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ON THE WEAK CONVERGENCE THEOREM FOR NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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Abstract. Assume that K is a closed convex subset of a uniformly convex Banach space E, and assume that $\{T(s)\}_{s>0}$ is a nonexpansive semigroup on K. By using the following implicit iteration sequence $\{x_n\}$ defined by

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n \cdot \frac{1}{t_n}\int_0^{t_n} T(s)x_n \, ds, \quad \forall n \geq 1,$$

the main purpose of this paper is to establish a weak convergence theorem for the nonexpansive semigroup $\{T(s)\}_{s>0}$ in uniformly convex Banach spaces without the Opial property. Our results are different from some recently announced results.

1. Introduction

Suppose that E is a real Banach space and that K is a nonempty subset of E. A 1-parameter family $\mathfrak{T} = \{T(t) : 0 \leq t < \infty\}$ of Lipschitz operators from K into itself is said to be a *semigroup of Lipschitz operators* on K if it satisfies the following conditions:

(i) $T(0)x = x$, $\forall x \in K$;
(ii) $T(t+s)x = T(t)T(s)(x)$ for each $t, s \geq 0$ and $x \in K$;
(iii) for each $x \in K$, the mapping $t \mapsto T(t)x$ is continuous on $[0, \infty)$;

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(iv) for $\tau > 0$ there exists $L_\tau > 0$ such that $\|T(t)x - T(t)y\| \leq L_\tau \|x - y\|$ for each $t \in [0, \tau]$ and $x, y \in K$.

A Lipschitz semigroup $\mathfrak{T}$ is said to be *nonexpansive* if $L_\tau = 1$ for all $\tau > 0$. Let $F(\mathfrak{T})$ be the common fixed point set of the semigroup $\mathfrak{T}$; that is, $F(\mathfrak{T}) = \{x \in K : T(s)x = x, \forall s > 0\}$.

The fixed point method has a large number of applications in many areas, such as in optimization theory, control theory, economics, and nonlinear analysis. Iterative approximation construction of fixed points is vigorously proposed and analyzed for various classes of maps in different spaces. We point out that an implicit process is generally desirable when no explicit scheme is available. Such a process is generally used as a "tool" to establish the convergence of an explicit scheme. Additionally, implicit algorithms provide better approximation of fixed points than explicit algorithms (see [12], [13]).

Iterative approximation techniques of fixed points of nonexpansive mappings (and of common fixed points of nonexpansive semigroups) is an important subject in nonlinear operator theory and its applications, in particular, in image recovery and signal processing (see [3], [9]).

Several authors (e.g., see Khan et al. [8], Yang et al. [14], and Zeidler [15]) discussed the weak or strong convergence of implicit iterative approximation for nonlinear mappings. Under a real uniformly convex Banach space with the Opial property or with a Fréchet differentiable norm, Khan et al. [8], Yang et al. [14], and Zeidler [15] studied the weak convergence in mean ergodic theorems. However, many important spaces like L_p for $1 \leq p \neq 2$ do not possess the Opial property.

Inspired and motivated by the above results, we aim in this paper to better understand the weak convergence of an implicit iteration approximation for nonexpansive semigroups $\{T(t) : 0 \leq t < \infty\}$ in uniformly convex Banach spaces without Opial property. Hence the present article opens a new research direction.

First, we will collect some well-known concepts and results. Let E be a Banach space with dimension $E \geq 2$, and let E^* be its dual. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \frac{1}{2}\|x + y\| : \|x\| = \|y\| = 1, \|x - y\| = \varepsilon\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

A sequence $\{x_n\}$ in a normed linear space E is said to converge weakly to x if

$$\lim_{n\to\infty} f(x_n) = f(x)$$

for every f in the dual space E^* of E. This relation is indicated by $x_n \rightharpoonup x$. Let $\varpi_w(\{x_n\}) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denote the weak limit set of $\{x_n\}$ and $x_{n_j} \subset \{x_n\}$.

A Banach space is said to have the Kadec–Klee property (see [6]) if, whenever $x \in \varpi_w(\{x_n\})$ with $\lim_{n\to\infty} \|x_n\| = \|x\|$, it then follows that $\lim_{n\to\infty} x_n = x$ strongly.

Lemma 1.1 ([4, Lemma 2.7]). *Suppose that K is a closed convex subset of a uniformly convex Banach space E. Let $\mathfrak{T} = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive* *semigroup on K such that $F(\mathfrak{T})$ is nonempty. Then for each $r > 0$ and $h > 0$,*

$$\lim_{t\to\infty} \sup_{x\in K\cap B_r} \Big\| \frac{1}{t}\int_0^t T(s)x\,ds - T(h)\frac{1}{t}\int_0^t T(s)x\,ds \Big\| = 0,$$

where B_r denotes the closed ball in E with center 0 and radius r.

Lemma 1.2 (see [10]). *Let E be a uniformly convex Banach space, and let a, b be two constants with $0 < a < b < 1$. Suppose that $\{\alpha_n\}$ is a real sequence in $[a, b]$ and that $\{\rho_n\}$, $\{\tau_n\}$ are two sequences in E such that*

$$\begin{cases} \limsup_{n\to\infty} \|\rho_n\| \le c, \\ \limsup_{n\to\infty} \|\tau_n\| \le c, \\ \lim_{n\to\infty} \|\alpha_n\rho_n + (1-\alpha_n)\tau_n\| = c. \end{cases}$$

Then $\lim_{n\to\infty} \|\rho_n - \tau_n\| = 0$, where $c \ge 0$ is some constant.

Lemma 1.3 ([7, Lemma 2]). *Assume that E is a real reflexive Banach space such that its dual E^* has the Kadec–Klee property. Let $\{x_n\}$ be a bounded sequence in E, with p_1, $p_2 \in \varpi_w(\{x_n\})$. Suppose that $\lim_{n\to\infty} \|\alpha x_n + (1-\alpha)p_1 - p_2\|$ exists for all $\alpha \in [0, 1]$. Then $p_1 = p_2$.*

Lemma 1.4 ([11, Lemma 2]). *Let two fixed real numbers be $q > 1$ and $D > 0$. Then a Banach space E is uniformly convex if and only if there is a strictly increasing, continuous, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\big\|\lambda x + (1-\lambda)y\big\|^q \le \lambda\|x\|^q + (1-\lambda)\|y\|^q - \omega_q(\lambda)g\big(\|x-y\|\big)$$

for all x, $y \in B_D$ and $\lambda \in [0, 1]$, where B_D is the closed ball with center zero and radius D, $\omega_q(\lambda) = \lambda(1-\lambda)^q + \lambda^q(1-\lambda)$.

Lemma 1.5 ([1, Theorem 2.3.7]). *Let E be a Banach space with modulus of convexity of δ_E. Then $\frac{\delta_E(s)}{s}$ is a nondecreasing function on $(0, 2]$.*

Lemma 1.6 (see [2]). *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let T be a nonexpansive mapping of K into itself with $F(T) \neq \phi$. Let $\{x_n\} \subset K$ be a sequence such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\{x_n\}$ converges weakly to z. Then z is a fixed point of T.*

2. Implicit iterative approximation

Let K be a closed convex subset of a uniformly convex Banach space E, and let $\mathfrak{T} = \{T(t) : 0 \le t < \infty\}$ be a nonexpansive semigroup on K. For $x_0 \in K$, $n \ge 1$, compute the implicit iteration process $\{x_n\}$ defined by the following formula:

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n \cdot \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds, \tag{2.1}$$

where $\{\alpha_n\} \subset (0, 1)$ is bounded away from 0 and 1 and $\{t_n\} \subset (0,\infty)$.

For $n \in \mathbb{N} = \{1, 2, \ldots\}$, $u, v \in K$, define

$$Q_n^v(u) = (1-\alpha_n)v + \alpha_n \cdot \frac{1}{t_n}\int_0^{t_n} T(s)u\,ds. \tag{2.2}$$

For $u_1, u_2 \in K$, we have

$$\begin{aligned}\big\|Q_n^v(u_1) - Q_n^v(u_2)\big\| &= \Big\|\alpha_n \cdot \frac{1}{t_n}\int_0^{t_n}\big(T(s)u_1 - T(s)u_2\big)\,ds\Big\| \\ &\le \alpha_n \cdot \frac{1}{t_n}\int_0^{t_n}\big\|T(s)u_1 - T(s)u_2\big\|\,ds \\ &\le \alpha_n\|u_1 - u_2\|.\end{aligned}$$

This implies that each $Q_n^v(u) : K \to K$ is a contraction. It follows from the Banach contraction principle that each x_n in (2.1) is uniquely defined.

The next two results deal with the general behavior of the implicit iterative processes of (2.1).

Lemma 2.1. *Let E be a Banach space, let K be a nonempty convex subset of E, and let a, b be two constants with $0 < a < b < 1$. Let $\mathfrak{T}$ be a nonexpansive semigroup on K, and let $F(\mathfrak{T}) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (2.1) with $a \le \alpha_n \le b$ for all $n \ge 1$, $t_n \to \infty$ as $n \to \infty$ and $p \in F(\mathfrak{T})$. Then*

(i) *there exists $c \ge 0$ such that $\lim_{n\to\infty}\|x_n - p\| = c$,*

(ii) *$\lim_{n\to\infty}\|x_n - T(s)x_n\| = 0$ for every $s > 0$.*

Proof. (i) It follows from (2.1) that

$$\begin{aligned}\|x_n - p\| &= \Big\|(1-\alpha_n)(x_{n-1} - p) + \alpha_n \cdot \frac{1}{t_n}\int_0^{t_n}\big(T(s)x_n - T(s)p\big)\,ds\Big\| \\ &\le (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \cdot \frac{1}{t_n}\int_0^{t_n}\big\|T(s)x_n - T(s)p\big\|\,ds \\ &\le (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n\|x_n - p\|. \qquad (2.3)\end{aligned}$$

It follows from (2.3) that $\|x_n - p\| \le \|x_{n-1} - p\|$. Therefore, there exists $c \ge 0$ such that $\lim_{n\to\infty}\|x_n - p\| = c$.

(ii) Let $K_1 = \{z \in K : \|z - p\| \le \tau_0\}$. Then K_1 is a nonempty bounded closed convex subset of K, and $T(s)$-invariant. Since $\{x_n\} \subset K_1$ and K_1 is bounded, there exists $\tau > 0$ such that $K_1 \subset B_\tau$. It follows from Lemma 1.1 that

$$\lim_{n\to\infty}\Big\|T(s)\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - \frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds\Big\| = 0. \qquad (2.4)$$

Since

$$\begin{aligned}\Big\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - p\Big\| &= \Big\|\frac{1}{t_n}\int_0^{t_n}\big(T(s)x_n - T(s)p\big)\,ds\Big\| \\ &\le \frac{1}{t_n}\int_0^{t_n}\big\|T(s)x_n - T(s)p\big\|\,ds \\ &\le \|x_n - p\|,\end{aligned}$$

then

$$\limsup_{n\to\infty}\Big\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - p\Big\| \le c. \qquad (2.5)$$

Since $\lim_{n\to\infty}\|x_n-p\| = \lim_{n\to\infty}\|\alpha_n(x_{n-1}-p)+(1-\alpha_n)(\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds-p)\| = c$ and observe $\lim_{n\to\infty}\|x_{n-1}-p\| = c$, it follows from (2.5) and Lemma 1.2 that

$$\lim_{n\to\infty}\Big\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - x_{n-1}\Big\| = 0. \tag{2.6}$$

Note that $x_n - x_{n-1} = (1-\alpha_n)(\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - x_{n-1})$. Therefore, by (2.6) we can conclude that

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{2.7}$$

Since

$$\begin{aligned}\lim_{n\to\infty}\Big\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - x_n\Big\| \le{}& \lim_{n\to\infty}\Big\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - x_{n-1}\Big\| \\ &+ \lim_{n\to\infty}\|x_{n-1}-x_n\|,\end{aligned}$$

it follows from (2.6) and (2.7) that we have

$$\lim_{n\to\infty}\Big\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - x_n\Big\| = 0. \tag{2.8}$$

Note that

$$\begin{aligned}&\big\|x_n - T(s)x_n\big\| \\ &\quad= \Big\|\Big(x_n - \frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds\Big) + \Big(\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - T(s)\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds\Big) \\ &\qquad+ \Big(T(s)\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - T(s)x_n\Big)\Big\| \\ &\quad\le \Big\|x_n - \frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds\Big\| + \Big\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - T(s)\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds\Big\| \\ &\qquad+ \Big\|T(s)\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - T(s)x_n\Big\| \\ &\quad\le 2\Big\|x_n - \frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds\Big\| \\ &\qquad+ \Big\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds - T(s)\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds\Big\|.\end{aligned}$$

Then it follows from (2.4) and (2.8) that $\lim_{n\to\infty}\|x_n - T(s)x_n\| = 0$. This completes the proof. □

3. Main results

In this section, we will first prove the demiclosed principle for a nonexpansive semigroup in uniformly convex Banach spaces. Since the dual of reflexive Banach spaces with the Opial property or a Fréchet differentiable norm has the Kadec–Klee property, our theorems generalize the known ones.

Lemma 3.1. *Let K be a closed and convex subset of a uniformly convex Banach space E, and let a, b be two constants with $0 < a < b < 1$. Let $\mathfrak{T}$ be a nonexpansive semigroup on K, $F(\mathfrak{T}) \neq \emptyset$, and let $\{x_n\}$ be defined by (2.1) with $a \leq \alpha_n \leq b$ for all $n \geq 1$, $t_n \to \infty$ as $n \to \infty$. Then for ω_1, $\omega_2 \in F(\mathfrak{T})$, the limit $\lim_{n\to\infty} \|tx_n + (1-t)\omega_1 - \omega_2\|$ exists for all $t \in [0,1]$.*

Proof. Let $a_n(t) = \|tx_n + (1-t)\omega_1 - \omega_2\|$. Then $\lim_{n\to\infty} a_n(0) = \|\omega_1 - \omega_2\|$ exists. It follows from Lemma 2.1(i) that $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - \omega_2\|$ exists. It now remains to prove the lemma for $t \in (0,1)$. Define

$$Q_n^{\omega}(u) = \alpha_n \cdot \frac{1}{t_n} \int_0^{t_n} T(s)u\, ds + (1-\alpha_n)\omega,$$

where $n \in \mathbb{N}$, $u \in K$, $\omega \in K$. Since $Q_n^{\omega} : K \to K$ is a contraction, by the Banach contraction principle there exists a unique $u_{n,\omega} \in K$ such that $Q_n^{\omega}(u_{n,\omega}) = u_{n,\omega}$. Thus, for any natural number n, we can define the mapping $A_n : K \to K$ by $A_n(\omega) = u_{n,\omega}$ for each $\omega \in K$. And we have $x_{n+1} = A_n(x_n)$ for all $n \in \mathbb{N}$. Note that $A_n(v_i) = u_i$ if and only if

$$u_i = \alpha_n \cdot \frac{1}{t_n} \int_0^{t_n} T(s)u_i\, ds + (1-\alpha_n)v_i.$$

For any v_1, $v_2 \in K$, we have

$$\begin{aligned}
\big\|A_n(v_1) - A_n(v_2)\big\| &= \|u_1 - u_2\| \\
&\leq \alpha_n \cdot \frac{1}{t_n}\bigg\| \int_0^{t_n} \big(T(s)u_1 - T(s)u_2\big)\, ds\bigg\| + (1-\alpha_n)\|v_1 - v_2\| \\
&\leq \alpha_n \cdot \frac{1}{t_n} \int_0^{t_n} \big\|T(s)u_1 - T(s)u_2\big\|\, ds + (1-\alpha_n)\|v_1 - v_2\| \\
&\leq \alpha_n\|u_1 - u_2\| + (1-\alpha_n)\|v_1 - v_2\|;
\end{aligned}$$

that is

$$\|u_1 - u_2\| \leq \alpha_n\|u_1 - u_2\| + (1-\alpha_n)\|v_1 - v_2\|,$$

which implies that

$$\big\|A_n(v_1) - A_n(v_2)\big\| = \|u_1 - u_2\| \leq \|v_1 - v_2\|.$$

Therefore, each A_n is a nonexpansive mapping. Let $\omega \in F(\mathfrak{T})$. This means $T(s)\omega = \omega$, and hence

$$\alpha_n \cdot \frac{1}{t_n} \int_0^{t_n} T(s)\omega\, ds + (1-\alpha_n)\omega = \omega.$$

This implies that $F(\mathfrak{T}) \subset \bigcap_{n=1}^{\infty} F(A_n)$. Set

$$U_{n,m} = A_{n+m-1} \circ A_{n+m-2} \circ \cdots \circ A_n \quad \text{for } m \geq 1.$$

Then $U_{n,m} : K \to K$ is a nonexpansive mapping, and $U_{n,m}(\omega) = \omega$, $U_{n,m}x_n = x_{n+m}$ for $\omega \in F(\mathfrak{T})$, m, $n \in \mathbb{N}$. If $\|x_n - \omega_1\| = 0$ for some $n_0 \in \mathbb{N}$, then we have $x_n = \omega_1$ for all $n \in \mathbb{N}$. Indeed, if $n < n_0$, then it follows from (2.1) that we have $x_{n_0-1} = x_{n_0-2} = \cdots = x_1 = \omega_1$. If $n > n_0$, since the sequence $\{\|x_n - \omega_1\|\}$ is nonincreasing, then we also have $x_n = \omega_1$. Thus we may assume that $\|x_n - \omega_1\| > 0$ for all $n \in \mathbb{N}$. Set

$$\begin{aligned} b_{n,m} &= U_{n,m}\big(tx_n + (1-t)\omega_1\big) - tU_{n,m}x_n - (1-t)U_{n,m}\omega_1, \\ L_n &= t(1-t)\|x_n - \omega_1\|, \\ W_{n,m} &= \frac{U_{n,m}\omega_1 - U_{n,m}(tx_n + (1-t)\omega_1)}{t\|x_n - \omega_1\|}, \\ D_{n,m} &= \frac{U_{n,m}(tx_n + (1-t)\omega_1) - U_{n,m}x_n}{(1-t)\|x_n - \omega_1\|}. \end{aligned}$$

Since $U_{n,m} : K \to K$ is a nonexpansive mapping, then we have $\|W_{n,m}\| \le 1$ and $\|D_{n,m}\| \le 1$. Note that E is a uniformly convex space, and we have

$$\begin{aligned} \big\|tx + (1-t)y\big\| &\le 1 - 2\min\{t, 1-t\}\delta_E\big(\|x-y\|\big) \\ &\le 1 - 2t(1-t)\delta_E\big(\|x-y\|\big) \end{aligned} \tag{3.1}$$

for all $t \in [0,1]$ and $x, y \in E$ such that $\|x\| \le 1$ and $\|y\| \le 1$. Therefore, it follows from (3.1) that

$$2t(1-t)\delta_E\big(\|W_{n,m} - D_{n,m}\|\big) \le 1 - \big\|tW_{n,m} + (1-t)D_{n,m}\big\|. \tag{3.2}$$

Note that

$$\begin{aligned} \big\|W_{n,m} - D_{n,m}\big\| &= \frac{\|b_{n,m}\|}{L_n}, \\ \big\|tW_{n,m} + (1-t)D_{n,m}\big\| &= \frac{\|U_{n,m}x_n - U_{n,m}\omega_1\|}{\|x_n - \omega_1\|} = \frac{\|x_{n+m} - \omega_1\|}{\|x_n - \omega_1\|}. \end{aligned}$$

Then it follows from (3.2) that

$$2L_n\delta_E\Big(\frac{\|b_{n,m}\|}{L_n}\Big) \le \|x_n - \omega_1\| - \|x_{n+m} - \omega_1\|. \tag{3.3}$$

It follows from Lemma 1.5 that $\frac{\delta_E(s)}{s}$ is nondecreasing. Note that $\lim_{n\to\infty}\|x_n - \omega_1\| = \lim_{n\to\infty}\|x_{n+m} - \omega_1\|$ and $\delta_E(0) = 0$. The continuity of δ_E gives from inequality (3.3) that $\liminf_n(\limsup_m \|b_{n,m}\|) = 0$ uniformly for all m; that is

$$\liminf_n\big(\limsup_m\big\|U_{n,m}\big(tx_n + (1-t)\omega_1\big) - tU_{n,m}x_n - (1-t)U_{n,m}\omega_1\big\|\big) = 0.$$

On the other hand, we have

$$\begin{aligned} a_{n+m}(t) &\le \big\|tx_{n+m} + (1-t)\omega_1 - \omega_2 \\ &\qquad + \big(U_{n,m}\big(tx_n + (1-t)\omega_1\big) - tU_{n,m}x_n - (1-t)U_{n,m}\omega_1\big)\big\| \\ &\qquad + \big\|-\big(U_{n,m}\big(tx_n + (1-t)\omega_1\big) + tU_{n,m}x_n + (1-t)U_{n,m}\omega_1\big)\big\| \\ &= \big\|U_{n,m}\big(tx_n + (1-t)\omega_1\big) - \omega_2\big\| \\ &\qquad + \big\|U_{n,m}\big(tx_n + (1-t)\omega_1\big) - tU_{n,m}x_n - (1-t)U_{n,m}\omega_1\big\| \\ &\le a_n(t) + \big\|U_{n,m}\big(tx_n + (1-t)\omega_1\big) - tU_{n,m}x_n - (1-t)U_{n,m}\omega_1\big\|. \end{aligned}$$

Therefore, $\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t)$. This implies that $\lim_{n\to\infty} a_n(t)$ exits for all $t \in [0,1]$. This completes the proof. □

Now, we prove the weak convergence of the implicit iterative processes (2.1) for nonexpansive semigroups.

Theorem 3.2. *Let E be a uniformly convex Banach space such that its dual E^* has the Kadec–Klee property and K is a nonempty closed convex subset of E. Let $\mathfrak{T}$ be a nonexpansive semigroup on K, and let a, b be two constants with $0 < a < b < 1$. Let $\{x_n\}$ be defined by (2.1) satisfying $\{\alpha_n\} \subset [a, b]$ for all $n \geq 1$ and $t_n \to \infty$ as $n \to \infty$. Then there exists a common fixed point $w \in F(\mathfrak{T})$ such that $x_n \rightharpoonup w$.*

Proof. By Lemma 2.1(i), we get that $\{x_n\}$ is bounded. Since E is a uniformly convex Banach space, $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_k}\}$. Assume that $\{x_{n_k}\}$ converges weakly to w for $k \to \infty$. Note that $\{x_n\} \subset K$ and K is weakly closed. Then $w \in K$. By Lemma 2.1(ii), we get $\lim_{n\to\infty} \|x_n - T(s)x_n\| = 0$. By Lemma 1.6, we have $w \in F(\mathfrak{T})$. Assume that $\{x_n\}$ does not converge weakly to w. Then $\exists\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some $p \neq w$. As in the case of w, we must have $p \in K$ and $p \in F(\mathfrak{T})$. It follows from Lemma 3.1 that $\lim_{n\to\infty} \|tx_n + (1-t)w - p\|$ exists for all $t \in [0, 1]$. By Lemma 1.3, we have that $p = w$. Hence $\{x_n\}$ converges weakly to w. This completes the proof. $\square$

Remark 3.3. The result of Theorem 3.2 is different from Theorem 3.4 of [5]. In this paper, T is a nonexpansive semigroup, but in Theorem 3.4 of [5] $T_1, T_2, \ldots, T_m$ are a finite family of asymptotically nonexpansive mappings. Additionally, the iterative process $\{x_n\}$ defined by (2.1) in this article is different from $\{x_n\}$ defined by (3.1) of [5].

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