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COMPUTATION OF RIEMANN MATRICES FOR THE HYPERBOLIC CURVES OF DETERMINANTAL POLYNOMIALS

MAO-TING CHIEN[1*] and HIROSHI NAKAZATO[2]

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Abstract. The numerical range of a matrix, according to Kippenhahn, is determined by a hyperbolic determinantal form of linear Hermitian matrices associated to the matrix. On the other hand, using Riemann theta functions, Helton and Vinnikov confirmed that a hyperbolic form always admits a determinantal representation of linear real symmetric matrices. The Riemann matrix of the hyperbolic curve plays the main role in the existence of real symmetric matrices. In this article, we implement computations of the Riemann matrix and the Abel–Jacobi variety of the hyperbolic curve associated to a determinantal polynomial of a matrix. Further, we prove that the lattice of the Abel–Jacobi variety is decomposed into the direct sum of two orthogonal lattices for some 4×4 Toeplitz matrices.

1. Introduction

Let T be an $n \times n$ complex matrix. The real ternary form $F_T(x,y,z)$ associated to T is defined by

$$F_T(x,y,z) = \det\bigl(zI_n + x\Re(T) + y\Im(T)\bigr),$$

where the two Hermitian matrices $\Re(T) = (T+T^*)/2$, $\Im(T) = (T-T^*)/(2i)$ correspond to the Cartesian decomposition $T = \Re(T) + i\Im(T)$. The form $F_T(x,y,z)$ is deeply related to the numerical range $W(T)$ of T defined by

$$W(T) = \{\xi^* T\xi : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}.$$

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*Corresponding author.
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The numerical radius of a matrix T or a Hilbert space operator T denoted by $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$ provides fruitful information for analyzing the operator T (see [17], [22]). Toeplitz [20] initiated the study of numerical range, and Hausdorff [11] proved its convexity. Kippenhahn [16] showed that $W(T)$ is the convex hull of the real part of the dual curve $G_T(X,Y,Z) = 0$ of the algebraic curve $F_T(x,y,z) = 0$. In [1], the classification of the curve $F_T(x,y,z) = 0$ is applied to categorize the shapes of the numerical ranges of 4×4 matrices. In [2] and [3], we discussed rational curves and elliptic curves in the context of numerical ranges. For any pair (x_0, y_0) of real numbers, the algebraic equation $F_T(x_0, y_0, z) = 0$ has n real solutions in z because $x_0\Re(T)+y_0\Im(T)$ is a Hermitian matrix. The polynomial $F_T(x,y,z)$ also satisfies $F_T(0,0,1) = 1$. A ternary form with these two properties is called *hyperbolic* with respect to $(0,0,1)$. Fiedler [8] conjectured that a ternary hyperbolic form $F(x,y,z)$ of degree n admits the representation $F(x,y,z) = F_T(x,y,z)$ for some $n\times n$ matrix T. Lax [18] conjectured a stronger requirement that the matrix T is a complex symmetric matrix. Helton and Vinnikov [14] confirmed that the Lax conjecture is true by using Riemann theta functions with g phases given by

$$\theta(\phi_1, \phi_2, \ldots, \phi_g : R) = \sum_{m_1,\ldots,m_g\in\mathbb{Z}} \exp\Big(2\pi\sqrt{-1}\Big(\sum_{i,j} r_{ij}m_im_j + \sum_i m_i\phi_i\Big)\Big),$$

where $R = (r_{ij})$ is a $g \times g$ Riemann matrix which will be described in Section 2. The confirmation of the Lax conjecture is useful for characterizing the shapes of the numerical ranges (see [13]). We have two main motivations for studying this Riemann matrix. One is to control the Riemann matrix R as an important parameter of the theta function when applying the Helton–Vinnikov theorem for the determinantal representation. The other is to recognize that the Riemann matrix R plays an essential part in classifying the curve $F_T(x,y,z) = 0$, which is efficient even when this curve has no singular points.

Let $F(x,y,z)$ be an irreducible ternary form of degree $n \geq 3$. A point $P = (x_0, y_0, z_o)$ of the complex projective curve

$$C_F = \big\{[x,y,z] \in \mathbb{CP}^2 : F(x,y,z) = 0\big\}$$

is called a *singular point* if the gradient of $F(x,y,z)$ at P is zero. For a singular point $P = (x_0,y_0,z_0) \neq (0,0,0)$, we can assume that $z_0 = 1$ by the exchange of the roles of the coordinates x, y, z if necessary. We assume that the multiplicity of the curve C_F at P is $m \geq 2$ and that the number of analytic branches of C_F around P is $s \geq 1$. The two polynomials

$$f(X,Y) = F(x_0+X, y_0+Y, 1), \qquad f_Y(X,Y) = F_Y(x_0+X, y_0+Y, 1)$$

are expressed by Taylor series as

$$f(X,Y) = \sum_{j+k\geq 2} a_{jk}X^jY^k, \qquad f_Y(X,Y) = \sum_{j+k\geq 2, k\geq 1} k a_{j,k}X^jY^{k-1}.$$

The Taylor series of these functions define an ideal (f, f_Y) of the ring $\mathbb{C}[[X,Y]]$ of formal power series in X, Y. The complex dimension of the quotient ring $\mathbb{C}[[X,Y]]/(f, f_Y)$ is finite, and is called the *local intersection number* of the curves

$F(x, y, 1) = 0$ and $F_y(x, y, 1) = 0$ at P. The δ-number of the curve C_F at P is defined as

$$\delta(P) = \frac{1}{2}\big(\dim \mathbb{C}[[X, Y]]/(f, f_Y) - m + s\big).$$

Let $Q_1, Q_2, \ldots, Q_\ell$ be the set of all singular points of the algebraic curve C_F. The *genus* $g(C_F)$ of the curve C_F is given by

$$g(C_F) = \frac{1}{2}(n-1)(n-2) - \sum_{j=1}^{\ell} \delta(Q_j).$$

An irreducible curve is called a *rational* (resp., *elliptic*) curve if its genus $g = 0$ (resp., $g = 1$).

We assume that the form $F(x, y, z)$ is irreducible. By using the resolution of singular points of the curve, we can construct a compact Riemann surface Γ which is a parameter space of the curve C_F. This Riemann surface is called the nonsingular model $\tilde{C}_F$ of the curve C_F. If the curve C_F has no singular points, the curve itself is a Riemann surface. We consider the canonical projection π of $\tilde{C}_F$ onto $C_F : F(x, y, z) = 0$. For a sufficiently small open neighborhood V of a singular point P of C_F, the set $\pi^{-1}(V)$ is composed of a finite number of connected open sets $V_1, V_2, \ldots, V_q$ of $\tilde{C}_F$ which form a multilevel crossing around P. The complex affine curve $F(x, y, 1) = 0$ is parameterized by two rational functions L_1, L_2 on $\tilde{C}_F$: $x = L_1(s), y = L_2(s)$. The nonsingular model $\tilde{C}_F$ is homeomorphic to one of the compact surfaces: the Riemann sphere $\mathbb{C} \cup \{\infty\}$, the torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, or a g-hole torus with $g \geq 2$. The genus g of the curve C_F coincides with the number of holes of the compact Riemann surface $\tilde{C}_F$. (For references on Riemann surfaces, see, e.g., [5], [4], [10].)

In this article, we use the complex analytic method to study invariants of the numerical ranges of matrices. The complex analytic technique is often involved in studying operators on a Hilbert space (see [6], [15]). We investigate some invariants of the Riemann surface $\tilde{C}_F$ of the algebraic curve associated to a matrix T. In particular, we compute the Riemann matrix of $\tilde{C}_F$. The Riemann matrix is a common invariant for the birationally equivalent class of algebraic curves, and hence the invariant for the curve C_F and its dual curve. The Riemann matrix generates a lattice of the Abel–Jacobi variety of $\tilde{C}_F$. We prove that the lattice is decomposed into the direct sum of two orthogonal sublattices for some 4×4 Toeplitz matrices.

2. Riemann matrix

Let Γ be a compact Riemann surface with genus g. We recall the definition of a Riemann matrix which adopts the standard notation used in [5]. For an arbitrary base point P_0 of Γ, define a cycle to be a continuous map $f : [0, 1] \to \Gamma$ with $f(0) = f(1) = P_0$. Two such maps f_1, f_2 produce a new map $f_2 \circ f_1$ by the rule

$$(f_2 \circ f_1)(t) = f_1(2t), \qquad (f_2 \circ f_1)(1/2 + t) = f_2(2t),$$

$0 \leq t \leq 1/2$. The inverse f^{-1} of f is given by $f^{-1}(t) = f(1 - t)$. Two maps f, g are identified if they are homotopic, that is, there is a continuous map

$F : [0, 1] \times [0, 1] \to \Gamma$ satisfying

$$F(0, t) = f(t), \qquad F(1, t) = g(t), \qquad F(s, 0) = F(s, 1) = P_0.$$

The fundamental group $\Pi_1(\Gamma)$ of Γ is the collection of cycles of Γ under the homotopic relation. We identify an element of $\Pi_1(\Gamma)$ with a closed, oriented, piecewise smooth curve on Γ. Let ω be an arbitrary $C^{(1)}$-differential 1-form on Γ. On a local coordinate $z = x + iy$ in an open neighborhood V of a point P of Γ, the differential form ω is expressed as $\omega = Q(z, \overline{z})\, dz + R(z, \overline{z})\, d\overline{z}$, by $C^{(1)}$-differentiable complex-valued functions Q, R. We define a differential 2-form $d\omega$ by

$$d\omega = \Big(\frac{\partial R}{\partial z} - \frac{\partial Q}{\partial \overline{z}}\Big)\, dz \wedge d\overline{z}.$$

If $d\omega = 0$, the differential form ω is closed. If there exists a holomorphic function U on an open neighborhood of an arbitrary point P of Γ satisfying $\omega = U_z\, dz$, then ω is called a *holomorphic* 1-*form* or an *abelian differential form of the first kind*. A holomorphic 1-form is a closed form.

Given two cycles a, b, the equation $\int_{a^{-1}\circ b^{-1}\circ a\circ b} \omega = 0$ holds for any holomorphic differential 1-form ω. Let N be the normal subgroup of $\Pi_1(\Gamma)$ generated by $\{a^{-1} \circ b^{-1} \circ a \circ b : a, b \in \Pi_1(\Gamma)\}$. The abelian group $H_1(\Gamma, \mathbb{Z}) = \Pi_1(\Gamma)/N$ is called the *1-dimensional homology group* (see [4]). A cycle c of Γ is homologous to zero if $\int_c \omega = 0$ for any holomorphic differential 1-form ω. For two cycles a and b with different base points, the intersection index $a \cdot b$ of two cycles counts the number of intersections, taking the orientation of the cycles into account. If two cycles a and b do not intersect transversally, their intersection index $a \cdot b$ is zero. If they intersect once, $a \cdot b$ is 1 if the outer product of the tangent vectors $t_a \times t_b$ points out from the surface, and -1 if the outer product points into the surface. The homology group $H_1(\Gamma) \cong \mathbb{Z}^{2g}$ has a canonical basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ satisfying the symplectic relations

$$a_i \cdot a_j = 0, \qquad b_i \cdot b_j = 0, \qquad a_i \cdot b_j = \delta_{ij},$$

where δ_{ij} is the Kronecker delta (see [10]). Any element c of $H_1(\Gamma)$ is expressed as

$$c = m_1 a_1 + \cdots + m_g a_g + m_{g+1} b_1 + \cdots + m_{2g} b_g$$

for some integers $m_1, \dots, m_{2g}$. The set of all holomorphic differential 1-forms is a g-dimensional complex vector space. Let $\{\omega_1, \dots, \omega_g\}$ be a basis for the holomorphic differential 1-form vector space. The period matrix Ω of Γ is a $g \times 2g$ complex matrix defined as $\Omega = [AB]$, where $A = (A_{ij})_{i,j=1}^{g}$ and $B = (B_{ij})_{i,j=1}^{g}$ are given by

$$A_{ij} = \oint_{a_j} \omega_i, \qquad B_{ij} = \oint_{b_j} \omega_i.$$

The matrix A is known to be invertible. The Riemann matrix of Γ is defined as $R = A^{-1}B$. The imaginary part matrix $(\Im R_{ij})_{i,j\le g}$ is positive definite. The Riemann matrix is closely related to the geometric and analytic structure of the Riemann surface under consideration (see [10]).

The period matrix Ω generates a nondegenerate lattice

$$\Lambda = \{m_1A_1 + \cdots + m_gA_g + m_{g+1}B_1 + \cdots + m_{2g}B_g : m_1, m_2, \ldots, m_{2g} \in \mathbb{Z}\},$$

where $A_1, \ldots, A_g$ and $B_1, \ldots, B_g$ are, respectively, the columns of A and B. This lattice is the period lattice associated to the basis $\{\omega_1, \ldots, \omega_g\}$ of the space of holomorphic 1-forms on Γ. Given a point $z_0 \in \Gamma$, the holomorphic mapping $J : \Gamma \to \mathbb{C}^g/\Lambda$ defined by

$$J(z) = \Big(\int_{z_0}^{z} \omega_1, \int_{z_0}^{z} \omega_2, \ldots, \int_{z_0}^{z} \omega_g\Big) \bmod \Lambda$$

is a holomorphic embedding of Γ onto $J(\Gamma) = \mathbb{C}^g/\Lambda$. The g-dimensional compact space $\mathbb{C}^g/\Lambda$ is called the *Abel–Jacobi variety* of Γ.

Suppose that Γ is a Riemann surface with genus g associated to an irreducible hyperbolic curve $F_T(x, y, 1) = 0$ defined by a complex $n \times n$ matrix T. It would be interesting to see whether there exists a canonical basis $\{a_1, a_2, \ldots, a_g, b_1, \ldots, b_g\}$ for the homology group $H_1(\Gamma)$ for which the Riemann matrix $R = A^{-1}B = (e_{g+1}, \ldots, e_{2g})^T$ and the basis $e_1 = (1, 0, \ldots, 0)^T, \ldots, e_g = (0, \ldots, 0, 1)^T$ span the lattice

$$\Sigma = \{m_1e_1 + \cdots + m_ge_g + m_{g+1}e_{g+1} + \cdots + m_{2g}e_{2g} : m_1, m_2, \ldots, m_{2g} \in \mathbb{Z}\}$$

so that Σ is decomposed into the direct sum of two orthogonal lattices Σ_1, Σ_2, where each of Σ_1 and Σ_2 is spanned by g elements of the column vectors $\{e_1, \ldots, e_g, e_{g+1}, \ldots, e_{2g}\}$ and $\Sigma_1 \cong \Sigma_2 \cong \mathbb{Z}^g$ as free abelian groups. In the case $g = 1$, the Riemann matrix R is none other than the τ-invariant of the elliptic curve $F_T(x, y, 1) = 0$. In [3], the authors showed that the τ-invariants of the elliptic curves for some 4×4 matrices are pure imaginary. In this article, we generalize the problem for $g \geq 2$, and we verify the orthogonal lattice decomposition in Sections 3 and 4 for some 4×4 Toeplitz matrices.

3. Quartic curve of genus 2

Using a graph-theoretic technique, C. Tretkoff and M. Tretkoff [21] provided an explicit method of constructing a canonical basis $\{a_j, b_j : j = 1, \ldots, g\}$ for the homology group $H_1(\Gamma) = H_1(\Gamma, \mathbb{Z})$ of a Riemann surface Γ. Deconinck and van Hoeji [5] gave an algorithm to compute the period matrix of the Riemann surface using the Tretkoff method, where the Maple algcurves package, a collection of Maple programs for computations with algebraic curves, is performed (see also [4]). In this article, we follow the algorithm in [5], and implement the algcurves package to find a canonical basis for the group $H_1(\Gamma)$ associated to 4×4 Toeplitz matrices. We provide some numerical values to explain the process of computing the Riemann matrix.

Consider a Toeplitz matrix

$$T = \begin{pmatrix} 0 & 2 & 2a & 2k \\ 0 & 0 & 2 & 2a \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.1}$$

for $a \in \mathbb{R}$, $k = a^2 - 1$. Then

$$F_T(x,y,z) = \big((4-4a^2)x^2 + 4a^3xz - (4+a^4)z^2\big)y^2 \\ + \big((4-4a^2)x^4 + 4a^3x^3z - (4+a^4)x^2z^2 + z^4\big).$$

If $a = 0$, then the curve C_F is a rational curve. We assume that $a > 0$. If $a = 1$, then the polynomial $F_T(x,y,z : a) = z(4x^3 + 4xy^2 - 5x^2z - 5y^2z + z^3)$ is reducible. Since

$$F(x,1,z) = \big(-(2+2a)x + (2+2a+a^2)z\big)\big(-(2-2a)x - (2-2a+a^2)z\big) \\ + (4-4a^2)x^4 + 4a^3x^3z - (4+a^4)x^2z^2 + z^4,$$

the point $(x,y,z) = (0,1,0)$ is an ordinary double point under the condition $0 < a < 1$ or $a > 1$. The roots of the polynomial

$$p(x) = \big((2a+2)x - (a^2+2a+2)\big)\big((2a-2)x - (a^2-2a+2)\big) \\ \times \big((2a+2)x^2 - a^2x - 1\big)\big((2a-2)x^2 - a^2x + 1\big)$$

are branch points of the Riemann surface. For $a = \sqrt{2}$, two of these points coincide. We treat the case $a > \sqrt{2}$. In this case, the curve $F_T(x,y,z) = 0$ is a hyperelliptic curve of genus 2. The roots of $p(x)$ are given by

$$r_1 = \frac{a^2 - \sqrt{a^4+8a+8}}{4(a+1)} < r_2 = \frac{a^2 - \sqrt{a^4-8a+8}}{4(a-1)} < r_3 = \frac{a^2 + \sqrt{a^4+8a+8}}{4(a+1)} \\ < r_4 = \frac{a^2-2a+2}{2(a-1)} < r_5 = \frac{a^2+2a+2}{2(a+1)} \\ < r_6 = \frac{a^2 + \sqrt{a^4-8a+8}}{4(a-1)}.$$

Theorem 3.1. *Let T be the 4×4 matrix defined in (3.1) with $a > \sqrt{2}$. Suppose that Γ is the Riemann surface associated to the irreducible hyperbolic form $F_T(x,y,z)$. Then there exists a canonical basis $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$ for the homology group $H_1(\Gamma)$ such that the lattice*

$$\Sigma = \{m_1e_1 + m_2e_2 + m_3e_3 + m_4e_4 : m_1, m_2, m_3, m_4 \in \mathbb{Z}\},$$

spanned by the Riemann matrix $R = A^{-1}B = (e_3, e_4)^T$ and the basis $\{e_1 = (1,0), e_2 = (0,1)\}$, is the direct sum of two mutually orthogonal lattices $\Sigma_1 = \{m_1e_1 + m_2e_2 : m_1, m_2 \in \mathbb{Z}\}$ and $\Sigma_2 = \{m_3e_3 + m_4e_4 : m_3, m_4 \in \mathbb{Z}\}$.

Proof. Consider the curve $f(x,y) = F_T(x,y,1) = 0$ which is described by

$$\big((2a+2)x - (a^2+2a+2)\big)\big((2a-2)x - (a^2-2a+2)\big)y^2 \\ = \big((2a-2)x^2 - a^2x + 1\big)\big(-(2a+2)x^2 + a^2x + 1\big).$$

The basis $\{\omega_1, \omega_2\}$ for the space of holomorphic differential 1-forms is given by

$$\omega_1 = -\frac{1}{2}\frac{dx}{((4a^2-4)x^2 - 4a^3x + (a^4+4))y}$$

and

$$\omega_2 = -\frac{1}{2}\frac{x\,dx}{((4a^2-4)x^2-4a^3x+(a^4+4))y}.$$

The base point x_0 is taken on the real line with $x_0 < r_1$, and hence the function $y = y(x)$ with $f(x, y(x)) = 0$ takes pure imaginary values at $x = x_0$. We define the two sheets of the surface Γ as $\Im(y_1(x_0)) < 0$ and $\Im(y_2(x_0)) > 0$.

Suppose that $\{a_1, a_2, b_1, b_2\}$ is a canonical basis for the homology group $H_1(\Gamma)$ produced by the algorithm in [5] and the algcurves implementation. We construct a new basis depending on the canonical basis as follows.

1. The cycle $\tilde{a}_1 = a_1$ starts on sheet 1, encircles branch point $r_1 = (a^2 - \sqrt{a^4+8a+8})/(4(a+1))$ to arrive at sheet 2, and encircles branch point $r_2 = (a^2 - \sqrt{a^4-8a+8})/(4(a-1))$ to arrive at sheet 1.
2. The cycle $\tilde{a}_2 = -a_2$ starts on sheet 1, encircles branch point $r_3 = (a^2 + \sqrt{a^4+8a+8})/(4(a+1))$ to arrive at sheet 2, and encircles branch point $r_4 = (a^2-2a+2)/(2(a-1))$ to arrive at sheet 1.
3. The cycle $\tilde{b}_2 = -b_2 - a_1$ starts on sheet 1, encircles branch point $r_4 = (a^2-2a+2)/(2(a-1))$ to arrive at sheet 2, and encircles branch point $r_5 = (a^2+2a+2)/(2(a+1))$ to arrive at sheet 1.
4. The cycle $\tilde{b}_3 = \tilde{b}_1 - \tilde{b}_2 = b_1 + b_2 + a_2$ starts on sheet 1, encircles branch point $r_2 = (a^2 - \sqrt{a^4-8a+8})/(4(a-1))$ to arrive at sheet 2, and encircles branch point $r_3 = (a^2 + \sqrt{a^4+8a+8})/(4(a+1))$ to arrive at sheet 1.

Consider the set $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$ in the group $H_1(\Gamma)$ given by

$$\begin{aligned} &\tilde{a}_1 = a_1, && \tilde{a}_2 = -a_2, \\ &\tilde{b}_1 = b_1 - a_1 + a_2, && \tilde{b}_2 = -b_2 - a_1. \end{aligned}$$

Then, the new basis $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$ of $H_1(\Gamma)$ satisfies the symplectic structure:

$$\begin{aligned} &\tilde{b}_1 \cdot \tilde{b}_2 = (b_1 - a_1 + a_2) \cdot \big(-(b_2 + a_1)\big) = -(b_1 \cdot a_1 + a_2 \cdot b_2) = 0, \\ &\tilde{b}_1 \cdot \tilde{b}_1 = (b_1 - a_1 + a_2) \cdot (b_1 - a_1 + a_2) = -b_1 \cdot a_1 - a_1 \cdot b_1 = 0, \\ &\tilde{b}_2 \cdot \tilde{b}_2 = (b_2 + a_1) \cdot (b_2 + a_1) = b_2 \cdot b_2 + a_1 \cdot a_1 + b_2 \cdot a_1 + a_1 \cdot b_2 = 0, \\ &\tilde{a}_1 \cdot \tilde{b}_1 = a_1 \cdot (b_1 - a_1 + a_2) = 1, \\ &\tilde{a}_2 \cdot \tilde{b}_1 = 0, \\ &\tilde{a}_1 \cdot \tilde{b}_2 = a_1 \cdot \big[-(b_2 + a_1)\big] = 0, \\ &\tilde{a}_2 \cdot \tilde{b}_2 = -a_2 \cdot \big[-(b_2 + a_2)\big] = 1, \\ &\tilde{a}_1 \cdot \tilde{a}_1 = \tilde{a}_2 \cdot \tilde{a}_2 = \tilde{a}_1 \cdot \tilde{a}_2 = \tilde{a}_2 \cdot \tilde{a}_1 = 0. \end{aligned}$$

Taking a base point $x_0 < r_1$, we define the branch $y_2(x)$ of the function $y = y(x)$ with $F_T(x, y(x), 1) = 0$ on sheet 2 in each interval $r_j < x < r_{j+1}$, $j = 1, 2, 3, 4$, according to the algorithm given in [5] on pages 33–34. We characterize $y_2(x)$ and the function $U(x) = -1/(((4a^2-4)x^2 - 4a^3x + (a^4+4))y_2(x))$ on each interval $r_j < x < r_{j+1}$ as follows.

On the interval $r_1 < x < r_2$, the functions $y_2(x)$ and $U(x)$ are real valued, $y_2(x) > 0$ and $U(x) < 0$. Moreover, $U(x) = -\frac{1}{\sqrt{V(x)}}$, where

$$\begin{aligned} V(x) = &\big((2a+2)x - (a^2+2a+2)\big)\big((2a-2)x - (a^2-2a+2)\big) \\ &\times \big((2a-2)x^2 - a^2x + 1\big)\big(-(2a+2)x^2 + a^2x + 1\big). \end{aligned}$$

On the interval $r_3 < x < r_4$, the functions $y_2(x)$ and $U(x)$ are real valued, $y_2(x) < 0$ and $U(x) > 0$.

On the interval $r_2 < x < r_3$, the functions $y_2(x)$ and $U(x)$ take pure imaginary values, $\Im(U(x)) < 0$ and $\Im(y_2(x)) < 0$.

On the interval $r_4 < x < r_5$, the functions $y_2(x)$ and $U(x)$ take pure imaginary values, $\Im(U(x)) > 0$ and $\Im(y_2(x)) < 0$.

According to the labeling of the branches of $y(x)$ in [5], we compute that

$$\tilde{a}_{11} = -\int_{a_1} \frac{1}{2}\frac{dx}{((4a^2-4)x^2 - 4a^3x + (a^4+4))y(x)}\,dx = -\int_{r_1}^{r_2} \frac{dx}{\sqrt{V(x)}} < 0$$

and

$$\tilde{a}_{12} = -\int_{\tilde{a}_2} \frac{1}{2}\frac{dx}{((4a^2-4)x^2 - 4a^3x + (a^4+4))y(x)}\,dx = \int_{r_3}^{r_4} \frac{dx}{\sqrt{V(x)}} > 0.$$

Similarly, we have

$$\tilde{b}_{12} = -\int_{b_2} \frac{1}{2}\frac{dx}{((4a^2-4)x^2 - 4a^3x + (a^4+4))y(x)} = i\int_{r_4}^{r_5} \frac{dx}{\sqrt{-V(x)}},$$

which is a pure imaginary number with $\Im(\tilde{b}_{12}) > 0$, and

$$\begin{aligned} b_{11} + b_{12} + a_{12} &= -\int_{b_1+b_2+a_2} \frac{1}{2}\frac{dx}{((4a^2-4)x^2 - 4a^3x + (a^4+4))y(x)} \\ &= -i\int_{r_2}^{r_3} \frac{dx}{\sqrt{-V(x)}}\,dx, \end{aligned}$$

which is also a pure imaginary number with $\Im(b_{11} + b_{12} + a_{12}) < 0$.

We proceed to compute the entries of the periodic matrix. The values of the integrals

$$\oint_{\tilde{a}_1} \omega_2 = -\frac{1}{2}\oint_{a_1} \frac{x\,dx}{[(4a^2-4)x^2 - 4a^3x + (a^4+4)]y(x)}\,dx = -\int_{r_1}^{r_2} \frac{x\,dx}{\sqrt{V(x)}}$$

and

$$\oint_{\tilde{a}_2} \omega_2 = -\frac{1}{2}\oint_{\tilde{a}_2} \frac{x\,dx}{[(4a^2-4)x^2 - 4a^3x + (a^4+4)]y(x)}\,dx = \int_{r_3}^{r_4} \frac{x\,dx}{\sqrt{V(x)}}$$

are real numbers. The values of the integrals

$$\oint_{\tilde{b}_2} \omega_2 = -\frac{1}{2}\oint_{\tilde{b}_2} \frac{x\,dx}{[(4a^2-4)x^2 - 4a^3x + (a^4+4)]y(x)}\,dx = i\int_{r_4}^{r_5} \frac{x\,dx}{\sqrt{-V(x)}}$$

and

$$\oint_{\tilde{b}_1-\tilde{b}_2} \omega_2 = -\frac{1}{2}\oint_{\tilde{b}_1-\tilde{b}_2} \frac{x\,dx}{[(4a^2-4)x^2-4a^3x+(a^4+4)]y(x)}\,dx = -i\int_{r_2}^{r_3} \frac{x\,dx}{\sqrt{-V(x)}}$$

are pure imaginary numbers, and hence $\oint_{\tilde{b}_1} \omega_2$ is also pure imaginary.

Let $\tilde{\Omega} = [\tilde{A}\tilde{B}]$ be the periodic matrix of Γ with respect to the new basis $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$. Then the matrix $\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}$ is invertible and its entries are real. Further, the entries of the matrix $\tilde{B} = \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \end{pmatrix}$ are pure imaginary, and hence the Riemann matrix $R = \tilde{A}^{-1}\tilde{B}$ is a complex symmetric matrix whose entries are pure imaginary.

Let $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$, and set $e_1 = (1,0)$, $e_2 = (0,1)$, $e_3 = (r_{11}, r_{12})$, $e_4 = (r_{21}, r_{22}) = (r_{12}, r_{22})$. The space $\mathbb{C}^g \cong \mathbb{R}^{2g}$ is equipped with the real part of the standard inner product. Then the lattice Σ spanned by the vectors e_1, e_2, e_3, e_4 is the direct sum of two orthogonal lattices $\Sigma_1 = \{ne_1 + me_2 : n, m \in \mathbb{Z}\}$ and $\Sigma_2 = \{ne_3 + me_4 : n, m \in \mathbb{Z}\}$. $\square$

We run the algcurves package for the case $a = 2$; the period matrix $[\tilde{A}\tilde{B}]$ constructed by Theorem 3.1 is approximately evaluated by

$$\tilde{A} \approx \begin{pmatrix} -0.195915 & 0.410645 \\ -0.0199325 & 0.382071 \end{pmatrix}, \qquad \tilde{B} \approx \begin{pmatrix} 0.170162i & 0.655061i \\ 0.553593i & 0.865335i \end{pmatrix}.$$

4. Quartic curve of genus 3

C. Tretkoff and M. Tretkoff [21, p. 501] mentioned that the Riemann matrix of the Fermat curve $x^4 + y^4 = 1$ is $\mathrm{diag}(i, 2i, i)$. Hence, the lattice Σ is spanned by six mutually orthogonal bases of the lattice. They also found that the Riemann matrix of the Klein–Hurwitz curve $-y^7 + x(1-x)^2 = 0$ is $(-1+\sqrt{7}i)/2I_3$. In this case, the lattice Σ for this real Riemann surface cannot be decomposed into the direct sum of two orthogonal lattices Σ_1, Σ_2; each one has three generators. However, we may find a suitable basis for the homology group $H_1(\Gamma)$ so that the Riemann surface generates a lattice which is the direct sum of two orthogonal lattices for some 4×4 Toeplitz matrices. (For the computation of the Riemann matrix and the split of the lattice Σ, we refer the reader to [7], [12], [19].)

Theorem 4.1. *Let T be the 4×4 nilpotent matrix given by*

$$T = \begin{pmatrix} 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the homology group $H_1(\Gamma)$ of the Riemann surface of the algebraic curve C_F of the polynomial $F_T(x, y, z)$ has a canonical basis $\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{b}_1, \tilde{b}_2, \tilde{b}_3\}$ for the homology group $H_1(\Gamma)$ such that the Riemann matrix $R = A^{-1}B$ of the period *matrix* $[AB]$ *is the form*

$$R = \begin{pmatrix} ir_{11} & ir_{12} & r_{13} \\ ir_{12} & ir_{22} & r_{23} \\ r_{13} & r_{23} & ir_{33} \end{pmatrix}$$

for some real numbers r_{ij}, $1 \leq i \leq j \leq 3$. *Furthermore, the lattice* Σ *generated by the six vectors* $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ *and* $e_4 = (ir_{11}, ir_{12}, r_{13}), e_5 = (ir_{12}, ir_{22}, r_{23}), e_6 = (r_{13}, r_{23}, ir_{33})$ *is decomposed into the direct sum of two mutually orthogonal lattices generated, respectively, by* $\{e_1, e_2, e_6\}$ *and* $\{e_3, e_4, e_5\}$.

Proof. It is easy to find that

$$F_T(x,y,z) = 16y^4 + (20x^2 - 12z^2)y^2 + 4x^4 - 12x^2z^2 + z^4.$$

The curve C_F of the polynomial $F_T(x,y,z)$ has no singular points with genus 3. From the partial derivative $F_y(x,y,1) = 8y(5x^2 + 8y^2 - 3)$, a basis for the space of holomorphic differential 1-forms on the Riemann surface Γ associated to C_F consists of

$$\omega_1 = \frac{1}{8}\frac{dx}{y(5x^2+8y^2-3)}, \qquad \omega_2 = \frac{1}{8}\frac{dx}{5x^2+8y^2-3},$$
$$\omega_3 = \frac{1}{8}\frac{x\,dx}{y(5x^2+8y^2-3)}$$

(see [4]). Our strategy is to find a basis $\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{b}_1, \tilde{b}_2, \tilde{b}_3\}$ of the homology group $H_1(\Gamma)$ so that the period matrix $[\tilde{A}\tilde{B}]$ is the form

$$\tilde{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & 0 & i\alpha_{23} \\ i\alpha_{31} & 0 & \alpha_{33} \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 0 & i\beta_{12} & \beta_{13} \\ i\beta_{21} & 0 & \beta_{23} \\ \beta_{31} & \beta_{32} & i\beta_{33} \end{pmatrix},$$

where α_{ij} and β_{ij} are real numbers. Then the form $\tilde{A}^{-1}\tilde{B}$ is the kind required.

The curve C_F has the following eight branch points:

$$P_1 : x = -i\sqrt{5/3} \approx -1.29099i, \qquad P_2 : x = -\frac{i}{\sqrt{3}} \approx -0.577350i,$$
$$P_3 : x = \frac{-2-\sqrt{2}}{2} \approx -1.70711, \qquad P_4 : x = \frac{-2+\sqrt{2}}{2} \approx -0.292893,$$
$$P_5 : x = \frac{2-\sqrt{2}}{2} \approx 0.292893, \qquad P_6 = \frac{2+\sqrt{2}}{2} \approx 1.70711,$$
$$P_7 : x = \frac{i}{\sqrt{3}} \approx 0.577350i, \qquad P_8 : x = i\sqrt{5/3} \approx 1.29099i.$$

We take a base point $x_0 = -2.27279$. The labels of the four branches of the function are based on the following labeling:

$$y_1(-2.27279) \approx -2.26981i, \qquad y_2(-2.27279) \approx -0.744951i,$$
$$y_3(-2.27279) \approx 0.744951i, \qquad y_4(-2.27279) \approx 2.26981i.$$

The values of $y_j(x_0)$ are pure imaginary, and labeled as

$$\Im\big(y_1(x_0)\big) < \Im\big(y_2(x_0)\big) < \Im\big(y_3(x_0)\big) < \Im\big(y_4(x_0)\big).$$

First, we define the cycles $\{a_1, a_2, a_3, b_1, c_6 - a_1, c_7\}$ in the following way:

1. The cycle a_1 starts on sheet 1, encircles branch point P_1 to arrive at sheet 2, and encircles branch point P_2 to arrive at sheet 1.
2. The cycle a_2 starts on sheet 1, encircles branch point P_4 to arrive at sheet 4, and encircles branch point P_5 to arrive at sheet 1.
3. The cycle a_3 starts on sheet 3, encircles branch point P_2 to arrive at sheet 4, and encircles branch point P_7 to arrive at sheet 3.
4. The cycle b_1 starts on sheet 1, encircles branch point P_1 to arrive at sheet 2, and encircles point P_8 to arrive sheet 1.
5. The cycle $c_6 - a_1$ starts on sheet 1, encircles branch point P_2 to arrive at sheet 2, encircles branch point P_3 to arrive at sheet 3, encircles branch point P_7 to arrive at sheet 4, and encircles branch point P_4 to arrive at sheet 1.
6. The cycle c_7 starts on sheet 1, encircles branch point P_1 to arrive at sheet 2, encircles branch point P_3 to arrive at sheet 3, encircles branch point P_8 to arrive at sheet 4, and encircles branch point P_4 to arrive at sheet 1.

The cycles b_2, b_3 are defined by $b_2 = b_1 - a_1 - 2(c_6 - a_1) + c_7$ and $b_3 = a_2 + (c_6 - a_1) + a_1 - c_7$. Then the set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ is a canonical basis for the homology group $H_1(\Gamma)$. We use another canonical basis $\{\tilde{a}_1 = a_1, \tilde{a}_2 = a_2, \tilde{a}_3 = a_3 - a_2, \tilde{b}_1 = b_1, \tilde{b}_2 = b_2 + b_3 - a_2, \tilde{b}_3 = b_3\}$. Then the cycles $\tilde{b}_2, b_3$ satisfy $\tilde{b}_2 = b_1 - (c_6 - c_1), b_3 = a_2 + (c_6 - a_1) + a_1 - c_7$.

We outline the method for computing the period matrix $[\tilde{A}\tilde{B}]$ by computing some specified entries.

The integrals over the cycle $\tilde{a}_1 = a_1$ give $\tilde{a}_{i1}$, the column one entries of $\tilde{A}$. On the line segment $\{iX : -\sqrt{5/3} < X < -1/\sqrt{3}\}$, the function $y(x)$ takes imaginary values. The branches $y_1(x), y_2(x)$ of the function on this line segment satisfy $y_2(x) = \overline{y_1(x)}$ and $\Im(y_1(x)) < 0 < \Im(y_2(x))$, and hence the four branches $y_j(x)$ are vertices of a rectangle with edges parallel to the real or imaginary axis and satisfy $y_1(x) + y_2(x) + y_3(x) + y_4(x) = 0$. The function $W(x) = 1/((5x^2 + 8y^2 - 3)y)$ satisfies $W_2(x) = \overline{W_1(x)}$ for $y = y_j(x)$, and $\Im(W_2(x)) < 0 < \Im(W_1(x))$. Then

$$\begin{aligned}
\tilde{a}_{11} &= \frac{1}{8}\oint_{\tilde{a}_1} \frac{dx}{(5x^2 + 8y^2 - 3)y} \\
&= \frac{1}{8}\int_{-\sqrt{5/3}i}^{-i/\sqrt{3}} \frac{dx}{(5x^2 + 8y_2^2 - 3)y_2} + \frac{1}{8}\int_{-i/\sqrt{3}}^{-\sqrt{5/3}i} \frac{dx}{(5x^2 + 8y_1^2 - 3)y_1} \\
&\approx (0.0778069 - 0.0066436i) + (0.0778069 + 0.0066436i) \\
&\approx 0.155614.
\end{aligned}$$

The function $U(x) = 1/(5x^2+8y^2-3)$ on the line segment takes pure imaginary values, and $\Im(U_2(x)) < 0 < \Im(U_1(x))$. Then

$$\begin{aligned}
\tilde{a}_{21} &= \frac{1}{8}\oint_{\tilde{a}_1} \frac{dx}{(5x^2+8y^2-3)} \\
&= \frac{1}{8}\int_{-\sqrt{5/3}i}^{-i/\sqrt{3}} \frac{dx}{(5x^2+8y_2^2-3)} + \frac{1}{8}\int_{-i/\sqrt{3}}^{-\sqrt{5/3}i} \frac{dx}{(5x^2+8y_1^2-3)} \\
&\approx 0.072851 + 0.072851 \\
&\approx 0.145702.
\end{aligned}$$

The function $V(x) = x/((5x^2+8y^2-3))$ on the line segment takes imaginary values, and $V_2(x) = -\overline{V_1(x)}$. These branches satisfy $\Re(V_2(x)) < 0 < \Re(V_1(x))$. Then

$$\begin{aligned}
\tilde{a}_{31} &= \frac{1}{8}\oint_{\tilde{a}_1} \frac{x\,dx}{(5x^2+8y^2-3)} \\
&= \frac{1}{8}\int_{-\sqrt{5/3}i}^{-i/\sqrt{3}} \frac{x\,dx}{(5x^2+8y_2^2-3)y_2} + \frac{1}{8}\int_{-i/\sqrt{3}}^{-\sqrt{5/3}i} \frac{x\,dx}{(5x^2+8y_1^2-3)y_1} \\
&\approx (-0.00568944 - 0.0669032i) + (0.00568944 - 0.0669032i) \\
&\approx -0.133806i.
\end{aligned}$$

Similarly, the integrals over the cycle $\tilde{b}_1 = b_1$ give $\tilde{b}_{i1}$, the column one entries of $\tilde{B}$. We compute that

$$\begin{aligned}
\tilde{b}_{11} &= \frac{1}{8}\int_{\tilde{b}_1} \frac{dx}{(5x^2+8y^2-3)y} \\
&= \frac{1}{8}\int_{-2.7279}^{-i/\sqrt{3}} \Big(\frac{1}{(5x^2+8y_1^2-3)y_1} - \frac{1}{(5x^2+8y_2^2-3)y_2}\Big)\,dx \\
&\quad - \frac{1}{8}\int_{-2.27279}^{i/\sqrt{3}} \Big(\frac{1}{(5x^2+8y_1^2-3)y_1} - \frac{1}{(5x^2+8y_1^2-3)y_2}\Big)\,dx \\
&\quad + \frac{1}{8}\int_{-\sqrt{5/3}i}^{-i/\sqrt{3}} \Big(\frac{1}{(5x^2+8y_2^2-3)y_2} - \frac{1}{(5x^2+8y_1^2-3)y_1}\Big)\,dx \\
&\quad - \frac{1}{8}\int_{\sqrt{5/3}i}^{i/\sqrt{3}} \Big(\frac{1}{(5x^2+8y_2^2-3)y_2} - \frac{1}{(5x^2+8y_1^2-3)y_1}\Big)\,dx \\
&= \frac{1}{4}\Re\Big(\int_{-2.27279}^{-i/\sqrt{3}} \Big(\frac{1}{(5x^2+8y_1^2-3)y_1} - \frac{1}{(5x^2+8y_2^2-3)y_2}\Big)\,dx\Big) \\
&\quad - \frac{1}{4}\Re\Big(\int_{-\sqrt{5/3}i}^{-i/\sqrt{3}} \Big(\frac{1}{(5x^2+8y_2^2-3)y_1} - \frac{1}{(5x^2+8y_1^2-3)y_2}\Big)\,dx\Big) \\
&\approx 2\Re(-0.155614 + i0.0721547) - 2\Re(-0.155614) \\
&= 0.
\end{aligned}$$

Next, we compute

$$
\begin{aligned}
\tilde{b}_{21} &= \frac{1}{8}\oint_{\tilde{b}_1} \frac{dx}{5x^2+8y^2-3} \\
&= \frac{1}{8}\int_{-2.27279}^{-i/\sqrt{3}} \frac{dx}{5x^2+8y_1^2-3} - \frac{1}{8}\int_{-2.27279}^{-i/\sqrt{3}} \frac{dx}{5x^2+8y_2^2-3} \\
&\quad + \frac{1}{8}\int_{-2.27279}^{i/\sqrt{3}} \frac{dx}{5x^2+8y_2^2-3} - \frac{1}{8}\int_{-2.27279}^{i/\sqrt{3}} \frac{dx}{5x^2+8y_1^2-3} \\
&\quad + \frac{1}{8}\int_{-\sqrt{5/3}i}^{-i/\sqrt{3}} \frac{dx}{5x^2+8y_2^2-3} - \frac{1}{8}\int_{-\sqrt{5/3}i}^{-i/\sqrt{3}} \frac{dx}{5x^2+8y_1^2-3} \\
&\quad - \frac{1}{8}\int_{\sqrt{5/3}i}^{i/\sqrt{3}} \frac{dx}{5x^2+8y_2^2-3} + \frac{1}{8}\int_{\sqrt{5/3}i}^{i/\sqrt{3}} \frac{dx}{5x^2+8y_1^2-3} \\
&= \frac{4i}{8}\Im\Big(\int_{-2.27279}^{-i/\sqrt{3}} \frac{dx}{5x^2+8y^2-3}\Big) - \frac{4i}{8}\Im\Big(\int_{-\sqrt{5/3}i}^{-i/\sqrt{3}} \frac{dx}{5x^2+8y_1^2-3}\Big) \\
&\approx 4i\Im(-0.0555593+0.053564i) - 2i\Im(-0.145702) \\
&= 0.214256i.
\end{aligned}
$$

From the definition, we have

$$
\begin{aligned}
\tilde{b}_{31} &= \frac{1}{8}\oint_{\tilde{b}_1} \frac{x\,dx}{(5x^2+8y^2-3)y}\,dx \\
&= \Big(\frac{1}{8}\int_{-2.27279}^{-\sqrt{5/3}i} \frac{x\,dx}{(5x^2+8y_1^2-3)y_1} - \frac{1}{8}\int_{-2.27279}^{-\sqrt{5/3}i} \frac{x\,dx}{(5x^2+8y_2^2-3)y_2}\Big) \\
&\quad + \Big(\frac{1}{8}\int_{-2.27279}^{\sqrt{5/3}i} \frac{x\,dx}{(5x^2+8y_2^2-3)y_2} - \frac{1}{8}\int_{-2.27279}^{\sqrt{5/3}i} \frac{x\,dx}{(5x^2+8y_1^2-3)y_1}\Big).
\end{aligned}
$$

On the line segment $\{x(t) = (-2.27279)(1-t)+i\sqrt{5/3}it : 0 < t < 1\}$, we consider the branches $y_1(x)$ and $y_2(x)$ of the function $y = y(x)$ satisfying $F_T(x, y(x), 1) = 0$. The curve $9x(t)^4+18x(t)^2+5$, $0 < t < 1$, passes through the half-line $\{x : x < 0\}$ at some point $t_0 \in (0.8302473, 0.8302474)$. Denote the branch of square roots a complex number

$$\sqrt{\lambda} = \sqrt{|\lambda|}\exp(i\operatorname{Arg}\lambda/2)$$

for $-\pi < \operatorname{Arg}\lambda \leq \pi$. We define

$$
\begin{aligned}
Y_2(x) &= \frac{1}{2\sqrt{2}}\sqrt{-5x^2+3+\sqrt{9x^4+18x^2+5}}, \\
Y_1(x) &= \frac{1}{2\sqrt{2}}\sqrt{-5x^2+3-\sqrt{9x^4+18x^2+5}}.
\end{aligned}
$$

On the line $(-2.27279, -\sqrt{5/3}i)$, the functions $y_2(x), y_1(x)$ for $0 < t < t_0$ are expressed, respectively, as

$$\frac{x}{(5x^2+8Y_2^2-3)Y_2}, \qquad \frac{x}{(5x^2+8Y_1^2-3)Y_1}.$$

The functions $y_1(x), y_2(x)$ are expressed, respectively, as

$$\frac{x}{(5x^2+8Y_2^2-3)Y_2}, \qquad \frac{x}{(5x^2+8Y_1^2-3)Y_1}$$

for $t_0 < t < 1$. Direct computations show that

$$\frac{1}{8}\int_{-2.27279}^{-\sqrt{5/3}i} \frac{x\,dx}{(5x^2+8y_1^2-3)y_1} - \frac{1}{8}\int_{-2.27279}^{-\sqrt{5/3}i} \frac{x\,dx}{(5x^2+8y_2^2-3)y_2}$$
$$\approx 0.133806 - 0.0562865i,$$

and

$$\frac{1}{8}\int_{-2.27279}^{\sqrt{5/3}i} \frac{x\,dx}{(5x^2+8y_2^2-3)y_2} - \frac{1}{8}\int_{-2.27279}^{\sqrt{5/3}i} \frac{x\,dx}{(5x^2+8y_1^2-3)y_1}$$
$$\approx 0.133806 + 0.0562865i.$$

Therefore,

$$\begin{aligned}\tilde{b}_{31} &= \frac{1}{8}\oint_{b_1} \frac{x\,dx}{(5x^2+8y^2-3)y} \\ &\approx (0.133806 - 0.0562865i) + (0.133806 + 0.0562865i) \approx 0.267613.\end{aligned}$$

The remaining entries of the period matrix $[\tilde{A}\tilde{B}]$ can be evaluated in a similar way. We obtain numerically the matrix

$$\tilde{A} = \begin{pmatrix} 0.155614 & -0.311228 & 0 \\ 0.145702 & 0 & -0.214256i \\ -0.133806i & 0 & 0.267613 \end{pmatrix}$$

and

$$\tilde{B} = \begin{pmatrix} 0 & -0.349582i & -0.155614 \\ 0.214256i & 0 & 0.145702 \\ 0.267613 & 0.267613 & 0.133806i \end{pmatrix}.$$

Thus the Riemann matrix is numerically given by

$$R = \tilde{A}^{-1}\tilde{B} = \begin{pmatrix} 1.69486i & 0.847432i & 0.152568 \\ 0.847432i & 1.54696i & 0.576284 \\ 0.152568 & 0.576284 & 0.576284i \end{pmatrix}.$$

We equip the space $\mathbb{C}^g \cong \mathbb{R}^{2g}$ with the real part of the standard inner product. Then it is easy to verify that

$$\begin{aligned}\langle e_1, e_3\rangle = \langle e_1, e_4\rangle = \langle e_1, e_5\rangle = 0,\\ \langle e_2, e_3\rangle = \langle e_2, e_4\rangle = \langle e_2, e_5\rangle = 0,\\ \langle e_6, e_3\rangle = \langle e_6, e_4\rangle = \langle e_6, e_4\rangle = 0.\end{aligned}$$

Hence the two lattices

$$\Sigma_1 = \{ne_1 + me_2 + \ell e_6 : n, m, \ell \in \mathbb{Z}\}, \qquad \Sigma_2 = \{ne_3 + me_4 + \ell e_5 : n, m, \ell \in \mathbb{Z}\}$$

are mutually orthogonal, and $\Sigma = \Sigma_1 \oplus \Sigma_2$. $\square$

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References

1. M. T. Chien and H. Nakazato, *Singular points of ternary polynomials associated with 4-by-4 matrices*, Electron. J. Linear Algebra **23** (2012), 755–769. Zbl 1253.15030. MR2992391. DOI 10.13001/1081-3810.1554. 153
2. M. T. Chien and H. Nakazato, *Determinantal representations of hyperbolic forms via weighted shift matrices*, Appl. Math. Comput. **258** (2015), 172–181. Zbl 1338.15048. MR3323060. DOI 10.1016/j.amc.2015.02.019. 153
3. M. T. Chien and H. Nakazato, *Elliptic modular invariants and numerical ranges*, Linear Multilinear Algebra **63** (2015), no. 8, 1501–1519. Zbl 1314.14056. MR3304989. DOI 10.1080/03081087.2014.947982. 153, 156
4. B. Deconinck and M. S. Patterson, "Computing with plane algebraic curves and Riemann surfaces: The algorithms of the Maple package 'algcurves' "in *Computational Approach to Riemann Surfaces*, Lecture Notes in Math. **2013**, Springer, Heidelberg, 2011, 67–123. Zbl 1213.14114. MR2905611. DOI 10.1007/978-3-642-17413-1_2. 154, 155, 156, 161
5. B. Deconinck and M. van Hoeji, *Computing Riemann matrices of algebraic curves*, Phys. D **152/153** (2001), 28–46. Zbl 1054.14079. MR1837895. DOI 10.1016/S0167-2789(01)00156-7. 154, 156, 158, 159
6. R. G. Douglas, T. Nakazi, and M. Seto, *Shift operators on the* $\mathbb{C}^2$*-valued Hardy space*, Acta Sci. Math. (Szeged) **73** (2007), no. 3–4, 729–744. Zbl 1164.47010. MR2380073. 154
7. C. J. Earle, "Some Jacobian varieties which split" in *Complex Analysis (Joensuu, 1978)*, Lecture Notes in Math. **747**, Springer, Berlin, 1979, 101–107. Zbl 0423.14018. MR0553033. 160
8. M. Fiedler, *Geometry of the numerical range of matrices*, Linear Algebra Appl. **37** (1981), 81–96. Zbl 0452.15024. MR0636211. DOI 10.1016/0024-3795(81)90169-5. 153
9. J. Frauendiener and C. Klein, *Computational approach to hyperelliptic Riemann surfaces*, Lett. Math. Phys. **105** (2015), no. 3, 379–400. Zbl 1317.14127. MR3312511. DOI 10.1007/s11005-015-0743-4. 166
10. R. C. Gunning, *Riemann Surfaces and Generalized Theta Functions*, Ergeb. Math. Grenzgeb. (3) **91**, Springer, Berlin, 1976. Zbl 0341.14013. MR0457787. 154, 155
11. F. Hausdorff, *Der Wertvorrat einer Bilinearform*, Math. Z. **3** (1919), no. 1, 314–316. JFM 47.0088.02. MR1544350. DOI 10.1007/BF01292610. 153
12. T. Hayashida and M. Nishi, *Existence of curves of genus two on a product of two elliptic curves*, J. Math. Soc. Japan **17** (1965), 1–16. Zbl 0132.41701. MR0201434. 160
13. J. W. Helton and I. M. Spitkovsky, *The possible shapes of numerical ranges*, Oper. Matrices **6** (2012), no. 3, 607–611. Zbl 1270.15014. MR2987030. DOI 10.7153/oam-06-41. 153
14. J. W. Helton and V. Vinnikov, *Linear matrix inequality representation of sets*, Comm. Pure Appl. Math. **60** (2007), no. 5, 654–674. Zbl 1116.15016. MR2292953. DOI 10.1002/cpa.20155. 153
15. K. Izuchi, T. Nakazi, and M. Seto, *Backward shift invariant subspaces in the bidisc, II*, J. Operator Theory **51** (2004), no. 2, 361–376. Zbl 1055.47009. MR2074186. 154
16. R. Kippenhahn, *Über den wertevorrat einer Matrix*, Math. Nachr. **6** (1951), 193–228. Zbl 0044.16201. MR0059242. 153

17. F. Kittaneh, M. S. Moslehian, and T. Yamazaki, *Cartesian decomposition and numerical radius inequalities*, Linear Algebra Appl. **471** (2015), 46–53. Zbl 1322.47008. MR3314323. DOI 10.1016/j.laa.2014.12.016. 153
18. P. D. Lax, *Differential equations, difference equations and matrix theory*, Comm. Pure Appl. Math. **11** (1958), 175–194. Zbl 0086.01603. MR0098110. 153
19. Y. Tashiro, S. Yamazaki, M. Ito, and T. Higuchi, *On Riemann's period matrix of* $Y^2 = X^{2n+1} - 1$, RIMS Kôkyûroku **963** (1996), 124–141. Zbl 0948.14501. MR1472044. 160
20. O. Toeplitz, *Das algebraische Analogon zu einem Satze von Fejér*, Math. Z. **2** (1918), no. 1–2, 187–197. JFM 46.0157.02. MR1544315. DOI 10.1007/BF01212904. 153
21. C. L. Tretkoff and M. D. Tretkoff, "Combinatorial group theory, Riemann surfaces and differential equations" in *Contributions to Group Theory*, Contemp. Math. **33**, Amer. Math. Soc., Providence, 1984, 467–519. Zbl 0557.30036. MR0767125. DOI 10.1090/conm/033/767125. 156, 160
22. T. Yamazaki, *On numerical range of the Aluthge transformation*, Linear Algebra Appl. **341** (2002), no. 1–3, 111–117. Zbl 1021.47004. MR1873612. DOI 10.1016/S0024-3795(01)00333-0. 153

[1]Department of Mathematics, Soochow University, Taipei 11102, Taiwan.
E-mail address: mtchien@scu.edu.tw

[2]Department of Mathematical Sciences, Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan.
E-mail address: nakahr@hirosaki-u.ac.jp