Ann. Funct. Anal. 8 (2017), no. 4, 512–519
http://dx.doi.org/10.1215/20088752-2017-0015
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

A QUANTITATIVE VERSION OF THE JOHNSON–ROSENTHAL THEOREM

DONGYANG CHEN*

Communicated by D. Leung

Abstract. Let X, Y be Banach spaces. We define

$$\alpha_Y(X) = \sup\big\{\|T^{-1}\|^{-1} : T : Y \to X \text{ is an isomorphism with } \|T\| \le 1\big\}.$$

If there is no isomorphism from Y to X, we set $\alpha_Y(X) = 0$, and

$$\gamma_Y(X) = \sup\big\{\delta(T) : T : X \to Y \text{ is a surjective operator with } \|T\| \le 1\big\},$$

where $\delta(T) = \sup\{\delta > 0 : \delta B_Y \subseteq TB_X\}$. If there is no surjective operator from X onto Y, we set $\gamma_Y(X) = 0$. We prove that for a separable space X, $\alpha_{l_1}(X^*) = \gamma_{c_0}(X)$ and $\alpha_{L_1}(X^*) = \gamma_{C(\Delta)}(X) = \gamma_{C[0,1]}(X)$.

1. Introduction and preliminaries

This note is motivated by much of the recent research on quantitative versions of various theorems and properties of Banach spaces (see [7] and its references). Our main goal in this note is to prove quantitative versions of two well-known theorems in the isomorphic theory of Banach spaces: the Johnson–Rosenthal theorem and the Bessaga–Pełczyński theorem. The Johnson–Rosenthal theorem [5, Theorem IV.3] says that, for a separable space X,

(1) c_0 is isomorphic to a quotient of X whenever X^* contains a (closed) subspace isomorphic to l_1,

Received Oct. 5, 2016; Accepted Jan. 6, 2017.
First published online Jun. 22, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46B15; Secondary 46C05.
Keywords. Johnson–Rosenthal theorem, Bessaga–Pełczyński theorem, quantitative versions, isomorphisms, Banach spaces.

(2) $C(\Delta)$ is isomorphic to a quotient of X whenever X^* contains a subspace isomorphic to L_1, where $\Delta = \{0,1\}^{\mathbb{N}}$ is the Cantor set.

To quantify the Johnson–Rosenthal theorem, we define a quantity measuring how well a Banach space is isomorphically embedded into another Banach space as follows: let X, Y be Banach spaces. We define

$$\alpha_Y(X) = \sup\big\{\|T^{-1}\|^{-1} : T : Y \to X \text{ is an isomorphism with } \|T\| \leq 1\big\}.$$

If there is no isomorphism from Y to X, we set $\alpha_Y(X) = 0$. Obviously, $\alpha_Y(X) = 1$ if and only if X contains almost-isometric copies of Y.

The classical Banach–Mazur distance $d(X, Y)$ between Banach spaces X and Y, which measures how well a Banach space is isomorphic to another Banach space, is defined as follows:

$$d(X, Y) = \inf\big\{\|T\|\|T^{-1}\| : T : X \to Y \text{ is a surjective isomorphism}\big\}.$$

If X is not isomorphic to Y, we set $d(X, Y) = \infty$. It should be mentioned that there are close relationships between quantity $\alpha_Y(X)$ and the Banach–Mazur distance $d(X, Y)$. For example, it is easy to see that

$$\alpha_Y(X) = \big[\inf\big\{d(Y, M) : M \text{ is a closed subspace of } X\big\}\big]^{-1}.$$

We also define a quantity measuring how well a Banach space is isomorphic to a quotient of another Banach space, as follows. Let X, Y be Banach spaces. We set

$$\gamma_Y(X) = \sup\big\{\delta(T) : T : X \to Y \text{ is a surjective operator with } \|T\| \leq 1\big\},$$

where $\delta(T) = \sup\{\delta > 0 : \delta B_Y \subseteq T B_X\}$. If there is no surjective operator from X onto Y, we set $\gamma_Y(X) = 0$. It is easy to see that $\gamma_Y(X) = 1$ if and only if Y is a $(1+\epsilon)$-(linear) quotient of X for every $\epsilon > 0$.

Then, using the above quantities, we quantify the Johnson–Rosenthal theorem as follows.

Theorem 1.1. *Let X be a separable Banach space. Then*

(a) $\alpha_{l_1}(X^*) = \gamma_{c_0}(X)$,
(b) $\alpha_{L_1}(X^*) = \gamma_{C(\Delta)}(X) = \gamma_{C[0,1]}(X)$.

The Bessaga–Pełczyński theorem [2] states that for a Banach space X, X^* contains a subspace isomorphic to c_0 if and only if X contains a complemented subspace isomorphic to l_1 if and only if X^* contains a subspace isomorphic to l_∞. To quantify Bessaga–Pełczyński theorem, we also need a quantity measuring how close a Banach space is to being isomorphic to a complemented subspace of another Banach space.

Let X, Y be Banach spaces. We set

$$\beta_Y(X) = \sup\big\{\big(\|A\|\|B\|\big)^{-1} : A : X \to Y, B : Y \to X \text{ are operators such that } AB = I_Y\big\}.$$

If there are no such operators A, B, then we set $\beta_Y(X) = 0$. Clearly, $\beta_Y(X) = 1$ if and only if for every $\epsilon > 0$ there exists a subspace M of X so that M is

$(1+\epsilon)$-isomorphic to Y and M is $(1+\epsilon)$-complemented in X. That is, $M = BY$ is complemented in X, and $BA : X \to M$ is the projection.

In this note, we quantify the Bessaga–Pełczyński theorem as follows.

Theorem 1.2. *Let X be a Banach space. Then*

$$\big(\alpha_{c_0}(X^*)\big)^2 \leq \beta_{l_1}(X) \leq \alpha_{l_\infty}(X^*) \leq \alpha_{c_0}(X^*).$$

Throughout this article, an *operator* will always mean a *bounded linear operator*. An operator $T : X \to Y$ is called an *isomorphism* if it is one-to-one and has closed range. If X is a Banach space, we denote by B_X its closed unit ball $\{x \in X : \|x\| \leq 1\}$. Also, $I_X : X \to X$ denotes the identity map. Our notation and terminology are standard, and we refer the readers to [1] and [8] for any unexplained terms.

2. Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. (a) Step 1: $\alpha_{l_1}(X^*) \leq \gamma_{c_0}(X)$.

Let $0 < c < \alpha_{l_1}(X^*)$ be arbitrary. Then there exists a sequence $(x_n^*)_n$ in X^* so that

$$c\sum_{k=1}^{n}|a_k| \leq \Big\|\sum_{k=1}^{n} a_k x_k^*\Big\| \leq \sum_{k=1}^{n}|a_k| \tag{2.1}$$

for all scalars $a_1, a_2, \ldots, a_n$ and all $n \in \mathbb{N}$. Since X is separable, by passing to subsequences, we may assume that $(x_n^*)_n$ is w^*-convergent. We set $f_n = x_{2n-1}^* - x_{2n}^*$ $(n \in \mathbb{N})$. According to (2.1), we get

$$2c\sum_{k=1}^{n}|a_k| \leq \Big\|\sum_{k=1}^{n} a_k f_k\Big\| \leq 2\sum_{k=1}^{n}|a_k|, \tag{2.2}$$

for all scalars $a_1, a_2, \ldots, a_n$ and all $n \in \mathbb{N}$.

Now let $g_n = \frac{f_n}{\|f_n\|}$ $(n \in \mathbb{N})$. Then $(g_n)_n$ is a w^*-null sequence in S_{X^*}. In view of (2.2), we get

$$c\sum_{k=1}^{n}|a_k| \leq \Big\|\sum_{k=1}^{n} a_k g_k\Big\| \leq \sum_{k=1}^{n}|a_k|, \tag{2.3}$$

for all scalars $a_1, a_2, \ldots, a_n$ and all $n \in \mathbb{N}$.

Define an operator $T : X \to c_0$ by $Tx = (\langle g_n, x\rangle)_n$ $(x \in X)$. Then $T^* e_n^* = g_n$ for all $n \in \mathbb{N}$. It follows from (2.3) that T^* is an isomorphism, and hence T is surjective. Let $Z = X/\ker(T)$. Define an operator $\widehat{T} : Z \to c_0$ by $\widehat{T}([x]) = Tx$ for $[x] \in Z$. Then $\widehat{T}$ is a surjective isomorphism. For each $n \in \mathbb{N}$, we choose $z_n \in Z$ with $\widehat{T} z_n = e_n$. Let $(z_n^*)_n$ denote the functionals biorthogonal to $(z_n)_n$. Since $(z_n)_n$ is a shrinking basis for Z, $(z_n^*)_n$ forms a basis for Z^*. Let $Q : X \to X/\ker(T)$ be the quotient mapping. It is easy to see that $Q^* z_n^* = g_n$ for each $n \in \mathbb{N}$. Since Q^* is an isometric embedding, it follows from (2.3) that for all scalars $b_1, b_2, \ldots, b_n$ and all $n \in \mathbb{N}$, we have

$$c\sum_{k=1}^{n}|b_k| \leq \Big\|\sum_{k=1}^{n} b_k z_k^*\Big\| \leq \sum_{k=1}^{n}|b_k|. \tag{2.4}$$

We claim that $cB_{c_0} \subseteq \widehat{T}B_Z$. Let $(t_n)_n \in B_{c_0}$. Set $z = \sum_{n=1}^{\infty} t_n z_n$. By the Hahn–Banach theorem, we choose $z^* \in S_{Z^*}$ with $\langle z^*, z\rangle = \|z\|$. According to (2.4), we get

$$\begin{aligned}\|z\| = \langle z^*, z\rangle &= \sum_{n=1}^{\infty} t_n \langle z^*, z_n\rangle \\ &\leq \sum_{n=1}^{\infty} \big|\langle z^*, z_n\rangle\big| \\ &\leq \frac{1}{c}\Big\|\sum_{n=1}^{\infty}\langle z^*, z_n\rangle z_n^*\Big\| \\ &= \frac{1}{c}.\end{aligned}$$

Let $\epsilon > 0$. Clearly, $B_Z \subseteq (1+\epsilon)QB_X$. By the claim, we get $\frac{c}{1+\epsilon}B_{c_0} \subseteq TB_X$. Consequently, $\gamma_{c_0}(X) \geq \frac{c}{1+\epsilon}$. Letting $\epsilon \to 0$, we get $\gamma_{c_0}(X) \geq c$. Since c is arbitrary, we conclude Step 1.

Step 2: $\gamma_{c_0}(X) \leq \alpha_{l_1}(X^*)$.

Fix any $0 < c < \gamma_{c_0}(X)$. Then there is an operator $T : X \to c_0$ so that $\|T\| \leq 1$ and $cB_{c_0} \subseteq TB_X$. This means that $\|T^* z\| \geq c\|z\|$ for all $z \in l_1$. Thus, $\alpha_{l_1}(X^*) \geq c$, and we are done.

(b) Step 1: $\alpha_{L_1}(X^*) \leq \gamma_{C(\Delta)}(X)$.

Fix any $0 < c < \alpha_{L_1}(X^*)$. Then there exist a subspace N of X^* and an isomorphism T from N onto L_1 such that

$$\|x^*\| \leq \|Tx^*\| \leq \frac{1}{c}\|x^*\|, \quad x^* \in N. \tag{2.5}$$

By the proof of [5, Theorem IV.3], we obtain a sequence $(x_n^*)_n$ in N such that $(Tx_n^*)_n$ is isometrically equivalent to the Haar basis $(h_n)_n$ for L_1. Moreover, the operator $S : X \to (\overline{\operatorname{span}}\{x_n^* : n \in \mathbb{N}\})^*$ defined by

$$\langle Sx, x^*\rangle = \langle x^*, x\rangle\big(x \in X, x^* \in \overline{\operatorname{span}}\{x_n^* : n \in \mathbb{N}\}\big)$$

satisfies the following properties:

(i) $SX = Z$, where $Z = \overline{\operatorname{span}}\{u_n : n \in \mathbb{N}\}$ and $(u_n)_n$ are the functionals biorthogonal to $(x_n^*)_n$,

(ii) $\overline{SB_X} = B_Z$.

Let $(h_n^*)_n$ denote the functionals biorthogonal to $(h_n)_n$. It is known that $(h_n^*)_n$ is 1-equivalent to the Haar basis of $C(\Delta)$. Let $U : \overline{\operatorname{span}}\{h_n^* : n \in \mathbb{N}\} \to C(\Delta)$ be a surjective linear isometry. By (2.5), we have

$$\Big\|\sum_{k=1}^{n} a_k x_k^*\Big\| \leq \Big\|\sum_{k=1}^{n} a_k Tx_k^*\Big\| = \Big\|\sum_{k=1}^{n} a_k h_k\Big\| \leq \frac{1}{c}\Big\|\sum_{k=1}^{n} a_k x_k^*\Big\|, \tag{2.6}$$

for all scalars $a_1, a_2, \ldots, a_n$ and all $n \in \mathbb{N}$.

Define an operator $V : \overline{\text{span}}\{x_n^* : n \in \mathbb{N}\} \to L_1$ by $Vx_n^* = h_n (n \in \mathbb{N})$. Then V is a surjective isomorphism and hence $V^* h_n^* = u_n$ for all $n \in \mathbb{N}$. According to (2.6), an easy computation shows

$$\Big\|\sum_{k=1}^{n} a_k h_k^*\Big\| \leq \Big\|\sum_{k=1}^{n} a_k u_k\Big\| \leq \frac{1}{c}\Big\|\sum_{k=1}^{n} a_k h_k^*\Big\|, \tag{2.7}$$

for all scalars $a_1, a_2, \ldots, a_n$ and all $n \in \mathbb{N}$.

Define an operator $A : Z \to \overline{\text{span}}\{h_n^* : n \in \mathbb{N}\}$ by $Au_n = h_n^* (n \in \mathbb{N})$. It follows from (2.7) that $cB_{\overline{\text{span}}\{h_n^*:n\in\mathbb{N}\}} \subseteq AB_Z$. Let $\epsilon > 0$. Similarly, by (ii), we get $B_Z \subseteq (1+\epsilon)SB_X$. Combining this together with the fact that U is a surjective linear isometry, we get $cB_{C(\Delta)} \subseteq (1+\epsilon)UASB_X$. Moreover, combining (ii) and (2.7), we get $\|UAS\| \leq 1$. Thus, we have $\gamma_{C(\Delta)}(X) \geq \frac{c}{1+\epsilon}$. Letting $\epsilon \to 0$, we get $\gamma_{C(\Delta)}(X) \geq c$. Since c is arbitrary, we finish the proof of Step 1.

Step 2: $\gamma_{C(\Delta)}(X) \leq \gamma_{C[0,1]}(X)$.

Let $0 < c < \gamma_{C(\Delta)}(X)$ be arbitrary. Then there exists an operator $T : X \to C(\Delta)$ with $\|T\| \leq 1$ so that $cB_{C(\Delta)} \subseteq TB_X$. By [1, Lemma 4.4.7], there is a continuous surjection $\psi : \Delta \to [0,1]$ so that we can find a norm 1 operator $R : C(\Delta) \to C[0,1]$ with $R(f \circ \psi) = f$ for $f \in C[0,1]$. This yields $B_{C[0,1]} \subseteq RB_{C(\Delta)}$. Consequently, $cB_{C[0,1]} \subseteq RTB_X$. In view of $\|R\| = 1$, we get $\gamma_{C[0,1]}(X) \geq c$. By the arbitrariness of c, Step 2 is concluded. □

Finally, since L_1 is linearly isometric to a subspace of $C[0,1]^*$, it is easy to verify that $\gamma_{C[0,1]}(X) \leq \alpha_{L_1}(X^*)$.

Corollary 2.1. *If c_0 is isomorphic to a quotient of a Banach space X, then c_0 is a $(1+\epsilon)$-quotient of X for every $\epsilon > 0$.*

Proof. First, we will consider the case where X is a separable Banach space having a quotient isomorphic to c_0. Then X^* contains a subspace isomorphic to l_1. It follows from James's l_1-distortion theorem that $\alpha_{l_1}(X^*) = 1$. According to Theorem 1.1(a), we get $\gamma_{c_0}(X) = 1$.

For the general case, suppose that the quotient space X/M is isomorphic to c_0. It follows from the separable case that $\gamma_{c_0}(X/M) = 1$. Let $\epsilon > 0$ be arbitrary. Then there exists an operator $T : X/M \to c_0$ with $\|T\| \leq 1$ so that $(1-\epsilon)B_{c_0} \subseteq TB_{X/M}$. Since $B_{X/M} \subseteq (1+\epsilon)Q_M B_X$, it follows that $(1-\epsilon)B_{c_0} \subseteq (1+\epsilon)TQ_M B_X$, where $Q_M : X \to X/M$ is the natural quotient map. Therefore, $\gamma_{c_0}(X) \geq \frac{1-\epsilon}{1+\epsilon}$. Letting $\epsilon \to 0$, we also get $\gamma_{c_0}(X) = 1$. The proof is completed. □

Combining the main results of [9, Theorem 3.4] and [10, Theorem 1], one can deduce the following corollary (see the proof of [6, Theorem 2.1]). But, this corollary is also an immediate consequence of Theorem 1.1(b).

Corollary 2.2. *If $C[0,1]$ (or $C(\Delta)$) is isomorphic to a quotient of a Banach space X, then $C[0,1]$ (resp., $C(\Delta)$) is a $(1+\epsilon)$-quotient of X for every $\epsilon > 0$.*

Proof. If $C[0,1]$ is isomorphic to a quotient of a separable Banach space X, then X^* contains a subspace isomorphic to L_1. It follows from [3, Theorem 4] that $\alpha_{L_1}(X^*) = 1$. By Theorem 1.1(b), we get $\gamma_{C[0,1]}(X) = 1$. For the general case, the argument is analogous to Corollary 2.1. □

Now we need a quantitative version of the *Bessaga–Pełczyński selection principle*. Its proof is identical to the standard gliding hump arguments (see [1]).

Lemma 2.3. *Let $(x_n)_n$ be a basis for a Banach space X, and let $(x_n^*)_n$ be the sequence of coefficient functionals. If $(y_n)_n$ is a seminormalized sequence in X satisfying $\lim_{n\to\infty}\langle x_k^*, y_n\rangle = 0$ for each $k \in \mathbb{N}$, then, for every $\epsilon > 0$, there exist a subsequence $(y_{k_n})_n$ of $(y_n)_n$ and a (skipped) block basic sequence $(z_n)_n$ with respect to $(x_n)_n$ such that*

$$(1-\epsilon)\Big\|\sum_{i=1}^{n} a_i z_i\Big\| \le \Big\|\sum_{i=1}^{n} a_i y_{k_i}\Big\| \le (1+\epsilon)\Big\|\sum_{i=1}^{n} a_i z_i\Big\|,$$

for all scalars $a_1, a_2, \ldots, a_n$ and all $n \in \mathbb{N}$. If every seminormalized (skipped) block basic sequence with respect to $(x_n)_n$ is C-complemented in X (where the constant C depends only on X), then $\overline{\operatorname{span}}\{y_{k_n} : n \in \mathbb{N}\}$ is $(C \cdot \frac{1+\epsilon}{1-\epsilon})$-complemented in X.

Proof of Theorem 1.2. We only prove $(\alpha_{c_0}(X^*))^2 \le \beta_{l_1}(X)$. Inequalities $\beta_{l_1}(X) \le \alpha_{l_\infty}(X^*)$ and $\alpha_{l_\infty}(X^*) \le \alpha_{c_0}(X^*)$ are straightforward.

Let $0 < c < \alpha_{c_0}(X^*)$ be arbitrary. Then there exists an operator $T : c_0 \to X^*$ with $\|T\| \le 1$ so that $c\|z\| \le \|Tz\|$ for all $z \in c_0$. This yields

$$cB_{l_1} \subseteq T^* B_{X^{**}} \subseteq \overline{T^* B_X}^{w^*}. \tag{2.8}$$

Let S be the restriction of T^* to X. Then $Sx = (\langle Te_n, x\rangle)_n$ for all $x \in X$.

Let $\epsilon > 0$. According to (2.8), we obtain a sequence $(x_n)_n$ in X so that $\|x_n\| \le \frac{1}{c}$ for each n and so that

$$\big|\langle e_n^* - Sx_n, e_k\rangle\big| < \frac{\epsilon}{2^n}, \quad k = 1, 2, \ldots, n; n = 1, 2, \ldots. \tag{2.9}$$

By (2.9), we get $\lim_{n\to\infty}\langle Sx_n, e_k\rangle = 0$ for each k and $1-\epsilon \le \|Sx_n\| \le \frac{1}{c}$ for all n.

It follows from Lemma 2.3 that there exist a subsequence $(Sx_{k_n})_n$ of $(Sx_n)_n$ and a block basic sequence $(u_n)_n$ of $(e_n^*)_n$ so that

$$(1-\epsilon)\Big\|\sum_{i=1}^{n} a_i u_i\Big\| \le \Big\|\sum_{i=1}^{n} a_i Sx_{k_i}\Big\| \le (1+\epsilon)\Big\|\sum_{i=1}^{n} a_i u_i\Big\| \tag{2.10}$$

for all scalars $a_1, a_2, \ldots, a_n$ and all $n \in \mathbb{N}$. Moreover, $\overline{\operatorname{span}}\{Sx_{k_n} : n \in \mathbb{N}\}$ is $\frac{1+\epsilon}{1-\epsilon}$-complemented in l_1. By (2.10), we have $\frac{1-\epsilon}{1+\epsilon} \le \|u_n\| \le \frac{1}{c(1-\epsilon)}$ for each n. Since $(u_n)_n$ is a block basic sequence of $(e_n^*)_n$, we get

$$\frac{1-\epsilon}{1+\epsilon}\sum_{i=1}^{n} |a_i| \le \Big\|\sum_{i=1}^{n} a_i u_i\Big\| \le \frac{1}{c(1-\epsilon)}\sum_{i=1}^{n} |a_i| \tag{2.11}$$

for all scalars $a_1, a_2, \ldots, a_n$ and all $n \in \mathbb{N}$. Combining (2.10) with (2.11), we get

$$\frac{(1-\epsilon)^2}{1+\epsilon}\sum_{i=1}^{n} |a_i| \le \Big\|\sum_{i=1}^{n} a_i Sx_{k_i}\Big\| \le \Big\|\sum_{i=1}^{n} a_i x_{k_i}\Big\| \le \frac{1}{c}\sum_{i=1}^{n} |a_i|, \tag{2.12}$$

for all scalars $a_1, a_2, \ldots, a_n$ and all $n \in \mathbb{N}$.

Let $M = \overline{\operatorname{span}}\{x_{k_n} : n \in \mathbb{N}\}$. It follows from (2.12) that $S|_M : M \to l_1$ is an isomorphism with $\|(S|_M)^{-1}\| \le \frac{1+\epsilon}{c(1-\epsilon)^2}$. Let P be a projection from l_1 onto

$\overline{\text{span}}\{Sx_{k_n} : n \in \mathbb{N}\}$ with $\|P\| \le \frac{1+\epsilon}{1-\epsilon}$. Then $(S|_M)^{-1}PS$ is a projection from X onto M with $\|(S|_M)^{-1}PS\| \le \frac{(1+\epsilon)^2}{c(1-\epsilon)^3}$. Define an operator $U : M \to l_1$ by $Ux_{k_n} = e_n^*$ $(n \in \mathbb{N})$. According to (2.12), U is a surjective isomorphism with $\|U\| \le \frac{1+\epsilon}{(1-\epsilon)^2}$.

Finally, we define operators $A : X \to l_1$ by $A = U(S|_M)^{-1}PS$ and $B : l_1 \to X$ by $Be_n^* = x_{k_n}$ $(n \in \mathbb{N})$. Then $AB = I_{l_1}$ and $\|A\|\|B\| \le \frac{(1+\epsilon)^3}{c^2(1-\epsilon)^5}$. Thus, we get

$$\beta_{l_1}(X) \ge \big(\|A\|\|B\|\big)^{-1} \ge \frac{c^2(1-\epsilon)^5}{(1+\epsilon)^3}.$$

Letting $\epsilon \to 0$, we get $\beta_{l_1}(X) \ge c^2$. The proof is completed. □

The following corollary is due to Dowling, Randrianantoanina, and Turett [4]. As an immediate application of Theorem 1.2, we give a short proof.

Corollary 2.4 ([4, Theorem 5]). *If a Banach space X contains a complemented subspace isomorphic to l_1, then, for every $\epsilon > 0$, there exists a subspace M of X so that M is $(1+\epsilon)$-isomorphic to l_1 and M is $(1+\epsilon)$-complemented in X.*

Proof. If X contains a complemented subspace isomorphic to l_1, it follows from the Bessaga–Pełczyński theorem [2] that X^* contains a subspace isomorphic to c_0. By James's c_0-distortion theorem, $\alpha_{c_0}(X^*) = 1$. According to Theorem 1.2, we get $\beta_{l_1}(X) = 1$. The proof is completed. □

Acknowledgments. This work was done during the author's visit to the Department of Mathematics at Texas A&M University. I would like to thank Professor W. B. Johnson for helpful discussions and comments. The author also thanks the anonymous referees for useful suggestions.

The author's work was partially supported by Natural Science Foundation of Fujian Province of China grant 2015J01026.

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School of Mathematical Sciences, Xiamen University, Xiamen, 361005, People's Republic of China.

E-mail address: cdy@xmu.edu.cn