Ann. Funct. Anal. 7 (2016), no. 1, 9–16
http://dx.doi.org/10.1215/20088752-3158134
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

HOMOGENEOUS SPACES AND SQUARE-INTEGRABLE REPRESENTATIONS

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Dedicated to Professor Anthony To-Ming Lau

Communicated by O. Christensen

Abstract. For a locally compact group G and a compact subgroup H of G, the square-integrable representations of group G and homogeneous space G/H are described. Also, the connection between the existence of admissible wavelets for locally compact groups and their homogeneous spaces is compared. Moreover, some properties of admissible wavelets and wavelet constants for homogeneous space G/H are investigated, when G is unimodular.

1. Introduction and preliminaries

For a locally compact group G, it is well known that a continuous unitary representation π of G is called *square integrable* if there exists a nonzero vector ζ in Hilbert space $\mathcal{H}$ such that

$$\int_G \left|\left\langle \pi(x)\zeta, \zeta \right\rangle\right|^2 d\lambda(x) < \infty,$$

in which λ is a left Haar measure. Such a unital vector ζ is called an *admissible vector*. The wavelet constant associated to the admissible wavelet is denoted by c_ζ and defined by

$$c_\zeta = \int_G \left|\left\langle \zeta, \pi(x)\zeta \right\rangle\right|^2 d\lambda(x).$$

Received Sep. 30, 2014; Accepted Dec. 22, 2014.
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2010 *Mathematics Subject Classification.* Primary 43A15; Secondary 43A85, 65T60.
Keywords. homogeneous space, square integrable, admissible wavelet, relatively invariant measure, strongly quasi-invariant measure.

(For more details about admissible wavelets on locally compact groups, see [15], [10] and [12]). The square-integrable representations of homogeneous spaces that admit G-invariant measure and relatively invariant measure have been studied in [2] and [6]. In this paper, we investigate the relation between square-integrable representations of locally compact group G and its homogeneous space G/H, in which H is a compact subgroup of G. To be more precise, we need to fix some notation and review some basic concepts (see also [4], [7], [8], [14]).

Let G be a locally compact group, and let H be a closed subgroup of G with left Haar measures λ and λ_H, respectively. Consider G/H as a homogeneous space on which G acts from the left. A Radon measure μ on G/H is said to be *G-invariant* if $\mu_x(yH) = \mu(yH)$ for all $x, y \in G$, where μ_x is defined by $\mu_x(E) = \mu(xE)$ (for Borel subsets E of G/H). It is well known that there is a G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$, where Δ_G, Δ_H are the modular functions on G and H, respectively. In this case, we have

$$\int_G f(x)\, d\lambda(x) = \int_{G/H} Pf(xH)\, d\mu(xH) = \int_{G/H}\int_H f(xh)\, d\lambda_H(h)\, d\mu(xH), \quad (1.1)$$

in which $Pf(xH) = \int_H f(xh)\, d\lambda_H(h)$ is a continuous linear map from $C_c(G)$ onto $C_c(G/H)$.

A Radon measure μ on G/H is called *strongly quasi-invariant* if there is a positive continuous function α of $G \times G/H$ such that $d\mu_x(yH) = \alpha(x, yH)d\mu(yH)$, for all $x, y \in G$. If the functions $\alpha(x, .)$ reduce to constants, then the measure μ is called a *relatively invariant measure*.

A rho-function for the pair (G, H) is defined to be a continuous positive function ρ from G which satisfies

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x) \quad (x \in G, h \in H).$$

In [11] it was shown that the existence of a homomorphism rho-function for the pair (G, H) is a necessary and sufficient condition for the existence of a relatively invariant measure μ on G/H.

In this paper, we introduce square-integrable representations of homogeneous space G/H equipped with a relatively invariant measure μ. The main aim of this paper is to compare square-integrable representations of locally compact groups and their homogeneous spaces. We show the relation between admissible wavelets on groups and their homogeneous spaces. Finally, we describe admissible wavelets and wavelet constants on homogeneous space G/H, when G is a unimodular group.

2. Main results

Let G be a locally compact group, let H be a closed subgroup of G, and let $\mathcal{H}$ be a Hilbert space. Now, we define representations of homogeneous spaces which will be needed in the definition of square-integrable representations of homogeneous spaces.

Definition 2.1. *A unitary representation of the homogeneous space* G/H is a map ϖ from G/H into the group $U(\mathcal{H})$, of all unitary operators on some nonzero Hilbert space $\mathcal{H}$, for which the map $xH \mapsto \langle \varpi(xH)\zeta, \xi\rangle$ from G/H into $\mathbb{C}$ is continuous, for each $\zeta, \xi \in \mathcal{H}$ and

$$\varpi(xyH) = \varpi(xH)\varpi(yH), \qquad \varpi(x^{-1}H) = \varpi(xH)^*,$$

for each $x, y \in G$. Moreover, a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be *invariant* with respect to ϖ if $\varpi(xH)\mathcal{M} \subseteq \mathcal{M}$, for all $x \in G$. A unitary representation ϖ is said to be *irreducible* if the only invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$.

In Theorem 2.5, we intend to establish a relation between square-integrable representations of locally compact groups and their homogeneous spaces. It is worthwhile to note that a unitary representation of homogenous space G/H is in one-to-one correspondence to a unitary representation of G, whose kernel π, which is denoted by N, contains H. We denote by q_N, q_H, p, the canonical mappings of G onto G/N, of G onto G/H, and of N onto N/H. Let λ_N and λ_H be the left Haar measures on N and H, respectively. Then there exists a G-invariant measure $\mu_{N/H}$ on N/H. On the other hand, if λ is a left Haar measure on G, then one can find a G-invariant measure $\mu_{G/N}$ on quotient group G/N and a relatively invariant measure $\mu_{G/H}$ on homogeneous space G/H which arises from rho-function ρ of G. Then it is easy to check that the mapping $(x, n) \mapsto q_H(xn)$ of $G \times N$ into G/H is continuous. Since $q_H(xnh) = q_H(xn)$, for all $h \in H$, this mapping defines a continuous mapping of $G \times (N/H)$ into G/H. Whence, for each fixed $x \in G$, the mapping ψ_x of N into G, such that $\psi_x(n) = xn$, defines a mapping ω_x of N/H into G/H in which

$$\omega_x\big(p(n)\big) = q_H\big(\psi_x(n)\big) = q_H(xn).$$

It is easy to show that $\psi_{xn} = \psi_x o \varrho_N(n)$, and therefore that $\omega_{xn} = \omega_x o \varrho_{N/H}(n)$, for all $n \in N$, in which $\varrho_N(n)(n') = nn'$. The following lemma shows that the map ω_x is proper.

Lemma 2.2. *Let E be a compact subset of G/H, and let K be a compact subset of G. Then $\bigcup_{x\in K} \omega_x^{-1}(E)$ is relatively compact in N/H. In particular, $\bigcup_{x\in K} \omega_x^{-1}(E)$ is contained in a compact subset of N/H.*

Proof. Let F be a compact subset of G such that $q_H(F) = E$. Let L be the set of $n \in N$ such that Kn intersects F. Then L is compact (see [3, Chapter III, Section 4.5, Theorem 1]). Let $n \in N$, such that $p(n) \in \bigcup_{x\in K} \omega_x^{-1}(E)$. Thus there exists $x \in K$ such that $\omega_x(p(n)) \in E$, that is, $q_H(xn) \in E$, and since $q_H(F) = E$, there exists $h \in H$, $xnh \in F$. Then $nh \in L$. So $p(nh) = p(n) \in p(L)$; that is, $\bigcup_{x\in K} \omega_x^{-1}(E) \subseteq p(L)$. □

Let $\mathcal{M}(N/H)$ and $\mathcal{M}(G/H)$ be complex measure spaces on homogeneous spaces N/H and G/H, respectively, as introduced in [4] and [14]. Lemma 2.2 shows that the mapping ω_x is proper. Then ω_x extends continuously to a map from $\mathcal{M}(N/H)$ into $\mathcal{M}(G/H)$ ([9, Section 4.5]). Now let $\varphi \in C_c(G/H)$. Define the function Ψ of G into $\mathcal{M}(G/H)$ such that

$$\Psi(x) = \big\langle \varphi, \omega_x(\mu_{N/H})\big\rangle = \int_{N/H} \varphi(\omega_x\big(p(n)\big)\, d\mu_{N/H}\big(p(n)\big).$$

The function Ψ is continuous with compact support. Moreover, since the measure $\mu_{N/H}$ is G-invariant, we have

$$\begin{aligned}
\Psi(xn) &= \big\langle \varphi, \omega_{xn}(\mu_{N/H})\big\rangle \\
&= \big\langle \varphi, \omega_x o \varrho_{N/H}(n)(\mu_{N/H})\big\rangle \\
&= \int_{N/H} \varphi\big(\omega_x o \varrho_{N/H}(n)\big(p(n')\big)\big)\, d\mu_{N/H}\big(p(n')\big) \\
&= \int_{N/H} \varphi\big(\omega_x(nn'H)\big)\, d\mu_{N/H}\big(p(n')\big) \\
&= \int_{N/H} \varphi\big(\omega_x(n'H)\big)\, d\mu_{N/H}\big(p(n')\big) \\
&= \big\langle \varphi, \omega_x(\mu_{N/H})\big\rangle \\
&= \Psi(x),
\end{aligned}$$

for $n \in N$. Then the mapping $\tilde{\Psi}$ of G/N into $\mathcal{M}(G/H)$, in which

$$\tilde{\Psi}\big(q_N(x)\big) = \big\langle \varphi, \omega_x(\mu_{N/H})\big\rangle, \tag{2.1}$$

is continuous with compact support, for all $\varphi \in C_c(G/H)$.

Proposition 2.3. *Let $\varphi \in C_c(G/H)$. Then*

$$\int_{G/N} \big\langle \varphi, \omega_x(\mu_{N/H})\big\rangle\, d\mu_{G/N}\big(q_N(x)\big) = \int_{G/H} \varphi\big(q_H(x)\big)\, d\mu_{G/H}\big(q_H(x)\big).$$

Proof. By (1.1), for $\varphi \in C_c(G/H)$, we have

$$\int_{G/H} \varphi\big(q_H(x)\big)\, d\mu_{G/H}\big(q_H(x)\big) = \int_G f(x)\, d\lambda(x),$$

where $\varphi = Pf$ and $f \in C_c(G)$. Also,

$$\begin{aligned}
&\int_{G/N} \big\langle \varphi, \omega_x(\mu_{N/H})\big\rangle\, d\mu_{G/N}\big(q_N(x)\big) \\
&= \int_{G/N}\int_{N/H} (Pf(\omega_x\big(p(n)\big)\, d\mu_{N/H}\big(p(n)\big)\, d\mu_{G/N}\big(q_N(x)\big) \\
&= \int_{G/N}\int_{N/H} (Pf\big(q_H(xn)\big)\, d\mu_{N/H}\big(p(n)\big)\, d\mu_{G/N}\big(q_N(x)\big) \\
&= \int_{G/N}\int_{N/H}\int_H L_x f(nh)\, d\lambda_H(h)\, d\mu_{N/H}\big(p(n)\big)\, d\mu_{G/N}\big(q_N(x)\big)
\end{aligned}$$

$$= \int_{G/N} \int_N f(xn)\, d\lambda_N(n)\, d\mu_{G/N}\big(q_N(x)\big)$$
$$= \int_G f(x)\, d\lambda(x),$$

in which $L_x f(n) = f(xn)$. $\square$

Corollary 2.4. (i) *Let φ be a $\mu_{G/H}$-integrable function on G/H. There exists a $\mu_{G/N}$-negligible subset E of G/N having the following property: if $x \in G$ is such that $q_N(x) \notin E$, then the function $\varphi o \omega_x$ on N/H is $\mu_{N/H}$-integrable. The integral $\int_{N/H} \varphi(\omega_x(p(n)))\, d\mu_{N/H}(p(n))$ is a $\mu_{G/N}$-integrable function, and*

$$\begin{aligned}\int_{G/H} &\varphi\big(q_H(x)\big)\, d\mu_{G/H}\big(q_H(x)\big) \\ &= \int_{G/N} d\mu_{G/N}\big(q_N(x)\big) \int_{N/H} \varphi\big(\omega_x\big(p(n)\big)\big)\, d\mu_{N/H}\big(p(n)\big).\end{aligned} \tag{2.2}$$

(ii) *Suppose that there exists a bounded positive measure $\mu_{G/H}$ on homogenous space G/H. Then there exists a bounded positive measure on homogeneous space N/H.*

Proof. (i) By Proposition 2.3 we have,

$$\begin{aligned}\int_{G/H} &\varphi\big(q_H(x)\big)\, d\mu_{G/H}\big(q_H(x)\big) \\ &= \int_{G/N} \big\langle \varphi, \omega_x(\mu_{N/H})\big\rangle\, d\mu_{G/N}\big(q_N(x)\big) \\ &= \int_{G/N} \int_{N/H} \Big(\varphi\big(\omega_x\big(p(n)\big)\, d\mu_{N/H}\big(p(n)\big)\big)\Big)\, d\mu_{G/N}\big(q_N(x)\big).\end{aligned}$$

(ii) The function 1 on G/H is $\mu_{G/H}$-integrable. By part (i), the function 1 on N/H is $\mu_{N/H}$-integrable. Thus $\mu_{N/H}$ is bounded. $\square$

As mentioned earlier, a unitary representation ϖ of homogeneous space G/H defines an associated representation π of G such that $\ker \pi = N$ contains H. Consider $\tilde{\pi}$ the representation of G/N by letting $\tilde{\pi}(xN) = \pi(x)$. An irreducible representation $\tilde{\pi}$ of G/N on $\mathcal{H}$ is said to be *square integrable* if there exists a nonzero element $\zeta \in \mathcal{H}$ such that

$$\int_{G/N} \big|\big\langle \zeta, \tilde{\pi}(xN)\zeta\big\rangle\big|^2\, d\mu_{G/N}(xN) < \infty. \tag{2.3}$$

If ζ satisfies (2.3), then it is called an *admissible vector*. An admissible vector $\zeta \in \mathcal{H}$ is said to be an *admissible wavelet* if $\|\zeta\| = 1$. In this case, we define the wavelet constant c_ζ as follows:

$$c_\zeta = \int_{G/N} \big|\big\langle \zeta, \tilde{\pi}(xN)\zeta\big\rangle\big|^2\, d\mu_{G/N}(xN).$$

The following theorem shows that the representation ϖ of G/H is square integrable if and only if the representation $\tilde{\pi}$ of quotient group G/N is square integrable.

Theorem 2.5. *Let ϖ be a unitary representation of G/H. The unitary representation ϖ of G/H is square integrable if and only if the representation $\tilde{\pi}$ of quotient group G/N is square integrable.*

Proof. Assume that the representation ϖ is irreducible. Let π be the associated representation of G, and let N be the kernel of representation π. The representation ϖ is irreducible if and only if π is irreducible or, equivalently, if the representation $\tilde{\pi}$ is irreducible. Then, by Corollary 2.4 and the fact that the G-invariant measure $\mu_{N/H}$ is finite (see [13]), we have

$$\int_{G/H} \varphi(xH)\, d\mu_{G/H}(xH) = \mu(N/H) \int_{G/N} \varphi(xN)\, d\mu_{G/N}(xN),$$

in which $\varphi(xH) = |\langle \varpi(xH)\zeta, \zeta\rangle|^2$. So, ϖ is a square-integrable representation of G/H if and only if π induces a representation $\tilde{\pi}$ of quotient group G/N where $\tilde{\pi}$ is square integrable. □

We may conclude that there exists an admissible wavelet for representation ϖ of homogeneous space G/H if and only if there exists an admissible wavelet for induced representation $\tilde{\pi}$ of quotient group G/N.

Corollary 2.6. *A unital vector ζ in Hilbert space $\mathcal{H}$ is an admissible wavelet for representation ϖ of homogeneous space G/H if and only if ζ in $\mathcal{H}$ is an admissible wavelet for representation $\tilde{\pi}$ of quotient group G/N.*

In the following, we obtain some interesting results on the admissible wavelets and wavelet constants for homogeneous space G/H, when G is unimodular group.

Theorem 2.7. *Let G be a unimodular group, and let ϖ be an irreducible representation of homogeneous space G/H. The set of all admissible wavelets for G/H is empty or is a Hilbert space $\mathcal{H}$.*

Proof. Unimodularity of group G implies that G/N is a unimodular group. Also, the representation $\tilde{\pi}$ of quotient group G/N is irreducible since ϖ of G/H is irreducible. By [1, Lemma 8.1.2], the set of admissible wavelets on G/N is empty or $\mathcal{H}$. Therefore, by Corollary 2.6, the set of admissible wavelets on G/H is empty or $\mathcal{H}$. □

Lemma 2.8. *Let G be a unimodular group, and let ξ, ζ be two admissible wavelets for a square-integrable representation ϖ of G/H on Hilbert spaces $\mathcal{H}$. Then*

$$c_{\xi,\zeta} = \langle \xi, \zeta\rangle c_\zeta,$$

in which $c_{\xi,\zeta}$ is a two-wavelet constant defined as

$$c_{\xi,\zeta} = \int_{G/N} \big\langle \xi, \tilde{\pi}(xN)\xi\big\rangle \big\langle \tilde{\pi}(xN)\zeta, \xi\big\rangle\, d\mu_{G/N}\big(q_N(x)\big).$$

Proof. Using the unimodularity of G/N, we have

$$\begin{aligned} c_{\xi,\zeta} &= \int_{G/N} \big\langle \xi, \tilde{\pi}(xN)\xi \big\rangle \big\langle \tilde{\pi}(xN)\zeta, \xi \big\rangle \, d\mu_{G/N}\big(q_N(x)\big) \\ &= \int_{G/N} \big\langle \tilde{\pi}(x^{-1}N)\xi, \xi \big\rangle \big\langle \zeta, \tilde{\pi}(x^{-1}N)\xi \big\rangle \, d\mu_{G/N}\big(q_N(x)\big) \\ &= \int_{G/N} \big\langle \tilde{\pi}(xN)\xi, \xi \big\rangle \big\langle \zeta, \tilde{\pi}(xN)\xi \big\rangle \, d\mu_{G/N}\big(q_N(x^{-1})\big) \\ &= \langle \xi, \zeta \rangle \int_{G/N} \big\langle \tilde{\pi}(xN)\xi, \xi \big\rangle \big\langle \xi, \tilde{\pi}(xN)\xi \big\rangle \, d\mu_{G/N}\big(q_N(x)\big) \\ &= \langle \xi, \zeta \rangle c_\zeta. \end{aligned}$$

□

The following theorem shows that the wavelet constants for two admissible wavelets for homogeneous space G/H is the same, when G is unimodular.

Theorem 2.9. *Let G be a unimodular group, and let ξ, ζ be two admissible wavelets for a square-integrable representation ϖ of G/H on Hilbert spaces $\mathcal{H}$. Then $c_\xi = c_\zeta$.*

Proof. By Theorem 2.8, $c_{\xi,\zeta} = \langle \xi, \zeta \rangle c_\zeta$ and $c_{\zeta,\xi} = \langle \zeta, \xi \rangle c_\xi$. But

$$\langle \xi, \zeta \rangle c_\zeta = c_{\xi,\zeta} = \overline{c_{\zeta,\xi}} = \overline{\langle \zeta, \xi \rangle} c_\xi = \langle \xi, \zeta \rangle c_\xi.$$

Thus $c_\xi = c_\zeta$, if $\langle \xi, \zeta \rangle \neq 0$. Now, suppose that $\langle \xi, \zeta \rangle = 0$. Let $\eta \in \mathcal{H}$ such that $\|\eta\| = 1$ and $\langle \eta, \zeta \rangle \neq 0$, $\langle \eta, \xi \rangle \neq 0$. Since G is unimodular, the vector η is an admissible wavelet for ϖ of G/H. Thus $c_\zeta = c_\eta$ and $c_\eta = c_\xi$. Hence $c_\xi = c_\zeta$. □

The following example supports our technical considerations.

Example 2.10. Consider the Euclidean group $G = SO(n) \times_\tau \mathbb{R}^n$ with group operations

$$(R_1, p_1) \cdot (R_2, p_2) = (R_1 R_2, R_1 p_2 + p_1), \qquad (R, p)^{-1} = (R^{-1}, -R^{-1} p).$$

Put $n = 2$ in G; that is, $G = SO(2) \times_\tau \mathbb{R}^2$ and $\mathcal{H} = L^2(S^1) \simeq L^2[-\pi, \pi]$. In this setting any $R \in SO(2)$ and $s \in S^1$ are given explicitly by

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$
$$s = \begin{pmatrix} \sin\gamma \\ \cos\gamma \end{pmatrix}.$$

The representation π on G, is defined as

$$\pi(\theta, p_1, p_2)\psi(\gamma) = e^{i(p_1 \sin\gamma + p_2 \cos\gamma)} \psi(\gamma - \theta),$$

for all $(\theta, p_1, p_2) \in G$, $\psi \in L^2(S^1)$. Since this representation of G is not square integrable (see [5]), we are looking for a suitable representation of its homogeneous space. Consider $H = \{(0, 0, p_2) \in G\}$. Thus the representation of G/H is square integrable (see [6]). By Theorem 2.5 the representation ϖ of homogeneous space G/H is square integrable if and only if the representation $\tilde{\pi}$ of quotient group G/N is square integrable, where $N = \ker \pi = \{(0, -p_2 \cot\gamma, p_2), p_2 \in \mathbb{R}\}$.

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