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CHARACTERIZATIONS AND APPLICATIONS OF THREE TYPES OF NEARLY CONVEX POINTS

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Abstract. By using some geometric properties and nested sequence of balls, we prove seven necessary and sufficient conditions such that a point x in the unit sphere of Banach space X is a nearly rotund point of the unit ball of the bidual space. For any closed convex set $C \subset X$ and $x \in X \setminus C$ with $P_C(x) \neq \emptyset$, we give a series of characterizations such that C is approximatively compact or approximatively weakly compact for x by using three types of nearly convex points. Furthermore, making use of an S point, we present a characterization such that the convex subset C is approximatively compact for some x in $X \setminus C$. We also establish a relationship between nested sequence of balls and the approximate compactness of the closed convex subset C for some $x \in X \setminus C$.

1. Introduction

For a Banach space X, let X^* be the dual of X, and let $S(X)$ and $B(X)$ stand for the unit sphere and closed unit ball of X, respectively. Let $B(x,r) = \{y \in X : \|y - x\| < r\}$ and let $\overline{B(x,r)}$ be the corresponding closed ball. We denote $D(x) = \{f \in S(X^*), f(x) = 1\}$ for any $x \in S(X)$. Let w and w^* stand for weak and weak* topology on X and X^*, respectively.

It is well known that if a Banach space X is a *generic continuity space* (GC space), then X is a *dual differentiability space* (DD space), and DD space plays an important role in the study of differentiability of convex functions and admits

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many important properties (for more details, see [5]). Giles et al. in [5, Theorem 1.13] proved that a Banach space X is a GC space if X can be equivalently renormed to be weakly locally uniformly rotund. Moreover, they showed that a Banach space X is a GC space if X admits an equivalent norm such that the following property holds: given $x \in S(X)$, for every $F \in X^{**} \setminus X$, $\|F\| = 1$, then $\|F + x\| < 2$. In [5, p. 420], the question of how to characterize this property is raised. In our terminology (see our Definition 1.1), the question of how to characterize a point $x \in S(X)$ to be a nearly rotund point of $B(X^{**})$ arises.

Definition 1.1. Let X be a Banach space, with $x \in S(X)$. Then we say that x is

(1) [1] a rotund point of $B(X^{**})$ if $x = x^{**}$ for any $x^{**} \in X^{**}$ satisfying $\|x^{**}\| = \|\frac{x+x^{**}}{2}\| = 1$;

(1′) a nearly rotund point of $B(X^{**})$ if $x^{**} \in X$ for any $x^{**} \in X^{**}$ satisfying $\|x^{**}\| = \|\frac{x+x^{**}}{2}\| = 1$;

(2) (see [2]) a WALUR point of $B(X)$ if, for any $\{x_n\}_{n=1}^{\infty} \subset B(X)$ and $\{x_m^*\}_{m=1}^{\infty} \subset B(X^*)$, the condition

$$\lim_m \lim_n x_m^* \Big(\frac{x + x_n}{2}\Big) = 1$$

implies that $x_n \xrightarrow{w} x$;

(2′) a CWALUR point of $B(X)$ if, for any $\{x_n\}_{n=1}^{\infty} \subset B(X)$ and $\{x_m^*\}_{m=1}^{\infty} \subset B(X^*)$, the condition

$$\lim_m \lim_n x_m^* \Big(\frac{x + x_n}{2}\Big) = 1$$

implies that $\{x_n\}_{n=1}^{\infty}$ is relatively weakly compact;

(3) (see [11], [12]) a nearly very convex point (resp., nearly strongly convex point) of $B(X)$ if, for any $\{x_n\}_{n=1}^{\infty} \subset B(X)$ such that $x^*(x_n) \to 1$ for some $x^* \in D(x)$, it holds that $\{x_n\}_{n=1}^{\infty}$ is relatively weakly compact (resp., relatively compact);

(4) (see [9], [10]) an S point (resp., weakly smooth or WS point) of X if, for any $\{x_n^*\}_{n=1}^{\infty} \subset S(X^*)$ such that $x_n^*(x) \to 1$, it holds that $\{x_n^*\}_{n=1}^{\infty}$ is relatively compact (resp., relatively weakly compact).

We say that a Banach space X has one of the above properties if every point of $S(X)$ has the same property.

Remark 1.2. That $x \in S(X)$ is a rotund point of $B(X^{**})$ implies that it is a nearly rotund point of $B(X^{**})$, but the converse is not generally true. For example, $X = (\mathbb{R}^2, \|\cdot\|_1)$ is reflexive and each $x \in S(X)$ is not a rotund point of $B(X)$, and hence not a rotund point of $B(X^{**})$. However, each $x \in S(X)$ is a nearly rotund point of $B(X^{**})$.

Definition 1.3 (see [2]). An unbounded nested sequence of balls in Banach space X is an increasing sequence $\{B_n = B(x_n, r_n)\}_{n=1}^{\infty}$ of open balls in X with $r_n \to \infty$. For a subset $C \subset X$, the metric projection $P_C : X \to 2^C$ is defined by $P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}$, where $d(x, C) = \inf\{\|x - y\|, y \in C\}$.

Definition 1.4 (see [12], [3]). Suppose that X is a Banach space, that C is a nonempty subset in X, and that $x \in X\backslash C$. Then C is considered to be approximatively weakly compact for x (resp., approximatively compact for x) if $\{x_n\}_{n=1}^{\infty} \subset C$ and $\lim_n \|x_n - x\| = d(x, C)$, and then $\{x_n\}_{n=1}^{\infty}$ is relatively weakly compact (resp., relatively compact).

Definition 1.5 (see [7, p. 285]). We have the following.

(1) A subset Φ of $B(X^*)$ is said to be a norming set for X if $\|x\| = \sup_{x^* \in \Phi} x^*(x)$ for all $x \in X$.

(2) A sequence $\{x_n\} \subset B(X)$ is said to be asymptotically normed by Φ if for any $\epsilon > 0$ there exist $x^* \in \Phi$ and $N \in \mathbb{N}$ such that $x^*(x_n) > 1 - \epsilon$ for all $n \geq N$.

In this article, we get seven necessary and sufficient conditions to guarantee that $x \in B(X^{**})$ is a nearly rotund point of $B(X^{**})$ in terms of some geometric properties and nested sequence of balls (Theorem 2.1, Theorem 2.2). By using nearly very convex point (resp., nearly strongly convex point, S point), we will give a series of characterizations such that a subset C in X with $P_C(x) \neq \emptyset$ for any $x \in X \setminus C$ is approximatively weakly compact for x (resp., approximatively compact for x) (see Theorems 3.1, 3.5, 3.8). Finally, by using nested sequence of balls, we give a characterization so that a subset $C \subset X$ is approximatively compact for some $x \in X \setminus C$ (see part (1) of Theorem 3.10).

2. The Characterizations of the Nearly Rotund Points

In this section, we present seven equivalent conditions to prove that a point $x \in S(X)$ is a nearly rotund point of $B(X^{**})$, and this solves the question proposed by Giles et al. in [5].

Theorem 2.1. *Let X be a Banach space, with $x \in S(X)$. Then the following are equivalent:*

(1) *x is a nearly rotund point of $B(X^{**})$;*

(2) *for any $\{x_n^{**}\}_{n=1}^{\infty} \subset B(X^{**})$, $\{x_m^*\}_{n=1}^{\infty} \subset B(X^*)$, the condition*

$$\lim_m \lim_n \Big(\frac{x + x_n^{**}}{2}\Big)(x_m^*) = 1$$

implies that all w^*-cluster points of $\{x_n^{**}\}_{n=1}^{\infty}$ belong to X;

(3) *for every unbounded nested sequence $\{B_n^*\}_{n=1}^{\infty}$ of balls such that x is bounded below on $\bigcup B_n^*$, if for any $\{y_n^{**}\}_{n=1}^{\infty} \subset S(X^{**})$ the sequence $\{\inf y_n^{**}(B_n^*)\}_{n=1}^{\infty}$ is also bounded below, then all w^*-cluster points of $\{y_n^{**}\}_{n=1}^{\infty}$ belong to X;*

(4) *for every unbounded nested sequence $\{B_n^*\}_{n=1}^{\infty}$ of balls in X^* such that x is bounded below on $\bigcup B_n^*$, if for any $\{y_n\}_{n=1}^{\infty} \subset S(X)$ the sequence $\{\inf y_n(B_n^*)\}_{n=1}^{\infty}$ is also bounded below, then $\{y_n\}_{n=1}^{\infty}$ is relatively weakly compact;*

(5) *for every unbounded nested sequence $\{B_n^*\}_{n=1}^{\infty}$ of balls in X^* such that x is bounded below on $\bigcup B_n^*$, if $x^{**} \in S(X^{**})$ is also bounded below on $\bigcup B_n^*$, then $x^{**} \in X$.*

Proof. The proof is similar to Theorem 6 in [2]. We will give their proof for the sake of completeness.

(1) $\Rightarrow$ (2) Let $\{x_n^{**}\}_{n=1}^{\infty} \subset B(X^{**})$, $\{x_m^*\}_{n=1}^{\infty} \subset B(X^*)$ such that

$$\lim_m \lim_n \Big(\frac{x + x_n^{**}}{2}\Big)(x_m^*) = 1.$$

Let y^{**} be a w^*-cluster point of $\{x_n^{**}\}_{n=1}^{\infty}$; then

$$\lim_m \Big(\frac{x + y^{**}}{2}\Big)(x_m^*) = 1,$$

and hence, $\|\frac{x+y^{**}}{2}\| = 1$. By (1), $y^{**} \in X$.

(2) $\Rightarrow$ (3) Let $\{B_n^*\}_{n=1}^{\infty}$ be an unbounded nested sequence of balls such that x is bounded below on $\bigcup B_n^*$. Suppose that $\{y_n^{**}\}_{n=1}^{\infty} \subset S(X^{**})$ such that $\{\inf y_n^{**}(B_n^*)\}_{n=1}^{\infty}$ is also bounded below. Suppose that $c \in \mathbb{R}$ is a common lower bound and $B_n^* = B(x_n^*, r_n)$. We may assume without loss of generality that $0 \in B_1^*$. Let $y_n^* = \frac{x_n^*}{r_n}$; then $\|y_n^*\| \leq 1$. The fact that $\{B_n^*\}_{n=1}^{\infty}$ is nested implies that $y_n^{**}(y_m^*) \geq 1 + \frac{c}{r_m}$ for all $n \geq m$. It follows that

$$\Big(\frac{x + y_n^{**}}{2}\Big)(y_m^*) \geq 1 + \frac{c}{r_m}, \quad \text{for all } n \geq m.$$

Since $r_m \to \infty$, we conclude with

$$\lim_m \lim_n \Big(\frac{x + y_n^{**}}{2}\Big)(y_m^*) = 1.$$

By (2), we obtain that all w^*-cluster points of $\{y_n^{**}\}_{n=1}^{\infty}$ belong to X.

(3) $\Rightarrow$ (4) This is clear.

(4) $\Rightarrow$ (5) Let $\{B_n^*\}_{n=1}^{\infty} = \{B(x_n^*, r_n)\}_{n=1}^{\infty}$ be an unbounded nested sequence of balls in X^* such that x and any $x^{**} \in S(X^{**}) \setminus X$ are both bounded below on $\bigcup B_n^*$ by some $c \in \mathbb{R}$. Hence, for all $n \geq 1$, $\inf x(B_n^*) = \langle x_n^*, x\rangle - r_n \geq c$. Assuming that $0 \in B_1^*$, for any $\epsilon > 0$ and $x^* \in S(X^*)$, by Goldstine's Theorem, there exists $\{x_n\}_{n=1}^{\infty} \subset B(X)$ such that

$$\big|x^*(x_n - x^{**})\big| < \frac{\epsilon}{2}, \qquad \big|x_n^*(x_n - x^{**})\big| < 1, \qquad x^{**}(x_n^*) - r_n \geq c.$$

Then

$$1 \geq \|x_n\| \geq \frac{x_n^*}{r_n}(x_n) \geq x^{**}\Big(\frac{x_n^*}{r_n}\Big) - \frac{1}{r_n} \geq 1 + \frac{c-1}{r_n}.$$

Hence, $\|x_n\| \to 1$. Putting $y_n = \frac{x_n}{\|x_n\|}$, it follows that $\{y_n\}_{n=1}^{\infty} \subset S(X)$. Moreover,

$$\begin{aligned} \inf x_n(B_n^*) &= x_n^*(x_n) - r_n\|x_n\| > x_n^*(x^{**}) - 1 - r_n\|x_n\| \\ &\geq (c-1) + r_n\big(1 - \|x_n\|\big) \geq c - 1. \end{aligned}$$

It follows that $\inf y_n(B_n^*) > \frac{c-1}{\|x_n\|}$, and hence, $\{\inf y_n(B_n^*)\}_{n=1}^{\infty}$ is also bounded below. By (4), we know that $\{y_n\}_{n=1}^{\infty}$ is relatively weakly compact. Therefore, there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty} \subset \{y_n\}_{n=1}^{\infty}$ and $y \in X$ such that $y_{n_k} \xrightarrow{w} y$, and so we may assure that x_{n_k} is also weakly convergent. Consequently, for sufficiently large k,

$$\left|x^*(y-x^{**})\right| < \left|x^*(x_{n_k}-y)\right| + \left|x^*(x_{n_k}-x^{**})\right| < \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

By the arbitrariness of ϵ and x^*, we know that $y=x^{**}\in X$, which is a contradiction.

(5) $\Rightarrow$ (1) For any $x^{**}\in S(X^{**})$ such that $\|x^{**}\| = \frac{\|x+x^{**}\|}{2} = 1$, taking $\{x_n^*\}_{n=1}^\infty \subset B(X^*)$ such that $(x+x^{**})(x_n^*)\to 2$, we can choose $\{\delta_n\}_{n=1}^\infty\subset\mathbb{R}$, $0<\delta_n<1$ satisfying $\sum_{n=1}^\infty \delta_n<1$ such that $x_n^*(x)>1-\delta_n$ and $x^{**}(x_n^*)>1-\delta_n$. Putting $B_n^*=B(\sum_{i=1}^n x_i^*, n+\sum_{i=1}^n\delta_i)$, then $\{B_n^*\}_{n=1}^\infty$ is an unbounded nested sequence of balls in X^*. Consequently,

$$\inf x(B_n^*) = x\Big(\sum_{i=1}^n x_i^*\Big) - n - \sum_{i=1}^n \delta_i = -\sum_{i=1}^n\big(1-x(x_i^*)+\delta_i\big) > -2\sum_{i=1}^n\delta_i > -2.$$

Similarly, $\inf x^{**}(B_n^*)>-2$, this shows that x and x^{**} are both bounded below on $\bigcup B_n^*$. By (5), we have that $x^{**}\in X$. $\square$

By using the preceding Theorem 2.1, we also obtain the following equivalent conditions to prove that a point $x\in S(X)$ is a nearly rotund point of $B(X^{**})$.

Theorem 2.2. *Let X be a Banach space, with $x\in S(X)$. The following are equivalent:*

(1) *x is a nearly rotund point of $B(X^{**})$;*
(2′) *x is a CWALUR point of $B(X)$;*
(3′) *for every $x^{**}\in S(X^{**})$, if there exists sequence $\{x_n^*\}_{n=1}^\infty\subset S(X^*)$ such that $\lim_n x^{**}(x_n^*)=\lim_n x_n^*(x)=1$, then $x^{**}\in X$;*
(4′) *for any $\{x_n^{**}\}_{n=1}^\infty\subset B(X^{**})$, if $\{\frac{x+x_n^{**}}{2}\}_{n=1}^\infty$ is asymptotically normed by $B(X^*)$, then all w^*-cluster points of $\{x_n^{**}\}_{n=1}^\infty$ belong to X.*

Proof. (1) $\Leftrightarrow$ (2′) Let $\{x_n\}_{n=1}^\infty\subset B(X)$ and let $\{x_m^*\}_{m=1}^\infty\subset B(X^*)$ such that

$$\lim_m\lim_n x_m^*\Big(\frac{x+x_n}{2}\Big)=1.$$

By the relatively weak*-compactness of $\{x_n\}_{n=1}^\infty\subset B(X^{**})$, we obtain a w^*-cluster point x^{**} of $\{x_n\}_{n=1}^\infty$. Then, it is easy to prove that $\lim_m\lim_n x_m^*(\frac{x+x^{**}}{2})=1$, which further implies that $x^{**}\in X$ by (1). Consequently, $\{x_n\}_{n=1}^\infty$ is relatively weakly compact. This shows x is a CWALUR point of $B(X)$.

Conversely, in order to complete (2′) $\Rightarrow$ (1), we just need to prove that (2′) $\Rightarrow$ (4) of Theorem 2.1. Let $\{B_n^*=B_n(x_n^*,r_n)\}_{n=1}^\infty$ be an unbounded nested sequence of balls in X^* such that x is bounded below on $\bigcup B_n^*$. Let $\{y_n\}_{n=1}^\infty\subset S(X)$ such that the sequence $\{\inf y_n(B_n^*)\}$ is also bounded below. Suppose that $c\in\mathbb{R}$ is a common lower bound. We may assume that $0\in B_1^*$. Putting $y_n^*=\frac{x_n^*}{r_n}$, the fact that $\{B_n^*\}_{n=1}^\infty$ is nested implies that $y_m^*(y_n)\ge 1+\frac{c}{r_m}$ and that $y_m^*(x)\ge 1+\frac{c}{r_m}$ for all $n\ge m$. It follows that $y_m^*(\frac{x+y_n}{2})\ge 1+\frac{c}{r_m}$. Since $r_m\to\infty$, we obtain $\lim_m\lim_n y_m^*(\frac{x+y_n}{2})=1$. By (2′), x is a CWALUR point of $B(X)$, so $\{y_n\}_{n=1}^\infty$ is relatively weakly compact.

(1) $\Leftrightarrow$ (3′) For every $x^{**} \in S(X^{**})$, we have $\{x_n^*\}_{n=1}^\infty \subset S(X^*)$ with

$$\lim_n x^{**}(x_n^*) = \lim_n x_n^*(x) = 1.$$

Since

$$1 = \lim_n \frac{x^{**}(x_n^*) + x_n^*(x)}{2} \leq \left\| \frac{x + x^{**}}{2} \right\| \leq 1,$$

we conclude that $\frac{\|x+x^{**}\|}{2} = 1$. By (1), we have $x^{**} \in X$.

Conversely, for any $x^{**} \in S(X^{**})$, $\|x^{**}\| = \frac{\|x+x^{**}\|}{2} = 1$, we can take $\{x_n^*\}_{n=1}^\infty \subset S(X^*)$ such that $\lim_n(\frac{x+x^{**}}{2})(x_n^*) = 1$. Clearly, $|x^{**}(x_n^*)| \leq 1$, $|x_n^*(x)| \leq 1$. Consequently, we obtain $\lim_n x^{**}(x_n^*) = \lim_n x_n^*(x) = 1$, and by (3′), we have $x^{**} \in X$.

(1) $\Leftrightarrow$ (4′) Suppose that $\{\frac{x+x_n^{**}}{2}\}_{n=1}^\infty$ is asymptotically normed by $B(X^*)$. Then, for any $m \geq 1$, there exists $\{x_m^*\}_{m=1}^\infty \subset B(X^*)$ and $N_m \in \mathbb{N}$ such that $x_m^*(\frac{x+x_n^{**}}{2}) > 1 - \frac{1}{m}$ for all $n \geq N_m$. Therefore,

$$\lim_m \lim_n x_m^* \Big(\frac{x + x_n^{**}}{2} \Big) = 1.$$

For any w^*-cluster point x^{**} of $\{x_n^{**}\}_{n=1}^\infty$, we have $\|\frac{x+x^{**}}{2}\| = 1$. By (1), we obtain $x^{**} \in X$.

Conversely, in order to complete (4′) $\Rightarrow$ (1), we just need to prove that (4′) $\Rightarrow$ (3) of Theorem 2.1. If $\{B_n^*\}_{n=1}^\infty = \{B_n(x_n^*, r_n)\}_{n=1}^\infty$ is an unbounded nested sequence of balls such that x is bounded below on $\bigcup B_n^*$, if for $\{y_n^{**}\}_{n=1}^\infty \subset S(X^{**})$ the sequence $\{\inf y_n^{**}(B_n^*)\}_{n=1}^\infty$ is also bounded below, and if $c \in \mathbb{R}$ is a common lower bound, then the sequence $\{\inf(\frac{x+y_n^{**}}{2})(B_n^*)\}_{n=1}^\infty$ is also bounded below by c. We may assume that $0 \in B_1^*$. Let $y_n^* = \frac{x_n^*}{r_n}$. The fact that $\{B_n^*\}_{n=1}^\infty$ is nested implies that $y_m^*(x) > 1 + \frac{c}{r_m}$ for all m and that $y_n^{**}(y_m^*) \geq 1 + \frac{c}{r_m}$ for all $n \geq m$. It follows that $(\frac{x+y_n^{**}}{2})(y_m^*) \geq 1 + \frac{c}{r_m}$ for all $n \geq m$. This means that $\{\frac{x+y_n^{**}}{2}\}_{n=1}^\infty$ is asymptotically normed by $B(X^*)$. By (4′), we obtain that all w^*-cluster points of $\{y_n^{**}\}_{n=1}^\infty$ belong to X. $\square$

3. Applications of Three Types of Nearly Convex Points

It is well known that if a closed convex set C in X is approximatively weakly compact for some $x \in X \setminus C$, then $P_C(x) \neq \emptyset$. However, its converse is not generally true (see [3, Example 2.3]). In this section, we will give a series of equivalent (or sufficient) conditions such that the reversed conclusion holds by using three types of nearly convex points and S points.

Fang and Wang in [4] proved that a Banach space X is nearly strongly convex if and only if every convex and proximinal subset of X is approximatively compact. Guirao and Montesinos in [6] improved Fang and Wang's result by showing that X is nearly strongly convex if and only if every proximinal hyperplane of X is approximatively compact. Inspired by these conclusions, we give the following theorems.

Theorem 3.1. *Let X be a Banach space, with $x \in X$. Then the following are equivalent.*

(1) *For every closed convex subset $C \subset X \setminus \{x\}$ and $c \in P_C(x)$, $\frac{c-x}{\|c-x\|}$ is a nearly very convex point.*
(2) *For every closed convex subset $C \subset X \setminus \{x\}$ and $P_C(x) \neq \emptyset$, C is approximatively weakly compact for x.*
(3) *For every closed subspace $Y \subset X \setminus \{x\}$ and $P_Y(x) \neq \emptyset$, Y is approximatively weakly compact for x.*
(4) *For every hyperplane $H \subset X \setminus \{x\}$ and $P_H(x) \neq \emptyset$, H is approximatively weakly compact for x.*

Proof. (1) $\Rightarrow$ (2) Assume that $\frac{c-x}{\|c-x\|}$ is a nearly very convex point, where $c \in P_C(x)$. Let $\{x_n\}_{n=1}^{\infty} \subset C$, $\lim_n \|x_n - x\| = \operatorname{dist}(x, C) = d$. By the *separation theorem*, there exists a $x^* \in S(X^*)$ such that

$$\sup\{x^*(y-x) : y \in \overline{B(x,d)}\} \leq \inf\{x^*(y-x) : y \in C\}.$$

Therefore,

$$d = \|c-x\| = x^*(c-x) \leq x^*(x_n - x) \leq \|x_n - x\| \to d,$$

so $x^* \in D(\frac{c-x}{\|c-x\|})$ and $x^*(\frac{x_n-x}{\|x_n-x\|}) \to 1$. Since $\frac{c-x}{\|c-x\|}$ is a nearly very convex point, $\{\frac{x_n-x}{\|x_n-x\|}\}_{n=1}^{\infty}$ is relatively weakly compact, and hence $\{x_n\}$ is relatively weakly compact.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (4) This is clear.

(4) $\Rightarrow$ (1) Let $c \in P_C(x)$ and $x^* \in S(X^*)$ such that $x^*(c-x) = \|c-x\|$. Let $H = \{y \in X : x^*(y-x) = \|c-x\|\}$, $\{x_n\}_{n=1}^{\infty}$ be a sequence in $B(X)$ such that $x^*(x_n) \to 1$. Put $y_n = \|c-x\|x_n + x + \lambda_n(c-x)$, where $\lambda_n = 1 - x^*(x_n)$, and then $x^*(y_n - x) = \|c-x\|$ for all $n \in \mathbb{N}$ and $\lim \lambda_n \to 0$. Moreover,

$$\begin{aligned}\|c-x\| = x^*(y_n - x) &\leq \|y_n - x\| \leq \|c-x\|\|x_n\| + |\lambda_n|\|c-x\| \\ &\leq \|c-x\| + |\lambda_n|\|c-x\| \to \|c-x\|.\end{aligned}$$

Hence $\|y_n - x\| \to \|c-x\|$, i.e., $\|x-y_n\| \to \operatorname{dist}(x,H)$. By (4), H is approximatively weakly compact for x, and we conclude that $\{y_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence, and then so does $\{x_n\}_{n=1}^{\infty}$. This proves that $\{x_n\}_{n=1}^{\infty}$ is relatively weakly compact. □

Lemma 3.2. *Let X be a Banach space, with $x \in S(X)$. Then x is a nearly very convex point if $x^{**} \in X$ for any $x^{**} \in S(X^{**})$ and $x^* \in S(X^*)$ with $x^{**}(x^*) = x^*(x) = 1$.*

Proof. Let$\{x_n\}_{n=1}^{\infty} \subset S(X)$ with $x^*(x_n) \to 1$ for some $x^* \in D(x)$. Let C be the set of all w^*-cluster points of $\{x_n\}_{n=1}^{\infty}$. For any $x^{**} \in C \setminus \{x_n\}_{n=1}^{\infty}$, there exists a subnet $\{x_{n(\alpha)}\}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n(\alpha)} \xrightarrow{w^*} x^{**}$. Clearly, $x^*(x_{n(\alpha)}) \to 1$. This shows that $x^{**}(x^*) = 1 = x^*(x)$. By Lemma 3.2, we know that $x^{**} \in X$. Hence, $\{x_n\}_{n=1}^{\infty}$ is relatively weakly compact. □

By equivalence of (1) and (3′) in Theorem 2.2 and Lemma 3.2, we obtain that if a point $x \in S(X)$ is a nearly rotund point of $S(X^{**})$, then x is a nearly very convex point of $B(X)$. Therefore, by Theorem 3.1, we can deduce the following corollary.

Corollary 3.3. *Let X be a Banach space. For any $x \in X$ and closed convex subset $C \subset X \setminus \{x\}$ with $c \in P_C(x)$, and if $\frac{c-x}{\|c-x\|}$ is a nearly rotund point of $B(X^{**})$, then C is approximatively weakly compact for x.*

It is easy to prove that $x^* \in D(\frac{c-x}{\|c-x\|})$ is a WS point, which implies that $\frac{c-x}{\|c-x\|}$ is nearly very convex point of X. By Theorem 3.1, we can obtain the following corollary.

Corollary 3.4. *For any $x \in X$ and closed convex subset $C \subset X \setminus \{x\}$ with $c \in P_C(x)$, if $x^* \in D(\frac{c-x}{\|c-x\|})$ is a WS point of X^*, then C is approximatively weakly compact for x.*

In a way similar to that used in Theorem 3.1, we can prove the following.

Theorem 3.5. *Let X be a Banach space, with $x \in X$. Then the following are equivalent.*

(1) *For every closed convex subset $C \subset X \setminus \{x\}$ and $c \in P_C(x)$, $\frac{c-x}{\|c-x\|}$ is a nearly strongly convex point.*
(2) *For every closed convex subset $C \subset X \setminus \{x\}$ and $P_C(x) \neq \emptyset$, C is approximatively compact for x.*
(3) *For every closed subspace $Y \subset X \setminus \{x\}$ and $P_Y(x) \neq \emptyset$, Y is approximatively compact for x.*
(4) *For every hyperplane $H \subset X \setminus \{x\}$ and $P_H(x) \neq \emptyset$, H is approximatively compact for x.*

Lemma 3.6 (see [8])**.** *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in Banach space X and let $\varepsilon > 0$ be a positive real number. If $\{x_n\}$ has no finite ε-net, then for any positive integers $n_0 > 0$, $k > 0$, and any $x \in X$, there exist $n_1, n_2, \ldots, n_{k+1} > n_0$ such that*

$$d\big(x_{n_1}, \operatorname{span}\{x\}\big) > \frac{\varepsilon}{2}, \qquad d\big(x_{n_{i+1}}, \operatorname{span}\{x, x_{n_1}, x_{n_2}, \ldots, x_{n_i}\}\big) > \frac{\varepsilon}{2},$$

for $i = 1, 2, \ldots, k-1$.

Lemma 3.7. *Let $x \in S(X)$. Then x is a nearly strongly convex point of X if and only if $x^* \in D(x)$ is an S point of X^**

Proof. Necessity case: Let $x \in S(X)$, $x^* \in D(x)$. Suppose that x^* is not an S point of X^*. Then there exists a sequence $\{F_n\}_{n=1}^{\infty} \subset S(X^{**})$ with $F_n(x^*) \to 1$ which is not relatively compact. By the preceding Lemma 3.6, there exist $F_1^{(n)}, F_2^{(n)}, \ldots, F_{n+1}^{(n)} \in \{F_n\}_{n=1}^{\infty}$ such that

$$d\big(F_{l+1}^{(n)}, \operatorname{span}\{F_1^{(n)}, \ldots, F_l^{(n)}\}\big) > \frac{\varepsilon}{2}, \quad l = 1, 2, \ldots, n; n = 1, 2, \ldots.$$

Let $E_n^{**} = \operatorname{span}\{F_1^{(n)}, F_2^{(n)}, \ldots, F_{(n+1)}^{(n)}\}$, $E^* = \operatorname{span}\{x^*\}$. By locally reflexive principle, there exists a linear mapping $T_n : E_n^{**} \to X$ satisfying

$$\Big(1-\frac{1}{n}\Big)\|F\| \le \big\|T_n(F)\big\| \le \Big(1+\frac{1}{n}\Big)\|F\|, \quad \forall F \in E_n^{**}, \tag{3.1}$$

$$x^*\big(T_n(F)\big) = F(x^*), \quad \forall F \in E_n^{**}, \tag{3.2}$$

$$T_n(x) = x, \quad \forall x \in X \cap E_n^{**}. \tag{3.3}$$

Set $x_l^{(n)} = \frac{T_n(F_l^{(n)})}{\|T_n(F_l^{(n)})\|}$, $l = 1, 2, \ldots, n+1$, $n = 1, 2, 3, \ldots$. Then $x_l^{(n)} \in S(X)$. In view of (3.1) and (3.2), we have

$$x^*(x_l^{(n)}) = x^*\Big(\frac{T_n(F_l^{(n)})}{\|T_n(F_l^{(n)})\|}\Big) = \frac{F_l^{(n)}(x^*)}{\|T_n(F_l^{(n)})\|} \to 1$$

as $n \to \infty$.

Moreover, for any $\alpha_1, \ldots, \alpha_l \in R$,

$$\begin{aligned}
&\|x_{l+1}^{(n)} - \alpha_1 x_1^{(n)} - \cdots - \alpha_l x_l^{(n)}\| \\
&= \Big\|\frac{T_n(F_{l+1}^{(n)})}{\|T_n(F_{l+1}^{(n)})\|} - \alpha_1\frac{T_n(F_1^{(n)})}{\|T_n(F_1^{(n)})\|} - \cdots - \alpha_l\frac{T_n(F_l^{(n)})}{\|T_n(F_l^{(n)})\|}\Big\| \\
&= \frac{1}{\|T_n(F_{l+1}^{(n)})\|}\Big\|T_n\Big(F_{l+1}^{(n)} - \alpha_1\frac{\|T_n(F_{l+1}^{(n)})\|}{\|T_n(F_1^{(n)})\|}F_1^{(n)} - \cdots - \alpha_l\frac{\|T_n(F_{l+1}^{(n)})\|}{\|T_n(F_l^{(n)})\|}F_l^{(n)}\Big)\Big\| \\
&\ge \frac{1-n^{-1}}{1+n^{-1}} d\big(F_{l+1}^{(n)}, \operatorname{span}\{F_1^{(n)}, \ldots, F_l^{(n)}\}\big).
\end{aligned}$$

Therefore, for n large enough,

$$d\big(x_{l+1}^{(n)}, \operatorname{span}\{x_1^{(n)}, \ldots, x_l^{(n)}\}\big) \ge \frac{\varepsilon}{2}.$$

Since for every $l, x^*(x_l^{(n)}) \to 1$, and since x is a nearly strongly convex point, $\{x_l^{(n)}\}_{n=1}^{\infty}$ is relatively compact. By the *diagonal process*, we can select a subsequence $\{n_k\} \subset \{n\}$ such that, for every l, we have that $\{x_l^{(n_k)}\}_{k\ge1}$ is convergent. Let $x_l^{(n_k)} \to x_l$, $l = 1, 2, \ldots$.

On one hand, noting that $x^*(x_l) = 1$ $(\forall l)$, we obtain that $\{x_l\}_{l\ge1}$ is relatively compact. On the other hand,

$$d\big(x_{l+1}, \operatorname{span}\{x_1, \ldots, x_l\}\big) \ge \liminf_{n\to\infty} d\big(x_{l+1}^{(n)}, \operatorname{span}\{x_1^{(n)}, \ldots, x_l^{(n)}\}\big) \ge \frac{\varepsilon}{2}.$$

This leads to a contradiction.

Sufficiency case: This is trivial. □

By Theorem 3.5 and Lemma 3.7, we have the following theorem.

Theorem 3.8. *Let X be a Banach space, with $x \in X$. For any closed convex subset $C \subset X \setminus \{x\}$ and $c \in P_C(x)$, we have that $x^* \in D(\frac{c-x}{\|c-x\|})$ is an S point of X^* if and only if C is approximatively compact for x.*

Finally, we give a characterization of closed convex subset C to be approximatively compact for some x in $X \setminus C$ by using nested sequence of balls.

The S point (resp., WS point) is said to be P-II point (resp., P-III point) in [1].

Lemma 3.9 ([1, Proposition 5.5]). *For a Banach space $X, x \in S(X)$ is a P-II point (resp., P-III point) if and only if for every straight unbounded nested sequence $\{B_n\}_{n=1}^{\infty}$ of balls in the direction of x, for any $\{y_n^*\}_{n=1}^{\infty} \subset S(X^*)$, if the sequence $\{\inf y_n^*(B_n\}_{n=1}^{\infty}$ is bounded below, then the sequence $\{y_n^*\}_{n=1}^{\infty}$ is relatively compact (resp., relatively weakly compact).*

By Lemma 3.9, Corollary 3.4, and Theorem 3.8, we can obtain the following results.

Theorem 3.10. *Let X be a Banach space, let $C \subset X$ be a closed convex subset, and let $x \in X \setminus C$ and $c \in P_C(x)$. Then we have the following.*

(1) *For every straight unbounded nested sequence $\{B_n^*\}_{n=1}^{\infty}$ of balls in the direction of $x^* \in D(\frac{c-x}{\|c-x\|})$, for any $\{y_n^{**}\}_{n=1}^{\infty} \subset S(X^{**})$, if the sequence $\{\inf y_n^{**}(B_n^*)\}_{n=1}^{\infty}$ is bounded below, then $\{y_n^{**}\}_{n=1}^{\infty}$ is relatively compact if and only if C is approximatively compact for x.*

(2) *For every straight unbounded nested sequence $\{B_n^*\}_{n=1}^{\infty}$ of balls in the direction of $x^* \in D(\frac{c-x}{\|c-x\|})$, for any $\{y_n^{**}\}_{n=1}^{\infty} \subset S(X^{**})$, if that the sequence $\{\inf y_n^{**}(B_n^*)\}_{n=1}^{\infty}$ is bounded below implies that $\{y_n^{**}\}_{n=1}^{\infty}$ is relatively weakly compact, then C is approximatively weakly compact for x.*

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