_+] &= 2iR_+,\\
[Z_{1,m},R_-] &= -2iR_-,\\
[R_+,R_-] &\doteqdot -4iZ_{1,m}\big(2\Omega+d(d-2)+2Z_{1,m}^2\big).
\end{aligned}
$$

Moreover, $Z_{1,m}$, R_+, and R_- commute with $\mathfrak{k}\cap\mathfrak{l}$. Now let ϖ be the highest weight of an irreducible representation of K such that $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ is nonzero. Then

$Z_{1,m}$, R_+, and R_- act on $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ and the above equivalences imply that

$$[Z_{1,m}, R_+] = 2iR_+,$$
$$[Z_{1,m}, R_-] = -2iR_-,$$
$$[R_+, R_-] = -4iZ_{1,m}\big(h(\varpi) + 2Z_{1,m}^2\big)$$

in $\mathrm{End}(V_\varpi^{\mathfrak{k}\cap\mathfrak{l}})$, where we define

$$h(\varpi) = 2c(\varpi) + d(d-2).$$

These facts lead us to consider the abstract commutation relations

$$\begin{aligned}[] [Z, R_+] &= 2iR_+, \\ [Z, R_-] &= -2iR_-, \\ [R_+, R_-] &= -4iZ(h + 2Z^2), \end{aligned} \tag{5.1}$$

where h is a constant. Let $\mathcal{H}$ be the complex associative algebra generated by Z, R_+, and R_- subject to the relations (5.1).

LEMMA 5.3

In $\mathcal{H}$ let R be either R_+ or R_-, and let c be the constant such that $[Z, R] = cR$. Define a sequence of polynomials in Z by $p_1(Z) = 1$ and $p_{k+1}(Z) = Z^k + (Z - c)p_k(Z)$ for $k \geq 1$. Then the degree of p_k is $k-1$, its leading coefficient is k, and $[Z^k, R] = cp_k(Z)R$ for all $k \geq 1$. There is a unique sequence of polynomials $q_k(Z)$ such that $q_k(0) = 0$ and $[q_k(Z), R] = Z^k R$ for all $k \geq 0$. In particular,

$$q_0(Z) = \frac{1}{c}Z,$$
$$q_1(Z) = \frac{1}{c}\Big(\frac{1}{2}Z^2 + \frac{c}{2}Z\Big),$$
$$q_2(Z) = \frac{1}{c}\Big(\frac{1}{3}Z^3 + \frac{c}{2}Z^2 + \frac{c^2}{6}Z\Big),$$
$$q_3(Z) = \frac{1}{c}\Big(\frac{1}{4}Z^4 + \frac{c}{2}Z^3 + \frac{c^2}{4}Z^2\Big).$$

Proof

The recurrence relation is derived by writing

$$cp_{k+1}(Z)R = [Z^{k+1}, R] = [Z^k Z, R] = Z^k[Z, R] + [Z^k, R]Z$$

and the recurrence relation makes the other claims about $p_k(Z)$ clear. The matrix of the linear map $Z^k \mapsto cp_k(Z)$ from $\mathbb{C}[Z]Z \to \mathbb{C}[Z]$ is upper triangular with the diagonal $(c, 2c, 3c, 4c, \ldots)$ and is hence invertible. The polynomial $q_k(Z)$ is the image of Z^k under the inverse map. By using the recurrence relation to compute the upper left 4-by-4 block in the matrix and then the inverse of this we find the stated values of $q_0(Z), \ldots, q_4(Z)$. $\square$

PROPOSITION 5.4

The element

$$\Lambda = -2Z^4 - 2(h-4)Z^2 + R_+R_- + R_-R_+$$

lies in the center of $\mathcal{H}$*.*

Proof

Let $c = 2i$ so that $[Z, R_+] = cR_+$. With this notation, we have $[R_+, R_-] = -2cZ \times (h + 2Z^2)$. Thus

$$\begin{aligned}
[R_+, R_+R_- + R_-R_+] &= R_+[R_+, R_-] + [R_+, R_-]R_+ \\
&= -2cR_+Z(h+2Z^2) - 2cZ(h+2Z^2)R_+ \\
&= -4cZ(h+2Z^2)R_+ - 2ch[R_+, Z] - 4c[R_+, Z^3] \\
&= -4cZ(h+2Z^2)R_+ + 2c^2hp_1(Z)R_+ + 4c^2p_3(Z)R_+ \\
&= \bigl(-4chZ - 8cZ^3 + 2c^2hp_1(Z) + 4c^2p_3(Z)\bigr)R_+.
\end{aligned}$$

We know that $p_1(Z) = 1$ and the recurrence relation in Lemma 5.3 gives

$$p_3(Z) = 3Z^2 - 3cZ + c^2.$$

By using these values, we find that

$$[R_+R_- + R_-R_+, R_+] = \bigl(8cZ^3 - 12c^2Z^2 + 4c(3c^2+h)Z - 2c^2(2c^2+h)\bigr)R_+.$$

The defining property of the polynomials $q_k(Z)$ now implies that if we let

$$u = 8cq_3(Z) - 12c^2q_2(Z) + 4c(3c^2+h)q_1(Z) - 2c^2(2c^2+h)q_0(Z),$$

then we will have

$$[u, R_+] = [R_+R_- + R_-R_+, R_+],$$

so that

$$\Lambda = -u + (R_+R_- + R_-R_+)$$

commutes with R_+. By using the values of $q_0(Z), \dots, q_3(Z)$ given in Lemma 5.3, we find that

$$u = 2Z^4 + 2(h+c^2)Z^2.$$

One reason for writing this computation in terms of c is that it may now be repeated with R_+ replaced by R_- and c replaced by $-c$ throughout. Since u depends on c only through c^2, the same value of u is found, and hence Λ also commutes with R_-. The relations $[Z, R_+] = 2iR_+$ and $[Z, R_-] = -2iR_-$ make it apparent that Λ commutes with Z, and this completes the proof. $\square$

As usual, it is more convenient to express R_+R_- and R_-R_+ in terms of Z and Λ by use of the third commutation relation. The result is that

$$R_+R_- = Z^4 - 4iZ^3 + (h-4)Z^2 - 2ihZ + \frac{1}{2}\Lambda, \tag{5.2}$$

$$R_-R_+ = Z^4 + 4iZ^3 + (h-4)Z^2 + 2ihZ + \frac{1}{2}\Lambda. \tag{5.3}$$

We assume henceforth that h is real. Let W be a finite-dimensional complex vector space with an Hermitian form $\langle\cdot,\cdot\rangle$, which we shall take to be linear in its first argument and conjugate linear in its second argument. Suppose that W is an $\mathcal{H}$-module; we shall systematically confuse elements of $\mathcal{H}$ with their images in $\mathrm{End}(W)$. We say that W is an Hermitian $\mathcal{H}$-module if $Z^* = -Z$ and $R_+^* = R_-$, where the star denotes Hermitian conjugate with respect to the given form on W. On an Hermitian $\mathcal{H}$-module, Z is diagonalizable with purely imaginary eigenvalues. The zero module $W = \{0\}$ is always present. As usual, we call an $\mathcal{H}$-module *irreducible* if it has no nonzero proper submodules and *simple* if it is irreducible and nonzero. An Hermitian $\mathcal{H}$-module is semisimple, because the orthogonal complement of a submodule is again a submodule.

It is convenient to introduce some further terminology for various properties that Hermitian $\mathcal{H}$-modules may enjoy. We call an Hermitian $\mathcal{H}$-module W *regular* if $h > 2\eta^2$ whenever $i\eta$ is an eigenvalue of Z on W. Note that regularity is equivalent to the condition that the Hermitian operator $h + 2Z^2$ be positive; in light of (5.1), this perhaps renders the condition more plausible. An Hermitian $\mathcal{H}$-module W is said to be *commensurable* if $\eta_1/\eta_2 \in \mathbb{Q}$ whenever $i\eta_1$ and $i\eta_2$ are nonzero eigenvalues of Z on W. The significance of this condition should become clearer below. An Hermitian $\mathcal{H}$-module is said to be *standard* if there is a polynomial $f \in \mathbb{R}[x]$ such that

$$[f(Z^2)R_+, f(Z^2)R_-] = -4iZ$$

in $\mathrm{End}(W)$ and the Hermitian operator $f(Z^2)$ on W is positive. The positive Hermitian operator $f(Z^2)$ then has a positive Hermitian square root $f(Z^2)^{1/2}$ and we define

$$Z_2 = \frac{1}{2}\big(f(Z^2)^{1/2}R_+f(Z^2)^{1/2} - f(Z^2)^{1/2}R_-f(Z^2)^{1/2}\big)$$

and

$$Z_3 = \frac{i}{2}\big(f(Z^2)^{1/2}R_+f(Z^2)^{1/2} + f(Z^2)^{1/2}R_-f(Z^2)^{1/2}\big).$$

Note that both Z_2 and Z_3 are skew-Hermitian operators. We may conjugate the above commutator relation by $f(Z^2)^{-1/2}$ to obtain

$$[f(Z^2)^{1/2}R_+f(Z^2)^{1/2}, f(Z^2)^{1/2}R_-f(Z^2)^{1/2}] = -4iZ$$

and it follows that $[Z_2, Z_3] = 2Z$. We also have $[Z, Z_2] = 2Z_3$ and $[Z, Z_3] = -2Z_2$, and hence the real span of Z, Z_2, and Z_3 in $\mathrm{End}(W)$ is a Lie algebra isomorphic to $\mathfrak{su}(2)$. In this way, a standard Hermitian $\mathcal{H}$-module becomes an Hermitian $\mathfrak{su}(2)$-module. The operator $f(Z^2)^{1/2}$ is invertible, and it follows that a subspace of W is an $\mathfrak{su}(2)$-submodule if and only if it is an $\mathcal{H}$-submodule. In particular, W is a simple $\mathcal{H}$-module if and only if it is a simple $\mathfrak{su}(2)$-module. Note that, in the definition of standard, f may depend on W. The existence of nonstandard modules, which is verified below, shows that this is unavoidable.

LEMMA 5.5

Let $n \geq 1$, and let $r_1, \dots, r_n$ be positive real numbers such that $r_k = r_{n-k+1}$ for $1 \leq k \leq n$. Then there exist positive real numbers $a_0, \dots, a_n$ such that $a_{k-1}a_k = r_k$ for $1 \leq k \leq n$ and $a_k = a_{n-k}$ for $0 \leq k \leq n$.

Proof

Let $\tau > 0$, and define

$$a_k = \tau^{(-1)^k} \prod_{j=1}^{k} r_j^{(-1)^{k-j}}$$

for $0 \leq k \leq n$. It is easy to check that $a_{k-1}a_k = r_k$ and so it remains to show that we may also arrange the symmetry condition on a_k by choosing τ appropriately. We have

$$\frac{a_{n-k}}{a_k} \cdot \frac{a_{n-k+1}}{a_{k-1}} = \frac{r_{n-k+1}}{r_k} = 1$$

for $1 \leq k \leq n$. If we can arrange that $a_0 = a_n$, then it would follow inductively from this that $a_k = a_{n-k}$ for all $0 \leq k \leq n$. If n is even, then p_j and p_{n-j+1} appear in the product defining a_n with opposite exponents and so $a_n = \tau = a_0$ for any τ. If n is odd, then $a_0 = \tau$ and $a_n = \tau^{-1}P$, where $P > 0$ denotes the remainder of the product defining a_n. We arrange that $a_0 = a_n$ by taking $\tau = \sqrt{P}$ in this case. □

In the following result, we completely determine the structure of simple Hermitian $\mathcal{H}$-modules, and classify them according to regularity, standardness, and, in some cases, commensurability. This information is quite a bit more than we strictly require for the purpose at hand. However, the author believes that it is helpful to clarify the relationship between $\mathcal{H}$-modules and $\mathfrak{su}(2)$-modules.

THEOREM 5.6

Let W be a simple Hermitian $\mathcal{H}$-module of dimension $n+1$, and denote by μ the largest eigenvalue of $-iZ$ on W. Let $\Delta = 2h - 4n^2 - 8n$. Then the isomorphism class of W is determined by n and μ. The possible values of μ and the conditions under which there is a corresponding module are

(1) $\mu = n$ *when* $h > 2n^2$ *and* $n \geq 1$,
(2) $\mu = n + (1/2)\sqrt{\Delta}$ *when* $2(n+1)^2 - 2 < h < 2(n+1)^2$ *and* $n \geq 1$,
(3) $\mu = n - (1/2)\sqrt{\Delta}$ *when* $2(n+1)^2 - 2 < h < 2(n+1)^2$ *and* $n \geq 1$,
(4) $\mu = 0$ *when* $n = 0$,
(5) $\mu = \sqrt{h/2}$ *when* $h > 0$ *and* $n = 0$,
(6) $\mu = -\sqrt{h/2}$ *when* $h > 0$ *and* $n = 0$.

Among these, the regular modules are those corresponding to (1), (2), (3), and, when $h > 0$, (4). The standard modules are those corresponding to (1) and (4). If h is an integer, then the commensurable modules are those corresponding to (1), *(4), (5), and (6). If h is an integer, then every simple, regular, commensurable module is standard.*

Proof
Let $v \in W$ be an eigenvector associated to μ, so that $Zv = i\mu v$. By Schur's lemma, Λ acts on W by a scalar, which we denote by λ. For $k \geq 0$, define $v_k = (R_-)^k v$, and note that $Zv_k = i(\mu - 2k)v_k$. Moreover, by (5.2), $R_+ v_k$ is a multiple of v_{k-1}. Let V be the subspace of W spanned by the set $\{v_k \mid k \geq 0\}$. The observations we have just made imply that V is an $\mathcal{H}$-submodule of W and so $V = W$. The map R_- is strictly triangular with respect to a basis drawn from this spanning set and so R_- is nilpotent. In particular, $(R_-)^{n+1} = 0$, which implies that $v_k = 0$ for $k \geq n+1$. It follows from this that the set $\{v_0, \dots, v_n\}$ must be a basis for W and, in particular, $v_n \neq 0$. From the choice of μ, we must have $R_+ v_0 = 0$ and so $R_- R_+ v_0 = 0$. From the construction of v_n, we have $R_- v_n = 0$ and so $R_+ R_- v_n = 0$. Let

$$\psi(x) = x^4 - 4x^3 - (h-4)x^2 + 2hx + \frac{1}{2}\lambda. \tag{5.4}$$

It follows from (5.2) and (5.3) that

$$R_+ R_- v_k = \psi(\mu - 2k) v_k,$$
$$R_- R_+ v_k = \psi(2k - \mu) v_k$$

for $k = 0, \dots, n$. In particular, we have $\psi(-\mu) = 0$ and $\psi(\mu - 2n) = 0$. Now

$$\psi(-\mu) - \psi(\mu - 2n) = 4(n+1)(\mu - n)(2\mu^2 - 4n\mu + 4n^2 + 4n - h)$$

and it follows that the permissible values of μ are $\mu_1 = n$,

$$\mu_2 = n + \frac{1}{2}\sqrt{\Delta},$$

and

$$\mu_3 = n - \frac{1}{2}\sqrt{\Delta}.$$

Note that μ_2 and μ_3 are only possible when $\Delta > 0$; we do not need to consider $\Delta = 0$, since in this case μ_2 and μ_3 reduce to μ_1.

To obtain an Hermitian module, we require that $R_-^* = R_+$. This implies that $R_+ R_-$ is a nonnegative operator whose kernel coincides with the kernel of R_-, which we know to be spanned by v_n. The equation

$$R_+ R_- v_k = \psi(\mu - 2k) v_k$$

then implies that $\psi(\mu - 2k) > 0$ for $0 \leq k \leq n-1$. Now $\psi(\mu - 2n) = 0$ and so this condition may be expressed by saying that $\varphi_k(\mu) = \psi(\mu - 2k) - \psi(\mu - 2n)$ is strictly positive for $0 \leq k \leq n-1$. On calculation, we find that

$$\varphi_{k-1}(\mu_1) = 4k(n-k+1)\big(h - 2n^2 + 4(k-1)(n-k)\big),$$
$$\varphi_{k-1}(\mu_2) = 4k(n-k+1)(2k - \sqrt{\Delta})\big(2(n-k+1) + \sqrt{\Delta}\big),$$

$$\varphi_{k-1}(\mu_3) = 4k(n-k+1)(2k+\sqrt{\Delta})\big(2(n-k+1)-\sqrt{\Delta}\big)$$

for $1 \le k \le n$. If $n = 0$, then no condition arises from the requirement that $\varphi_{k-1}(\mu_j) > 0$ for $1 \le k \le n$. If $n \ge 1$, then these expressions make it clear that the positivity condition is equivalent to $h > 2n^2$ for μ_1 and to $\sqrt{\Delta} < 2$ for μ_2 and μ_3. One checks that the inequalities $0 < \Delta < 4$ are equivalent to

$$2(n+1)^2 - 2 < h < 2(n+1)^2.$$

This shows that the list of possible largest eigenvalues of $-iZ$ on an Hermitian module and the conditions under which each may occur is exhaustive when $n \ge 1$. The case $n = 0$ is easily analyzed directly, since R_+ and R_- must both be the zero operator in this case. One finds that the list is exhaustive in this case also.

Now suppose that μ is one of the largest eigenvalues of $-iZ$ on the list, and suppose that the relevant condition is satisfied. Define

$$\lambda = -2\big(\mu^4 + 4\mu^3 - (h-4)\mu^2 - 2h\mu\big),$$

and let $\psi(x)$ be the polynomial (5.4) for this value of λ. The identities

$$\psi(x) - \psi(-x) = 4x(h - 2x^2) \tag{5.5}$$

and

$$\psi(x) - \psi(x+2) = 4x(h - 2x^2) \tag{5.6}$$

are easily verified. By construction, $\psi(-\mu) = 0$ and, by the method used to determine the admissible μ and the condition for admissibility, $\psi(\mu - 2n) = 0$ and $b_k = \psi(\mu - 2k + 2) > 0$ for $1 \le k \le n$. Let W be an $(n+1)$-dimensional complex vector space with basis $v_0, \dots, v_n$, and introduce the Hermitian form $\langle \cdot, \cdot \rangle$ on W such that

$$\langle v_k, v_l \rangle = \delta_{k,l} \prod_{j=1}^{k} b_j.$$

Let Z, R_+, and R_- in $\mathrm{End}(W)$ be such that $Zv_k = i(\mu - 2k)v_k$, $R_+v_0 = 0$, $R_+v_k = b_k v_{k-1}$ for $1 \le k \le n$, $R_-v_k = v_{k+1}$ for $0 \le k \le n-1$, and $R_-v_n = 0$. One can check that these definitions make W into an irreducible Hermitian $\mathcal{H}$-module. The main point is that

$$\begin{aligned}
b_1 &= \psi(\mu) = \psi(\mu) - \psi(-\mu) = 4\mu(h - 2\mu^2), \\
b_{k+1} - b_k &= \psi(\mu - 2k) - \psi(\mu - 2k + 2) = 4(\mu - 2k)\big(h - 2(\mu - 2k)^2\big), \\
-b_n &= -\psi(\mu - 2n + 2) = \psi(\mu - 2n) - \psi(\mu - 2n + 2) \\
&= 4(\mu - 2n)\big(h - 2(\mu - 2n)^2\big)
\end{aligned}$$

for $1 \le k \le n-1$, by (5.5), (5.6), $\psi(-\mu) = 0$, and $\psi(\mu - 2n) = 0$. These identities are equivalent to $[R_+, R_-] = -4iZ(h + 2Z^2)$ and the remaining requirements are more easily verified. This shows that every admissible pair (n, μ) on the list does, in fact, correspond to a simple Hermitian $\mathcal{H}$-module. This concludes the proof of the classification part of the theorem.

It is apparent that the modules in cases (5) and (6) are irregular, that the module in case (4) is regular if and only if $h > 0$, and that the module in case (1) is always regular. The eigenvalues of Z on the module in case (2) are $i(n + (1/2)\sqrt{\Delta} - 2k)$ for $0 \le k \le n$ and if $i\eta$ is one of these numbers then

$$\begin{aligned} 2\eta^2 &\le 2\Big(n + \frac{1}{2}\sqrt{\Delta}\Big)^2 \\ &= h + 2n(\sqrt{\Delta} - 2) \\ &< h, \end{aligned}$$

because the admissibility condition in this case implies that $\sqrt{\Delta} < 2$. Thus these modules are always regular. A similar computation shows that the modules in case (3) are also always regular.

If the module W is standard, then the spectrum of Z on W is the same as the spectrum of Z on the simple $\mathfrak{su}(2)$-module of dimension $n+1$. This shows that the only cases in which the module can possibly be standard are (1) and (4). It is clear that the module in (4) is standard, and so we are required to show that the module in (1) is also standard. When $\mu = \mu_1$, we have

$$b_k = 4k(n-k+1)\big(h - 2n^2 + 4(k-1)(n-k)\big)$$

for $1 \le k \le n$, as we saw above. Let us write these numbers in the form $b_k = c_k r_k$ with $c_k = 4k(n-k+1)$ and $r_k = h - 2n^2 + 4(k-1)(n-k)$. By inspection, the positive real numbers r_k satisfy the symmetry condition $r_k = r_{n-k+1}$. According to Lemma 5.5, we may find positive real numbers $a_0, \dots, a_n$ such that $a_{k-1}a_k = r_k^{-1}$ for $1 \le k \le n$ and $a_k = a_{n-k}$ for $0 \le k \le n$. Define $A \in \operatorname{End}(W)$ by $Av_k = a_k v_k$ for $0 \le k \le n$. This is a positive Hermitian operator. A brief calculation shows that

$$\begin{aligned} [AR_+, AR_-]v_k &= (a_{k+1}a_k b_{k+1} - a_{k-1}a_k b_k)v_k \\ &= (r_{k+1}^{-1} b_{k+1} - r_k^{-1} b_k)v_k \\ &= (c_{k+1} - c_k)v_k \\ &= 4(n-2k)v_k \\ &= -4iZv_k \end{aligned}$$

for $1 \le k \le n-1$. Similarly,

$$[AR_+, AR_-]v_0 = a_0 a_1 b_1 v_0 = c_1 v_0 = 4nv_0 = -4iZv_0$$

and

$$[AR_+, AR_-]v_n = -a_{n-1}a_n b_n v_n = -c_n v_n = -4nv_n = -4iZv_n$$

and we conclude that

$$[AR_+, AR_-] = -4iZ.$$

It remains to confirm that there is a polynomial $f \in \mathbb{R}[x]$ such that $f(Z^2) = A$. This equation is equivalent to $f(-(n-2k)^2) = a_k$ for $0 \le k \le n$, and this is possible precisely because $a_k = a_{n-k}$ for $0 \le k \le n$. Thus the modules in case (1) are standard.

Now suppose that h is an integer. It is evident that the modules in cases (1), (4), (5), and (6) are all commensurable. We must confirm that the modules in cases (2) and (3) are not commensurable. Since $h \in \mathbb{Z}$, the inequality $2(n+1)^2 - 2 < h < 2(n+1)^2$ that holds in these cases implies that $h = 2(n+1)^2 - 1$. It follows that $\Delta = 2$ and so $i(n \pm (1/2)\sqrt{2})$ and $i(n - 2 \pm (1/2)\sqrt{2})$ are both eigenvalues of Z on the module (with the same choice of sign in both expressions). One confirms that the ratio of these numbers is never rational and hence the modules in cases (2) and (3) are not commensurable.

The last claim in the statement follows on comparing the lists of regular, commensurable, and standard simple modules when $h \in \mathbb{Z}$. This completes the proof. $\square$

One consequence of Theorem 5.6 is that, for a fixed value of h, there are only a finite number of isomorphism classes of simple Hermitian $\mathcal{H}$-modules. For $n \ge 0$, we let $\mathbf{M}_n$ denote the standard simple Hermitian $\mathcal{H}$-module of dimension $n+1$ when this module exists.

LEMMA 5.7

Let V_ϖ be an irreducible $\mathfrak{k}$-module such that $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}} \ne \{0\}$, and let $i\eta$ be an eigenvalue of $Z_{1,m}$ on $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$. Then $\eta^2 \le c(\varpi)$ with equality if and only if $\varpi = 0$.

Proof

Let $\langle\cdot,\cdot\rangle$ be an invariant Hermitian form on V_ϖ, and take $v \in V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ to be a $Z_{1,m}$-eigenvector with eigenvalue $i\eta$. Then

$$\begin{aligned}
\big(c(\varpi) - \eta^2\big)\|v\|^2 &= \big\langle \big(c(\varpi) - \eta^2\big)v, v\big\rangle \\
&= \langle (\Omega + Z_{1,m}^2)v, v\rangle \\
&= \langle (-\Xi_1 - \Xi_2 + \Omega_0)v, v\rangle \\
&= \langle (-\Xi_1 - \Xi_2)v, v\rangle \\
&= -\sum_{p=2}^{m-1} \langle (Z_{1,p}^2 + Z_{p,m}^2)v, v\rangle \\
&= \sum_{p=2}^{m-1} (\|Z_{1,p}v\|^2 + \|Z_{p,m}v\|^2).
\end{aligned}$$

This identity shows that $c(\varpi) - \eta^2 \ge 0$. If equality holds, then $Z_{1,p}v = 0$ and $Z_{p,m}v = 0$ for $2 \le p \le m-1$. Thus v is annihilated by the algebra generated by $\mathfrak{k}\cap\mathfrak{l}$, $Z_{1,p}$, and $Z_{p,m}$ for $2 \le p \le m-1$, which is $\mathfrak{k}$. It follows that equality implies that V_ϖ is the trivial representation. $\square$

Let $m \geq 4$, and let V_ϖ be an irreducible $\mathfrak{k}$-module such that $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}} \neq \{0\}$. It follows from the constructions above that $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ becomes an Hermitian $\mathcal{H}$-module with $h = h(\varpi)$ if we let Z act by $Z_{1,m}$, and R_+ and R_- act by $\Xi + 2i\Upsilon_0$ and $\Xi - 2i\Upsilon_0$, respectively. If $\varpi = 0$, then $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}} \cong \mathbf{M}_0$ as an $\mathcal{H}$-module, so assume henceforth that $\varpi \neq 0$. Recall that $h(\varpi) = 2c(\varpi) + d(d-2)$, an integer. Since $d(d-2) \geq 0$, it follows from Lemma 5.7 that $2\eta^2 < h(\varpi)$ whenever $i\eta$ is an eigenvalue of $Z_{1,m}$ on $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$. Thus every $\mathcal{H}$-submodule of $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ is regular. The eigenvalues of $Z_{1,m}$ on V_ϖ are all integral multiples of i and so $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ is also a commensurable $\mathcal{H}$-module. Thus every $\mathcal{H}$-submodule of $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ is commensurable. It follows from these observations and Theorem 5.6 that $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ is isomorphic to a direct sum of standard $\mathcal{H}$-modules. By comparing the spectrum of $Z_{1,m}$ on $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ as stated in Lemma 5.1 with the spectrum of $Z_{1,m}$ on a standard $\mathcal{H}$-module, we conclude that this direct sum can only have one term. This demonstrates the following.

COROLLARY 5.8

Suppose that $m \geq 4$. Let V_ϖ be an irreducible $\mathfrak{k}$-module such that $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}} \neq \{0\}$, and let $\dim(V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}) = n+1$. Then $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}} \cong \mathbf{M}_n$ as an $\mathcal{H}$-module with $h = h(\varpi)$.

As we remarked previously, if Corollary 5.8 were our only goal then the argument here could be shortened somewhat.

LEMMA 5.9

We have

$$\Upsilon_z \otimes 1 = \big(\omega_0 + (z + z_0)T\big) \otimes 1$$

in $\mathcal{M}(d\chi)$.

Proof

Recall that $\omega_0 = \sum_{j=1}^d Y_j X_j$. Since $Y_j = Z_{j+1,m} + \bar{X}_j$, $X_j = Z_{1,j+1} + \bar{Y}_j$, $d\chi(\bar{X}_j) = d\chi(\bar{Y}_j) = 0$, and $[X_j, \bar{X}_j] = 0$, we have

$$\begin{aligned}
\omega_0 \otimes 1 &= \sum_{j=1}^d Y_j X_j \otimes 1 \\
&= \sum_{j=1}^d (Z_{j+1,m} + \bar{X}_j) X_j \otimes 1 \\
&= \sum_{j=1}^d Z_{j+1,m} X_j \otimes 1 \\
&= \sum_{j=1}^d Z_{j+1,m} Z_{1,j+1} \otimes 1 \\
&= \Upsilon \otimes 1
\end{aligned}$$

in $\mathcal{M}(d\chi)$. We also have $T = Z_{1,m} + \bar{T}$ and $d\chi(\bar{T}) = 0$, so that $T \otimes 1 = Z_{1,m} \otimes 1$ in $\mathcal{M}(d\chi)$. The claim follows from these identities. $\square$

We are now ready to apply the results of [7] to get information about the K-finite solutions to the Heisenberg ultrahyperbolic equation. Although it was assumed in [7] that the character χ was real-valued, this was only used to ensure that χ was trivial on $(K \cap L)^\circ$. This property holds in the present case also, and hence the results of [7] are applicable here without modification even when $z \notin \mathbb{R}$. To apply the results of [7] to a conformally invariant system, we must find elements of $\mathcal{U}(\mathfrak{k})$ that map to the operators in the system under the composition

$$\mathcal{U}(\mathfrak{k}) \to \mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{k}\cap\mathfrak{l})} \mathbb{C} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\chi} \to \mathcal{U}(\mathfrak{n}) \to \mathbb{D}[N].$$

Lemma 5.9 and (3.1) show that $\Upsilon_z \in \mathcal{U}(\mathfrak{k})$ corresponds to the ultrahyperbolic operator $\square_z$ in this sense.

Let V_ϖ be an irreducible representation of K with $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}} \neq \{0\}$. The group $K \cap L$ acts on $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ and we have

$$V_\varpi^{\mathfrak{k}\cap\mathfrak{l}} = \bigoplus_\eta V_\varpi^{(K\cap L,\eta)},$$

where the summands are the $(K \cap L)$-eigenspaces and the sum is over the characters of the component group of $K \cap L$. If η is such a character, then we define

$$\mathbb{M}_\eta(\varpi) = \{v \in V_\varpi^{(K\cap L,\eta)} \mid \Upsilon_{\bar{z}} v = 0\},$$

where $\bar{z}$ denotes the complex conjugate of z. Let $\chi = \chi(z, \varepsilon_1, \varepsilon_2)$. Then [7, Theorem 2.6] states that, as representations of K, we have

$$\mathrm{HC}\big(\Gamma(\mathcal{L}_\chi)^{\square_z}\big) \cong \bigoplus_\varpi V_\varpi \otimes \overline{\mathbb{M}_\chi(\varpi)},$$

where the left-hand side denotes the Harish-Chandra module underlying the smooth representation of G on the space $\Gamma(\mathcal{L}_\chi)^{\square_z}$ of solutions to the Heisenberg ultrahyperbolic equation, the sum is over the weights identified in Lemma 5.1, and the second factor in the tensor product on the right-hand side is the complex-conjugate vector space to $\mathbb{M}_\chi(\varpi)$. Concretely, if $\xi_1 \otimes \xi_2 \in V_\varpi \otimes \overline{\mathbb{M}_\chi(\varpi)}$ and we fix a K-invariant Hermitian form $\langle\cdot,\cdot\rangle_\varpi$ on V_ϖ, then the matrix coefficient

$$\psi_\varpi(\xi_1, \xi_2)(k) = \langle \xi_1, k\xi_2 \rangle_\varpi$$

is an element of $\Gamma(\mathcal{L}_\chi)^{\square_z}$. Here we are identifying the space $K/(K \cap L)$ with the space $G/\bar{Q}$ via the map induced by the inclusion $K \to G$, and correspondingly regarding $\mathcal{L}_\chi$ as a line bundle over $K/(K \cap L)$. It will be convenient also to consider the space

$$\mathbb{M}(\varpi) = \{v \in V_\varpi^{\mathfrak{k}\cap\mathfrak{l}} \mid \Upsilon_{\bar{z}} v = 0\},$$

which is the direct sum of the various spaces $\mathbb{M}_\eta(\varpi)$. Since every character of the component group of $K \cap L$ is the restriction to $K \cap L$ of one of the characters

$\chi(z,\varepsilon_1,\varepsilon_2)$, we have

$$\bigoplus_{\varepsilon_1,\varepsilon_2} \mathrm{HC}\big(\Gamma(\mathcal{L}_{\chi(z,\varepsilon_1,\varepsilon_2)})^{\square_z}\big) \cong \bigoplus_{\varpi} V_\varpi \otimes \overline{\mathbb{M}(\varpi)}$$

as representations of K.

To determine the space $\mathbb{M}(\varpi)$, we have to consider a certain tridiagonal determinant. It is convenient to introduce a succinct notation for tridiagonal matrices, and so we write

$$\begin{bmatrix} & b_1 & & b_2 & \cdots & b_{n-2} & & b_{n-1} & \\ a_1 & & a_2 & & \cdots & & a_{n-1} & & a_n \\ & c_1 & & c_2 & \cdots & c_{n-2} & & c_{n-1} & \end{bmatrix}$$

for the n-by-n tridiagonal matrix with $a_1,\dots,a_n$ on the diagonal, $b_1,\dots,b_{n-1}$ on the superdiagonal, and $c_1,\dots,c_{n-1}$ on the subdiagonal. For $n\geq 0$, we define an $(n+1)$-by-$(n+1)$ tridiagonal matrix $S_n(r,s,w)$ by

$$S_n(r,s,w)$$
$$= \begin{bmatrix} & u_1(r,s) & & u_2(r,s) & & \cdots & & u_{n-1}(r,s) & & u_n(r,s) & \\ nw & & (n-2)w & & (n-4)w & \cdots & -(n-4)w & & -(n-2)w & & -nw \\ & -u_n(r,s) & & -u_{n-1}(r,s) & & \cdots & & -u_2(r,s) & & -u_1(r,s) & \end{bmatrix},$$

where

$$u_k(r,s) = kr + \binom{k}{2} s$$

for $1\leq k\leq n$. In the following result, we shall evaluate $\det S_n(r,s,w)$ and find it to be factorizable; that is, the determinant is a product of linear forms in r, s, and w. The first significant example of such a factorizable tridiagonal determinant is that published without proof by Sylvester [17] in 1854. Askey [1] has developed a theory of such determinants based upon families of polynomials orthogonal with respect to a measure of finite support. The determinant $\det S_n(r,s,w)$ does not seem to fall immediately into a family covered by Askey's theory, although it is somewhat reminiscent of the determinant that Askey associates to the dual Hahn polynomials. Rather than attempting to manipulate the present determinant into a form to which Askey's theory applies, we give a direct proof of the required evaluation. It is based on the following elementary identity, to which many other factorizable tridiagonal determinants may also be reduced.

LEMMA 5.10

For $n\geq 0$, we have

$$\det \begin{bmatrix} & a_1 & & \cdots & & a_n & \\ 0 & & 0 & \cdots & 0 & & 0 \\ & b_1 & & \cdots & & b_n & \end{bmatrix}$$
$$= \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{(n+1)/2} \prod_{j=0}^{(n-1)/2} a_{2j+1} b_{2j+1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof
By using Laplace's expansion in the first row and then in the first column, the identity is easily proved by induction. □

PROPOSITION 5.11
If n is even, then $\det S_n(r,s,w)=0$. *If n is odd, then*

$$\det S_n(r,s,w)=(-1)^{(n+1)/2}C_n^2\prod_{j=0}^{(n-1)/2}\big(w^2-(r+js)^2\big),$$

where

$$C_n=\frac{n!}{2^{(n-1)/2}\big(\frac{n-1}{2}\big)!}.$$

Proof
Let $\mathcal{P}_n$ be the space of homogeneous polynomials of degree n in the variables z_1 and z_2, and take $z_1^n, z_1^{n-1}z_2,\dots,z_1z_2^{n-1},z_2^n$ as the standard basis of $\mathcal{P}_n$. With respect to the standard basis, the matrix of the operator $z_1\frac{\partial}{\partial z_1}-z_2\frac{\partial}{\partial z_2}$ is $\mathrm{diag}(n, n-2,\dots,-(n-2),-n)$, the matrix of the operator $z_1\frac{\partial}{\partial z_2}-z_2\frac{\partial}{\partial z_1}$ is

$$\begin{bmatrix} & 1 & & 2 & \cdots & & n & \\ 0 & & 0 & & \cdots & 0 & & 0 \\ & -n & & -(n-1) & \cdots & & -1 & \end{bmatrix},$$

and the matrix of the operator $(1/2)z_1z_2\big(\frac{\partial^2}{\partial z_2^2}-\frac{\partial^2}{\partial z_1^2}\big)$ is

$$\begin{bmatrix} & \binom{1}{2} & & \binom{2}{2} & \cdots & & \binom{n}{2} & \\ 0 & & 0 & & \cdots & 0 & & 0 \\ & -\binom{n}{2} & & -\binom{n-1}{2} & \cdots & & -\binom{1}{2} & \end{bmatrix}.$$

Thus $S_n(r,s,w)$ is the matrix of the operator

$$D=\frac{1}{2}sz_1z_2\Big(\frac{\partial^2}{\partial z_2^2}-\frac{\partial^2}{\partial z_1^2}\Big)+(wz_1-rz_2)\frac{\partial}{\partial z_1}+(rz_1-wz_2)\frac{\partial}{\partial z_2}$$

with respect to the standard basis. We now introduce new variables ζ_1 and ζ_2 by $z_1=\zeta_1+\zeta_2$ and $z_2=\zeta_1-\zeta_2$. Then $\zeta_1=(z_1+z_2)/2$ and $\zeta_2=(z_1-z_2)/2$ and so

$$\frac{\partial}{\partial z_1}=\frac{1}{2}\Big(\frac{\partial}{\partial\zeta_1}+\frac{\partial}{\partial\zeta_2}\Big),$$
$$\frac{\partial}{\partial z_2}=\frac{1}{2}\Big(\frac{\partial}{\partial\zeta_1}-\frac{\partial}{\partial\zeta_2}\Big),$$
$$\frac{\partial^2}{\partial z_2^2}-\frac{\partial^2}{\partial z_1^2}=-\frac{\partial^2}{\partial\zeta_1\,\partial\zeta_2}.$$

After some calculation, these identities show that

$$D=\frac{1}{2}s(\zeta_2^2-\zeta_1^2)\frac{\partial^2}{\partial\zeta_1\,\partial\zeta_2}+(w+r)\zeta_2\frac{\partial}{\partial\zeta_1}+(w-r)\zeta_1\frac{\partial}{\partial\zeta_2}$$

in the new variables. The matrix of D with respect to the basis $\zeta_1^n, \zeta_1^{n-1}\zeta_2, \ldots, \zeta_1\zeta_2^{n-1}, \zeta_2^n$ is

$$\begin{bmatrix} & a_1 & & \cdots & & a_n & \\ 0 & & 0 & \cdots & 0 & & 0 \\ & b_1 & & \cdots & & b_n & \end{bmatrix},$$

where $a_j = j(w - r - ((n-j)/2)s)$ and $b_j = (n-j+1)(w + r + ((j-1)/2)s)$. The evaluation now follows from Lemma 5.10 on noting that, when n is odd, C_n is simply the product of the odd integers from 1 up to n. $\square$

THEOREM 5.12

Let $m \geq 4$, and let $\varpi = a\lambda_1 + b\lambda_2$ be one of the weights described in Lemma 5.1. Then $\dim \mathbb{M}(\varpi) \leq 1$. We have $\dim \mathbb{M}(\varpi) = 1$ if and only if either $a - |b|$ is even or $a - |b|$ is odd and $z = \pm(z_0 + |b| + 2j)$ with $0 \leq j \leq (a - |b| - 1)/2$.

Proof

We are required to determine the null space of the operator $\Upsilon_{\bar{z}}$ in $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$. The Casimir eigenvalue associated with ϖ is

$$c(\varpi) = (\varpi, \varpi + 2\rho) = a^2 + ad + b^2 + bd - 2b$$

and so $V_\varpi^{\mathfrak{k}\cap\mathfrak{l}}$ is an $\mathcal{H}$-module with

$$h = h(\varpi) = 2c(\varpi) + d(d-2).$$

By Corollary 5.8, this $\mathcal{H}$-module is isomorphic to $\mathbf{M}_n$ with $n = a - |b|$. (The absolute value sign is only necessary when $m = 4$.) Thus this space has an ordered basis $v_0, \ldots, v_n$ such that $Z_{1,m}v_k = i(n-2k)v_k$ for $0 \leq k \leq n$, $R_- v_k = v_{k+1}$ for $0 \leq k \leq n-1$, $R_- v_n = 0$, $R_+ v_0 = 0$, and $R_+ v_k = b_k v_{k-1}$ for $1 \leq k \leq n$, where

$$b_k = 4k(n-k+1)\big(h(\varpi) - 2n^2 + 4(k-1)(n-k)\big).$$

We have

$$4i\Upsilon_{\bar{z}} = R_+ - R_- - 2iwZ_{1,m},$$

where we have written $w = -2\bar{z}$. The matrix of this operator with respect to the basis $v_0, \ldots, v_n$ is

$$A = \begin{bmatrix} & b_1 & & \cdots & & b_n & \\ 2nw & & 2(n-2)w & \cdots & -2(n-2)w & & -2nw \\ & -1 & & \cdots & & -1 & \end{bmatrix}.$$

It is evident that the last n rows of this matrix are linearly independent, and so the nullity of A is at most 1. This establishes the first claim.

The case where $m = 4$ and $b < 0$ requires separate treatment, so let us assume for the moment that either $m \geq 5$ or $m = 4$ and $b \geq 0$. We substitute the value of $h(\varpi)$ and $n = a - b$ into the above expression for b_k to find that

$$b_k = 4k(n-k+1)(d + 2b + 2k - 2)(d + 2a - 2k).$$

This may also be written as $b_k = 4u_k u_{n-k+1}$ with $u_k = k(d+2b+2k-2)$. In the exceptional case that $m=4$ and $b<0$ we have $n=a+b$ and the corresponding factorization turns out to be $b_k = 4u_k u_{n-k+1}$ with $u_k = k(2|b|+2k)$. Since $d=2$ in this case, we may express everything uniformly by writing $b_k = 4u_k u_{n-k+1}$ with $u_k = k(d+2|b|+2k-2)$. Note that u_k can never be zero. Let

$$g = \operatorname{diag}(1, 2u_n, 2^2 u_n u_{n-1}, \ldots, 2^n u_n u_{n-1} \cdots u_2 u_1).$$

One checks that

$$\frac{1}{2} g A g^{-1} = \begin{bmatrix} & u_1 & & \cdots & & u_n & \\ nw & & (n-2)w & \cdots & -(n-2)w & & -nw \\ & -u_n & & \cdots & & -u_1 & \end{bmatrix},$$

and this is the matrix $S_n(d+2|b|, 4, w)$ in the notation that we introduced above. Thus the nullity of A is 1 precisely when the determinant of $S_n(d+2|b|, 4, w)$ vanishes. By Proposition 5.11, this always occurs when $n = a-|b|$ is even, and it occurs when $n = a-|b|$ is odd precisely when $w = \pm(d+2|b|+4j)$ for some $0 \le j \le (a-|b|-1)/2$. Since $w = -2\bar{z}$ and $z_0 = d/2$, this last condition is equivalent to the one presented in the statement. $\square$

In Theorem 5.12 it was assumed that $m \ge 4$. We now deal with the remaining case, when $m=3$, by using the same techniques. However, the answer is different, and the difference arises because, as we see in the proof of Theorem 5.13, Corollary 5.8 does not extend to $m=3$.

THEOREM 5.13

Let $m=3$, and let $a \in \mathbb{N}$. Then $1 \le \dim \mathbb{M}(a\lambda_1) \le 2$. The value of $\dim \mathbb{M}(a\lambda_1)$ is 2 if a is odd and $z = \pm(1/2+2j)$ with $0 \le j \le (a-1)/2$ or if a is even and $z = \pm(3/2+2j)$ with $0 \le j \le (a-2)/2$. Otherwise, the value of $\dim \mathbb{M}(a\lambda_1)$ is 1.

Proof

The irreducible representations of K have highest weights $a\lambda_1$ with $a \in \mathbb{N}$, and $V_{a\lambda_1}^{\mathfrak{k}\cap\mathfrak{l}} = V_{a\lambda_1}$ for all a, since $\mathfrak{k}\cap\mathfrak{l} = \{0\}$. We have $c(a\lambda_1) = a(a+1)$ and so $h(a\lambda_1) = 2a^2+2a-1$. The spectrum of $Z_{1,3}$ on $V_{a\lambda_1}$ is $\{ik \mid -a \le k \le a\}$. Provided that $a \ge 1$, we have $h(a\lambda_1) > 2a^2$, and so $V_{a\lambda_1}$ is a direct sum of regular, commensurable $\mathcal{H}$-modules. From Theorem 5.6 and the nature of the spectrum of $Z_{1,3}$ on $V_{a\lambda_1}$, it follows that $V_{a\lambda_1} \cong \mathbf{M}_{a-1} \oplus \mathbf{M}_a$ as $\mathcal{H}$-modules when $a \ge 1$. Moreover, it is clear that $V_0 \cong \mathbf{M}_0$ as an $\mathcal{H}$-module. It now suffices to determine the null space of $\Upsilon_{\bar{z}}$ on $\mathbf{M}_a$ and on $\mathbf{M}_{a-1}$ with $h = h(a\lambda_1)$. Note that $\Upsilon_{\bar{z}} = R_+ - R_- - 2iwZ_{1,3}$ with $w = -2\bar{z}$. The matrix A of this operator with respect to the standard basis of $\mathbf{M}_n$ is the same as in the proof of Theorem 5.12, and it follows that the null space has dimension either 0 or 1 for all n. To decide which, we must consider the cases $n=a$ and $n=a-1$ separately. When $n=a$, a calculation shows that

$$b_k = 4k(n-k+1)\big(h(a\lambda_1) - 2n^2 + 4(k-1)(n-k)\big)$$

factors as $b_k = 4u_k u_{a-k+1}$ with $u_k = k(2k-1) = k + 4\binom{k}{2}$. Thus $(1/2)A$ is conjugate to the matrix $S_a(1,4,w)$. By Proposition 5.11, this matrix is singular when a is even, or when a is odd and $w = \pm(1+4j)$ with $0 \le j \le (a-1)/2$. The latter condition is equivalent to $z = \pm(1/2+2j)$ with $0 \le j \le (a-1)/2$. When $n = a-1$, b_k factors as $b_k = 4u_k u_{a-k}$ with $u_k = k(2k+1) = 3k + 4\binom{k}{2}$. Thus $(1/2)A$ is conjugate to the matrix $S_{a-1}(3,4,w)$. By Proposition 5.11, this matrix is singular when a is odd, or when a is even and $w = \pm(3+4j)$ with $0 \le j \le (a-2)/2$. The latter condition is equivalent to $z = \pm(3/2+2j)$ with $0 \le j \le (a-2)/2$. These observations combine to verify the statement. $\square$

Note that, in light of the discussion following Lemma 5.9, Theorems 5.12 and 5.13 complete the determination of the K-finite solution space $\mathrm{HC}(\Gamma(\mathcal{L}_\chi)^{\square_z})$ as a representation of K.

6. An analogue of Fritz John's theorem

The aim of this section is to obtain a result that may be regarded as an analogue of Fritz John's famous result [6, Theorem 6] identifying the solution space of the Euclidean ultrahyperbolic equation with the image of a certain integral transform. To avoid complications of detail, we restrict to the case $m \ge 5$ for this result, although much of the preliminary work is carried out for $m \ge 3$.

Let R (respectively, $\bar{R}$) be the standard block upper-triangular subgroup (respectively, block lower-triangular subgroup) of G with blocks of sizes $(d,2)$, let $H = R \cap \bar{R}$, and let U (respectively, $\bar{U}$) be the unipotent radical of R (respectively, $\bar{R}$). For $A \in \mathrm{Mat}(d,2)$ and $B \in \mathrm{Mat}(2,d)$, we let

$$u(A) = \begin{pmatrix} I_d & A \\ 0 & I_2 \end{pmatrix}$$

and

$$\bar{u}(B) = \begin{pmatrix} I_d & 0 \\ B & I_2 \end{pmatrix}.$$

These maps provide coordinates on U and $\bar{U}$, respectively. For $s \in \mathbb{C}$ and $\varepsilon \in \{\pm\}$, we define an analytic character $\nu(s,\varepsilon) : H \to \mathbb{C}^\times$ by

$$\nu(s,\varepsilon)\big(\mathrm{diag}(h_1,h_2)\big) = |\det(h_2)|_\varepsilon^s,$$

and regard $\nu(s,\varepsilon)$ as a character of $\bar{R}$ by extending it to be trivial on $\bar{U}$. There is a homogeneous line bundle $\mathcal{E}_\nu \to G/\bar{R}$ associated to $\nu = \nu(s,\varepsilon)$ such that the space of smooth sections $\Gamma(\mathcal{E}_\nu)$ may be identified with the space of smooth functions $f : G \to \mathbb{C}$ such that $f(g\bar{r}) = \nu(\bar{r})f(g)$ for $g \in G$ and $\bar{r} \in \bar{R}$. With the left-translation action of G, $\Gamma(\mathcal{E}_\nu)$ is a model of the smooth induced representation $\mathrm{Ind}(G,\bar{R},\nu^{-1})$. We denote this representation of G by σ_ν and the corresponding derived representation of $\mathfrak{g}$ by Σ_s.

It is necessary to identify two specific K-types in $\Gamma(\mathcal{E}_\nu)$ explicitly. To this end, we define a function $M : G \to \mathrm{Mat}(m,2)$ by

$$M(g) = g\begin{pmatrix} 0_d \\ I_2 \end{pmatrix}.$$

It is apparent from the definition that $M(g_1 g_2) = g_1 M(g_2)$ for all $g_1, g_2 \in G$, and it is easy to check that $M(g\bar{r}) = M(g)r_2$ when

$$\bar{r} = \begin{pmatrix} r_1 & 0 \\ A & r_2 \end{pmatrix} \in \bar{R}.$$

For $s \in \mathbb{C}$ let $F_s : G \to \mathbb{C}$ be

$$F_s(g) = \left|\det\bigl(M(g)^\top M(g)\bigr)\right|^{s/2}.$$

For $Z \in \mathfrak{k}$, let $\psi(\cdot, Z) : G \to \mathbb{C}$ be

$$\psi(g, Z) = \mathrm{tr}\bigl(M(g)^\top Z M(g) J\bigr),$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and for $s \in \mathbb{C}$ let $\psi_s(\cdot, Z) : G \to \mathbb{C}$ be

$$\psi_s(g, Z) = \psi(g, Z) F_{s-1}(g).$$

LEMMA 6.1

For all $s \in \mathbb{C}$, $F_s \in \Gamma(\mathcal{E}_{\nu(s,+)})$ and, for all $s \in \mathbb{C}$ and $Z \in \mathfrak{k}$, $\psi_s(\cdot, Z) \in \Gamma(\mathcal{E}_{\nu(s,-)})$. The function F_s is K-invariant and we have

$$\sigma_{\nu(s,-)}(k)\psi_s(\cdot, Z) = \psi_s\bigl(\cdot, \mathrm{Ad}(k)Z\bigr)$$

for all $k \in K$. The map $Z \mapsto \psi_s(\cdot, Z)$ embeds the adjoint representation of K into $\Gamma(\mathcal{E}_{\nu(s,-)})$.

Proof

The statements about F_s are well known. Granting them, it is clear that $\psi_s(\cdot, Z)$ is smooth. For $\bar{r} \in \bar{R}$ as above, we have

$$\begin{aligned}
\psi_s(g\bar{r}, Z) &= \mathrm{tr}\bigl(r_2^\top M(g)^\top Z M(g) r_2 J\bigr) F_{s-1}(g\bar{r}) \\
&= \mathrm{tr}\bigl(M(g)^\top Z M(g) r_2 J r_2^\top\bigr) F_{s-1}(g\bar{r}) \\
&= \det(r_2)|\det(r_2)|^{s-1}\,\mathrm{tr}\bigl(M(g)^\top Z M(g) J\bigr) F_{s-1}(g) \\
&= |\det(r_2)|_-^s \psi_s(g, Z),
\end{aligned}$$

where we have used the identity $r_2 J r_2^\top = \det(r_2) J$ from the second line to the third. It is easy to check the K-equivariance of $\psi_s(\cdot, Z)$. The last thing to verify is that $\psi_s(\cdot, Z)$ is not identically zero. This may be done by computing that $\psi_s(e, Z_{m-1,m}) = -2$ for all $s \in \mathbb{C}$. (Note that this is not quite sufficient when $m = 4$, since the adjoint representation of K is reducible in this case, but it is easy to verify directly that neither summand lies in the kernel of the map.) $\square$

Let

$$w_1 = \begin{pmatrix} 0 & I_{m-1} \\ (-1)^{m+1} & 0 \end{pmatrix}.$$

Then $w_1 \in K$ and

$$w_1^{-1} = \begin{pmatrix} 0 & (-1)^{m+1} \\ I_{m-1} & 0 \end{pmatrix}.$$

For $f \in \Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$, let $Tf : G \to \mathbb{C}$ be defined by

$$(Tf)(g) = \int_{\mathbb{R}^d} f\big(gw_1^{-1}u(0,\xi)\big)\, d\mu(\xi),$$

where μ denotes the Lebesgue measure on $\mathbb{R}^d$, when this integral converges.

LEMMA 6.2

Suppose that $\mathrm{re}(s) < -d$, *and let* $f \in \Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$. *Then the integral defining* $(Tf)(g)$ *converges absolutely and locally uniformly in* g. *We have*

$$(TF_s)(e) = \pi^{d/2}\frac{\Gamma(-(s+d)/2)}{\Gamma(-s/2)}.$$

In addition, if we let $Z = \mathrm{Ad}(w_1^{-1})Z_{m-1,m}$, *then*

$$\big(T\psi_s(\cdot, Z)\big)(e) = -2\pi^{d/2}\frac{\Gamma(-(s-1+d)/2)}{\Gamma(-(s-1)/2)}.$$

Proof

Since $w_1 \in K$,

$$(TF_s)(e) = \int_{\mathbb{R}^d} F_s\big(u(0,\xi)\big)\, d\mu(\xi)$$

and a brief calculation reveals that this is

$$(TF_s)(e) = \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^{s/2}\, d\mu(\xi).$$

In spherical coordinates, it becomes

$$(TF_s)(e) = \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \int_0^\infty (1+\rho^2)^{s/2}\rho^{d-1}\, d\rho$$

and the substitution $\tau = 1/(1+\rho^2)$ reduces the integral in this expression to Euler's beta integral, and hence yields the given evaluation. It is well known that if $f \in \Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$, then $|f|$ is bounded, locally uniformly in g, by a multiple of F_s. This gives the convergence statement. With Z as in the statement, we have

$$\begin{aligned}
\psi_s\big(w_1^{-1}u(0,\xi), Z\big) &= \big(\sigma_{\nu(s,-)}(w_1)\psi_s\big)\big(u(0,\xi), Z\big) \\
&= \psi_s\big(u(0,\xi), \mathrm{Ad}(w_1)Z\big) \\
&= \psi_s\big(u(0,\xi), Z_{m-1,m}\big) \\
&= \psi\big(u(0,\xi), Z_{m-1,m}\big)F_{s-1}\big(u(0,\xi)\big).
\end{aligned}$$

One finds that $\psi(u(0,\xi), Z_{m-1,m}) = -2$ for all ξ, and this gives the remaining statement. □

LEMMA 6.3

Suppose that $\mathrm{re}(s) < -d$, *and let* $f \in \Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$. *Then* $Tf \in \Gamma(\mathcal{L}_{\chi(z,\varepsilon,\varepsilon)})$ *with* $z = s+1+d/2$. *The map* $T : \Gamma(\mathcal{E}_{\nu(s,\varepsilon)}) \to \Gamma(\mathcal{L}_{\chi(z,\varepsilon,\varepsilon)})$ *is* G*-intertwining.*

Proof
We have

$$w_1 \bar{n}(x,y,t) w_1^{-1} u(0,\xi) = u\big(0, \xi + (-1)^{m+1} y\big) \bar{r}$$

with

$$\bar{r} = \begin{pmatrix} I_d & 0 & 0 \\ x & 1 & x\xi + (-1)^{m+1} t \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\bar{n} = \bar{n}(x,y,t)$, and let $\nu = \nu(s,\varepsilon)$. It follows from the above identity that

$$(Tf)(g\bar{n}) = \int_{\mathbb{R}^d} f\big(g w_1^{-1} u(0, \xi + (-1)^{m+1} y) \bar{r}\big)\, d\mu(\xi).$$

Now $\nu(\bar{r}) = 1$, and this and a change of variable in the integral shows that $(Tf)(g\bar{n}) = (Tf)(g)$. Now let $l = \mathrm{diag}(a_1, h, a_2) \in L$. We have

$$w_1 l w_1^{-1} u(0,\xi) = u(0, a_1^{-1} h\xi)\, \mathrm{diag}(h, a_2, a_1)$$

and so

$$\begin{aligned}
(Tf)(gl) &= \int_{\mathbb{R}^d} f\big(g w_1^{-1} u(0, a_1^{-1} h\xi)\, \mathrm{diag}(h, a_2, a_1)\big)\, d\mu(\xi) \\
&= |a_1 a_2|_\varepsilon^s \int_{\mathbb{R}^d} f\big(g w_1^{-1} u(0, a_1^{-1} h\xi)\big)\, d\mu(\xi) \\
&= |a_1 a_2|_\varepsilon^s |a_1|^d |\det(h)|^{-1} \int_{\mathbb{R}^d} f\big(g w_1^{-1} u(0,\xi)\big)\, d\mu(\xi) \\
&= |a_1 a_2|_\varepsilon^s |a_1|^d |a_1 a_2| (Tf)(g) \\
&= |a_1|_\varepsilon^{s+d+1} |a_2|_\varepsilon^{s+1} (Tf)(g) \\
&= \chi(z,\varepsilon,\varepsilon)(l)(Tf)(g)
\end{aligned}$$

with $z = s+1+d/2$. These calculations verify that Tf has the correct transformation on the right under $\bar{Q}$ to define an element of $\Gamma(\mathcal{L}_{\chi(z,\varepsilon,\varepsilon)})$ and the G-intertwining property is immediate from the definition of T. The only remaining point is the smoothness of Tf. However, the convergence statement in Lemma 6.2 and the usual argument based upon the invariance of $\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$ under $\Sigma_s(\mathfrak{g})$ imply that Tf is indeed smooth. □

Given Lemma 6.3, we may consider the dependence of Tf on s by looking at the restriction of the functions f and Tf to K as usual. From this perspective, the proof of Lemma 6.2 implies that the convergence of the integral defining Tf is also locally uniform in s. Our next task is to extend the operator T to spaces $\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$ for which s does not necessarily satisfy the condition $\mathrm{re}(s) < -d$. As a preliminary to this, we must find explicit expressions for certain of the operators $\Sigma_s(X)$ with $X \in \mathfrak{g}$. We write these operators as elements of $\mathbb{D}[U]$ by making use of the coordinates $\zeta_1, \ldots, \zeta_d$, $\xi_1, \ldots, \xi_d$ on $U = \{u(\zeta, \xi) \mid \zeta, \xi \in \mathrm{Mat}(d,1)\}$.

LEMMA 6.4

Let $B = \left(\begin{smallmatrix} b_1 & \cdots & b_d \\ c_1 & \cdots & c_d \end{smallmatrix}\right) \in \mathrm{Mat}(2,d)$, *and let* $X = \left(\begin{smallmatrix} 0 & 0 \\ B & 0 \end{smallmatrix}\right) \in \bar{\mathfrak{u}}$. *Then*

$$\Sigma_s(X) = -s(b\zeta + c\xi) + \sum_{j=1}^{d} \Big(\big((b\zeta)\zeta_j + (c\zeta)\xi_j\big) \frac{\partial}{\partial \zeta_j} + \big((b\xi)\zeta_j + (c\xi)\xi_j\big) \frac{\partial}{\partial \xi_j} \Big).$$

Proof

Let $A = (\zeta, \xi) \in \mathrm{Mat}(d,2)$. Then

$$\begin{aligned} e^{-\tau X} u(A) &= \begin{pmatrix} I & 0 \\ -\tau B & I \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & A(I - \tau BA)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \star & 0 \\ -\tau B & I - \tau BA \end{pmatrix}, \end{aligned}$$

and so if $f \in \Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$, then

$$f\big(e^{-\tau X} u(A)\big) = |\det(I - \tau BA)|_{\varepsilon}^{s} f\big(u(A(I - \tau BA)^{-1})\big).$$

Thus

$$\begin{aligned} \big(\Sigma_s(X) \bullet f\big)\big(u(A)\big) &= \frac{d}{d\tau}\Big|_{\tau=0} |\det(I - \tau BA)|_{\varepsilon}^{s} f\big(u(A(I - \tau BA)^{-1})\big) \\ &= -s\,\mathrm{tr}(BA) f\big(u(A)\big) + \frac{d}{d\tau}\Big|_{\tau=0} f\big(u(A(I - \tau BA)^{-1})\big). \end{aligned}$$

We have

$$A(I - \tau BA)^{-1} = \frac{1}{\det(I - \tau BA)} \big(\zeta + \tau((c\zeta)\xi - (c\xi)\zeta), \xi + \tau((b\xi)\zeta - (b\zeta)\xi)\big)$$

and it follows that

$$\begin{aligned} &\frac{d}{d\tau}\Big|_{\tau=0} f\big(u(A(I - \tau BA)^{-1})\big) \\ &\quad = \sum_{j=1}^{d} \Big(\big(\mathrm{tr}(BA)\zeta_j + (c\zeta)\xi_j - (c\xi)\zeta_j\big) \frac{\partial f}{\partial \zeta_j} \\ &\qquad + \big(\mathrm{tr}(BA)\xi_j + (b\xi)\zeta_j - (b\zeta)\xi_j\big) \frac{\partial f}{\partial \xi_j} \Big). \end{aligned}$$

Now $\mathrm{tr}(BA) = b\zeta + c\xi$, and by combining this with the above expressions, the desired evaluation is obtained. □

For $f \in \Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$ and $z \in \mathbb{C}$ with $\mathrm{re}(z) < -d - \mathrm{re}(s)$, we define

$$\Phi(z,f) = \int_{\mathbb{R}^d} f\big(u(0,\xi)\big) F_z\big(u(0,\xi)\big)\, d\mu(\xi).$$

Note that $fF_z \in \Gamma(\mathcal{E}_{\nu(s+z,\varepsilon)})$ and $\Phi(z,f) = T(fF_z)(w_1)$, so that the convergence of this integral is guaranteed by Lemma 6.2. If $\mathrm{re}(s) < -d$, then we may take $z = 0$, and the expression

$$T(f)(g) = \Phi\big(0, \sigma_{\nu(s,\varepsilon)}(w_1 g^{-1}) f\big) \tag{6.1}$$

allows us to recover T from Φ. For issues of continuity, we consider $\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$ with the usual smooth topology.

PROPOSITION 6.5

Let $s \in \mathbb{C}$. Then there are $u_s \in \mathcal{U}(\mathfrak{g})$ and polynomials $p_0(z,s)$, $p_2(z,s)$, and $p_4(z,s)$ such that

$$\begin{aligned}&(z+s+d+1)(z+s+d+2)\Phi(z+2,f)\\ &\qquad = \Phi\big(z, \Sigma_s(u_s)f\big) + p_0(z,s)\Phi(z,f) + p_2(z,s)\Phi(z-2,f) + p_4(z,s)\Phi(z-4,f)\end{aligned}$$

for all $f \in \Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$ and all $z \in \mathbb{C}$ with $\mathrm{re}(z+2) < -d - \mathrm{re}(s)$. Let

$$\gamma(z) = \frac{1}{\Gamma\big(-\frac{z+s+d}{2}\big)\Gamma\big(-\frac{z+s-1+d}{2}\big)}.$$

The map $(z,f) \mapsto \gamma(z)\Phi(z,f)$ extends to an entire family of continuous functionals on $\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$.

Proof

By taking $b = 0$ and $c = e_l$ in Lemma 6.4, we find that the operator

$$D_l = -s\xi_l + \sum_{j=1}^{d} \Big(\zeta_l \xi_j \frac{\partial}{\partial \zeta_j} + \xi_l \xi_j \frac{\partial}{\partial \xi_j}\Big)$$

lies in $\Sigma_s(\mathfrak{g})$ for all $1 \le l \le d$. Thus the operator

$$D = \sum_{l=1}^{d} D_l^2$$

lies in $\Sigma_s(\mathcal{U}(\mathfrak{g}))$ and so preserves the space $\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$. Let

$$\mathbb{E}_\xi = \sum_{j=1}^{d} \xi_j \frac{\partial}{\partial \xi_j}$$

be the Euler operator with respect to ξ. For $P \in \mathbb{D}[U]$, let $\tilde{P} \in \mathbb{D}[\mathbb{R}^d]$ be the pullback of P with respect to the map $\xi \mapsto u(0,\xi)$. By inspection of the operators D_l and D, we find that $\tilde{D}_l = \xi_l(\mathbb{E}_\xi - s)$ and

$$\tilde{D} = \sum_{l=1}^{d} \big(\xi_l(\mathbb{E}_\xi - s)\big)^2.$$

To see the second of these claims, note that ζ_l commutes with the second term in the sum defining D_l and so every term in D_l^2 that involves the derivative $\partial/\partial\zeta_j$ also has ζ_l as a factor on the left when written in normal order, and thus pulls back to zero. It follows that

$$\begin{aligned}\tilde{D} &= \sum_{l=1}^{d} \xi_l(\mathbb{E}_\xi - s)\xi_l(\mathbb{E}_\xi - s)\\ &= \sum_{l=1}^{d} (\mathbb{E}_\xi - s - 1)\xi_l(\mathbb{E}_\xi - s - 1)\xi_l\\ &= \sum_{l=1}^{d} (\mathbb{E}_\xi - s - 1)(\mathbb{E}_\xi - s - 2)\xi_l^2.\end{aligned}$$

It is easy to check that $\mathbb{E}_\xi^{\mathrm{adj}} = -(\mathbb{E}_\xi + d)$, where adj denotes the adjoint, and so

$$\begin{aligned}\tilde{D}^{\mathrm{adj}} &= \sum_{l=1}^{d} \xi_l^2(-\mathbb{E}_\xi - d - s - 2)(-\mathbb{E}_\xi - d - s - 1)\\ &= \sum_{l=1}^{d} \xi_l^2(\mathbb{E}_\xi + s + d + 2)(\mathbb{E}_\xi + s + d + 1).\end{aligned}$$

For brevity, we define $\tilde{F}_z$ by $\tilde{F}_z(\xi) = F_z(u(0,\xi))$ and let $\lambda = s + d + 1$. As we saw above, $\tilde{F}_z(\xi) = (1 + \|\xi\|^2)^{z/2}$ and so

$$\mathbb{E}_\xi \bullet \tilde{F}_z = z\tilde{F}_z - z\tilde{F}_{z-2}.$$

This, in turn, yields

$$\begin{aligned}&(\mathbb{E}_\xi + \lambda + 1)(\mathbb{E}_\xi + \lambda) \bullet \tilde{F}_z\\ &\qquad = (z+\lambda)(z+\lambda+1)\tilde{F}_z - z(2z + 2\lambda - 1)\tilde{F}_{z-2} + z(z-2)\tilde{F}_{z-4},\end{aligned}$$

and a further calculation, relying on the identity $\|\xi\|^2\tilde{F}_z = \tilde{F}_{z+2} - \tilde{F}_z$, then shows that

$$\begin{aligned}\tilde{D}^{\mathrm{adj}} \bullet \tilde{F}_z &= (z+\lambda)(z+\lambda+1)\tilde{F}_{z+2}\\ &\qquad - (3z^2 + 4z\lambda + \lambda^2 + \lambda)\tilde{F}_z + z(3z + 2\lambda - 3)\tilde{F}_{z-2} - z(z-2)\tilde{F}_{z-4}.\end{aligned}$$

Let us denote the coefficients of $\tilde{F}_z$, $\tilde{F}_{z-2}$, and $\tilde{F}_{z-4}$ in this expression by $-p_0(z,s)$, $-p_2(z,s)$, and $-p_4(z,s)$, respectively. Let u_s be the element of $\mathcal{U}(\mathfrak{g})$ such that $D = \Sigma_s(u_s)$, and assume for the moment that $\mathrm{re}(z)$ is sufficiently negative so as to justify the integration by parts required in the following calculation. Then

$$\begin{aligned}\Phi\big(z, \Sigma_s(u_s)f\big) &= \int_{\mathbb{R}^d} (D \bullet f)\big(u(0,\xi)\big)\tilde{F}_z(\xi)\,d\mu(\xi)\\ &= \int_{\mathbb{R}^d} \tilde{D} \bullet \big(f(u(0,\xi))\big)\tilde{F}_z(\xi)\,d\mu(\xi)\\ &= \int_{\mathbb{R}^d} f\big(u(0,\xi)\big)(\tilde{D}^{\mathrm{adj}} \bullet \tilde{F}_z)(\xi)\,d\mu(\xi),\end{aligned}$$

and, by introducing the evaluation of $\tilde{D}^{\mathrm{adj}} \bullet \tilde{F}_z$ found above, this gives

$$\Phi\big(z, \Sigma_s(u_s)f\big) = (z+\lambda)(z+\lambda+1)\Phi(z+2,f) \\ - p_0(z,s)\Phi(z,f) - p_2(z,s)\Phi(z-2,f) - p_4(z,s)\Phi(z-4,f).$$

This identity is equivalent to the stated one. It has been derived when $\mathrm{re}(z)$ is sufficiently negative, but both sides are analytic functions of z in the region $\mathrm{re}(z+2) < -d - \mathrm{re}(s)$, and so the identity holds in the entire region, as claimed.

With γ as in the statement, one verifies that

$$\gamma(z+2) = \frac{1}{4}(z+s+d+1)(z+s+d+2)\gamma(z).$$

If we let $\Phi^{\mathrm{n}}(z,f) = \gamma(z)\Phi(z,f)$ be the normalized version of Φ, then the recurrence relation that we derived above may be written as

$$4\Phi^{\mathrm{n}}(z+2,f) = \Phi^{\mathrm{n}}\big(z, \Sigma_s(u_s)f\big) + p_0(z,s)\Phi^{\mathrm{n}}(z,f) \\ + p_2(z,s)\frac{\gamma(z)}{\gamma(z-2)}\Phi^{\mathrm{n}}(z-2,f) + p_4(z,s)\frac{\gamma(z)}{\gamma(z-4)}\Phi^{\mathrm{n}}(z-4,f).$$

The quotients $\gamma(z)/\gamma(z-2)$ and $\gamma(z)/\gamma(z-4)$ are polynomials in z and s. In this form, the recurrence relation allows us to continue $\Phi^{\mathrm{n}}(z,f)$ to the entire z-plane inductively. Since $\Sigma_s(u_s)$ is a continuous operator on $\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$ and the functional is evidently continuous when $\mathrm{re}(z)$ is sufficiently negative, the continuity of the resulting extension follows. $\square$

For any $s \in \mathbb{C}$, we may now define an operator T^{n} on $\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$ by

$$T^{\mathrm{n}}(f)(g) = \Phi^{\mathrm{n}}\big(0, \sigma_{\nu(s,\varepsilon)}(w_1 g^{-1})f\big).$$

When $\mathrm{re}(s) < -d$, T^{n} is a nonzero multiple of T. In the sense that we mentioned above (via the restriction-to-K technique), T^{n} depends holomorphically on s. It follows as usual from this and Lemma 6.3 that $T^{\mathrm{n}} : \Gamma(\mathcal{E}_{\nu(s,\varepsilon)}) \to \Gamma(\mathcal{L}_{\chi(z,\varepsilon,\varepsilon)})$, where $z = s + 1 + d/2$, is an intertwining operator for all s.

THEOREM 6.6

Let $m \geq 5$. Suppose that $z \in \mathbb{C} - (z_0 + \mathbb{Z})$, and suppose that $\varepsilon \in \{\pm\}$. Let $s = z - z_0 - 1$. Then

$$\Gamma(\mathcal{L}_{\chi(z,\varepsilon,-\varepsilon)})^{\square_z} = \{0\}$$

and

$$T^{\mathrm{n}} : \Gamma(\mathcal{E}_{\nu(s,\varepsilon)}) \to \Gamma(\mathcal{L}_{\chi(z,\varepsilon,\varepsilon)})^{\square_z}$$

is an isomorphism of Fréchet spaces.

Proof

We first observe that it is sufficient to verify the statements at the level of Harish-Chandra modules. That is, it is sufficient to show that

$$\mathrm{HC}\big(\Gamma(\mathcal{L}_{\chi(z,\varepsilon,-\varepsilon)})^{\square_z}\big) = \{0\}$$

and that

$$T^{\mathrm{n}}: \mathrm{HC}\big(\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})\big) \to \mathrm{HC}\big(\Gamma(\mathcal{L}_{\chi(z,\varepsilon,\varepsilon)})^{\square_z}\big)$$

is an algebraic isomorphism. This follows from standard facts about the relationship between smooth representations, such as those of the smooth principal series, and their underlying Harish-Chandra modules (see, e.g., [18, Theorem 11.6.7]).

By hypothesis, $z \notin z_0 + \mathbb{Z}$ and, since $2z_0 = d \in \mathbb{Z}$, this implies that $z \notin -z_0 + \mathbb{Z}$ also. Thus, by Theorem 5.12, the space $\mathbb{M}(a\lambda_1 + b\lambda_2)$ is nonzero if and only if $a - b$ is even, in which case the dimension of $\mathbb{M}(a\lambda_1 + b\lambda_2)$ is 1. (Recall that a and b are restricted by the inequalities $a \geq b \geq 0$.) Thus, as representations of K,

$$\bigoplus_{\varepsilon_1,\varepsilon_2} \mathrm{HC}\big(\Gamma(\mathcal{L}_{\chi(z,\varepsilon_1,\varepsilon_2)})^{\square_z}\big) \cong \bigoplus_{a-b \text{ even}} V_{a\lambda_1+b\lambda_2}. \tag{6.2}$$

On the other hand, $\mathfrak{k} \cap \mathfrak{h} \cong \mathfrak{so}(m-2) \oplus \mathfrak{so}(2)$, embedded in the standard way in $\mathfrak{k}$, and it follows from this and Lemma 5.1 that

$$\bigoplus_{\varepsilon} \mathrm{HC}\big(\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})\big) \cong \bigoplus_{a-b \text{ even}} V_{a\lambda_1+b\lambda_2} \tag{6.3}$$

as representations of K as well. The trivial representation V_0 occurs in $\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,+)}))$; indeed we wrote the corresponding function F_s explicitly above. The adjoint representation of K is isomorphic to $V_{\lambda_1+\lambda_2}$. As we saw in Lemma 6.1, this summand occurs in $\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,-)}))$, and the map $Z \mapsto \psi_s(\cdot, Z)$ realizes the embedding. In the notation of Proposition 6.5,

$$\gamma(0) = \frac{1}{\Gamma\big(-\frac{s+d}{2}\big)\Gamma\big(-\frac{s-1+d}{2}\big)},$$

and it follows from this, Lemma 6.2, equation (6.1), the definition of T^{n}, and the identity principle that

$$(T^{\mathrm{n}}F_s)(e) = \pi^{d/2}\frac{1}{\Gamma\big(-\frac{s}{2}\big)\Gamma\big(-\frac{s-1+d}{2}\big)} \tag{6.4}$$

for all $s \in \mathbb{C}$. Similarly, with $Z = \mathrm{Ad}(w_1^{-1})Z_{m-1,m}$, we have

$$\big(T^{\mathrm{n}}\psi_s(\cdot, Z)\big)(e) = -2\pi^{d/2}\frac{1}{\Gamma\big(-\frac{s-1}{2}\big)\Gamma\big(-\frac{s+d}{2}\big)} \tag{6.5}$$

for all $s \in \mathbb{C}$. The assumption that $z \notin z_0 + \mathbb{Z}$ and the relationship between z and s imply that $s \notin \mathbb{Z}$. It follows that neither (6.4) nor (6.5) vanishes. Thus the images of the summands V_0 and $V_{\lambda_1+\lambda_2}$ in (6.3) under $T^{\mathrm{n}}: \mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})) \to \mathrm{HC}(\Gamma(\mathcal{L}_{\chi(z,\varepsilon,\varepsilon)}))$ are always nonzero.

By Lemma 5.1, we have

$$\bigoplus_{\varepsilon_1,\varepsilon_2} \mathrm{HC}\big(\Gamma(\mathcal{L}_{\chi(z,\varepsilon_1,\varepsilon_2)})\big) \cong \bigoplus_{a \geq b \geq 0} (a-b+1)V_{a\lambda_1+b\lambda_2} \tag{6.6}$$

as representations of K. In particular, V_0 and $V_{\lambda_1+\lambda_2}$ occur with multiplicity 1 in (6.6). They also occur in (6.2), and it follows from this and the conclusions of the previous paragraph that $T^{\mathrm{n}}(V_0) \subset \mathrm{HC}(\Gamma(\mathcal{L}_{\chi(z,+,+)})^{\square_z})$ and $T^{\mathrm{n}}(V_{\lambda_1+\lambda_2}) \subset$

$\mathrm{HC}(\Gamma(\mathcal{L}_{\chi(z,-,-)})^{\square_z})$. We have already observed that $s \notin \mathbb{Z}$ and we are assuming that $m \geq 5 > 4$. By [5, Theorem 3.4.1] and the observations made in the first paragraph on [5, page 291], it follows that $\Gamma(\mathcal{E}_{\nu(s,\varepsilon)})$ is an irreducible representation of G. We draw two conclusions from this and what we have already deduced. First, the subspace V_0 of $\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,+)}))$ generates the Harish-Chandra module and so $T^{\mathrm{n}}\big(\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,+)}))\big) \subset \mathrm{HC}(\Gamma(\mathcal{L}_{\chi(z,+,+)})^{\square_z})$. Similarly, the subspace $V_{\lambda_1+\lambda_2}$ of $\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,-)}))$ generates the Harish-Chandra module and so $T^{\mathrm{n}}\big(\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,-)})) \subset \mathrm{HC}(\Gamma(\mathcal{L}_{\chi(z,-,-)})^{\square_z})\big)$. Second, the restriction of T^{n} to both $\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,+)})$ and $\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,-)})$ is injective, and so these modules are embedded in

$$\bigoplus_{\varepsilon_1,\varepsilon_2} \mathrm{HC}\big(\Gamma(\mathcal{L}_{\chi(z,\varepsilon_1,\varepsilon_2)})^{\square_z}\big). \tag{6.7}$$

By comparing (6.2) and (6.3), it follows that

$$T^{\mathrm{n}}\big(\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,+)}))\big) = \mathrm{HC}\big(\Gamma(\mathcal{L}_{\chi(z,+,+)})^{\square_z}\big),$$

that $T^{\mathrm{n}}\big(\mathrm{HC}(\Gamma(\mathcal{E}_{\nu(s,-)}))\big) = \mathrm{HC}(\Gamma(\mathcal{L}_{\chi(z,-,-)})^{\square_z})$, and, since these two submodules exhaust all the K-types in (6.7), that $\mathrm{HC}(\Gamma(\mathcal{L}_{\chi(z,\varepsilon,-\varepsilon)})^{\square_z}) = \{0\}$ for $\varepsilon \in \{\pm\}$. This completes the proof. $\square$

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Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078, USA; akable@math.okstate.edu