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NEARLY RELATIVELY COMPACT PROJECTIONS IN OPERATOR ALGEBRAS

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ABSTRACT. Let A be a C^*-algebra, and let A^{**} be its enveloping von Neumann algebra. Akemann suggested a kind of noncommutative topology in which certain projections in A^{**} play the role of open sets, and he used two operator inequalities in connection with compactness. Both of these inequalities are equivalent to compactness for a closed projection in A^{**}, but only one is equivalent to relative compactness for a general projection. A third operator inequality, also related to compactness, was used by the author. The study of all three inequalities can be unified by considering a numerical invariant which is equivalent to the distance of a projection from the set of relatively compact projections. Tomita's concept of regularity of projections seems relevant, and so we give some results and examples on regularity. We also include a few related results on semicontinuity.

1. INTRODUCTION

A projection in A^{**} is called *open* if it is the support projection of a hereditary C^*-subalgebra of A, and p is *closed* if $1-p$ is open. Let $Q(A)$, the quasi-state space of A, be $\{f \in A^* : f \geq 0 \text{ and } \|f\| \leq 1\}$, and let $S(A)$, the state space of A, be $\{f \in Q(A) : \|f\| = 1\}$. For a projection p in A^{**}, let $F(p) = \{f \in Q(A) : f(1-p) = 0\}$. Then p is closed if and only if $F(p)$ is weak* closed (see Effros [13]), and p is called *compact* if $F(p) \cap S(A)$ is weak* closed. For every projection p in A^{**}, there is a smallest closed projection $\overline{p}$ such that $\overline{p} \geq p$, where $\overline{p}$ is called the *closure* of p, and p is called *relatively compact* if $\overline{p}$ is compact. For any subset

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S of A^{**}, S_{sa} denotes $\{x \in S : x = x^*\}$ and S_+ denotes $\{x \in S : x \geq 0\}$. If A has a unit, then every projection in A^{**} is relatively compact. Therefore, our concern is with nonunital C^*-algebras.

Consider the following properties for a projection p in A^{**}:

(1) $\exists a \in A_{sa}$ such that $p \leq a \leq 1$,
(2) $\exists a \in A_{sa}$ such that $p \leq a$,
(3) $\exists a \in A_{sa}$ such that $p \leq pap$.

Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and any of the properties for $\overline{p}$ implies the same property for p. Akemann [4] showed that for p closed each of (1) and (2) is equivalent to compactness and for general p, (1) is equivalent to relative compactness, but for general p, (2) does not imply relative compactness. We showed in [8] that for p open and A σ-unital, (3) is equivalent to the property that every closed subprojection of p is compact. We will show below that half of this result is true for general A—unfortunately, nothing in this article "explains" the result.

The original goal of this work was to find all possible answers to: Which of (1), (2), (3) are true for p and which are true for $\overline{p}$? There are, in fact, six possible answers, but it is better to organize the subject differently. If a nonzero p satisfies (3), let $\alpha(p) = \inf\{\|a\| : a \in A_{sa} \text{ and } p \leq pap\}$. Otherwise, let $\alpha(p) = \infty$. Also, let $\alpha(0) = 1$. Clearly, $1 \leq \alpha(p) \leq \infty$ and $\alpha(p) \leq \alpha(\overline{p})$. More generally, $p_1 \leq p_2 \Rightarrow \alpha(p_1) \leq \alpha(p_2)$, so that $\alpha(p)$ is some kind of measure of how large p is. Property (2) will be shown equivalent to "$\alpha(p) = 1$," and hence (1) is equivalent to "$\alpha(\overline{p}) = 1$." Thus all the information for our original goal is contained in the pair $(\alpha(p), \alpha(\overline{p}))$. We will give enough examples to show that every pair (s, t) such that $1 \leq s \leq t \leq \infty$ is $(\alpha(p), \alpha(\overline{p}))$ for some p and A.

Let RC be the set of relatively compact projections in A^{**}, let ORC be the set of open relatively compact projections, and let CRC be the set of compact projections (which, of course, is the same as the set of closed relatively compact projections). Then for any projection p in A^{**}, $\operatorname{dist}(p, \mathrm{RC}) = [1 - \alpha(p)^{-1}]^{1/2}$, where the distance is with respect to the metric induced by the norm. Also, if p is open, then $\operatorname{dist}(p, \mathrm{RC}) = \operatorname{dist}(p, \mathrm{ORC})$; and if p is closed, then $\operatorname{dist}(p, \mathrm{RC}) = \operatorname{dist}(p, \mathrm{CRC})$. Now CRC is a norm-closed set because of the semicontinuity characterization of compactness (see [7, Definition-Lemma 2.47(iv)]). Thus $\operatorname{dist}(p, \mathrm{CRC}) = 0$ implies that $p \in \mathrm{CRC}$. Neither RC nor ORC need be closed, since Akemann's counterexample in [4] showing (2) $\not\Rightarrow$ (1) uses an open projection. Thus in some sense our results "explain" Akemann's results that (2) $\Rightarrow$ (1) for closed projections but not for general projections.

A projection p in A^{**} will be called *nearly relatively compact* if $\operatorname{dist}(p, \mathrm{RC}) < 1$. By our results proved below, this is equivalent to "$\alpha(p) < \infty$" or "p satisfies (3)." We will not define "nearly compact." The reader might think this should mean "$\operatorname{dist}(p, \mathrm{CRC}) < 1$"; but we think a better meaning for this term would be "closed and nearly relatively compact." While we discuss this point at some length below, we do not consider the issue to be completely settled.

There are other natural interpretations of $\alpha(p)$ which are included, together with the main results, in Section 2, except for the examples, which are in Section 3. Section 4 contains some results and examples on regularity of projections and its relation to the above. Section 5 contains special results on open projections, Section 6 contains results on $\alpha(p_1 \vee p_2)$, and Sections 7, 8, and 9 contain miscellaneous related results, remarks, and examples.

2. Interpretations of $\alpha(p)$

Theorem 2.1. *If p is a projection in A^{**}, then $\alpha(p) = 1$ if and only if $p \le a$ for some a in A_{sa}.*

Proof. We rely on a result of Akemann (see [2, Theorem 1.2]), which states in slightly different words: if A is a C^*-subalgebra of B and c is a positive element of $\mathrm{her}_B(A)$, the hereditary C^*-subalgebra of B generated by A, then $\forall\varepsilon > 0$, $\exists a \in A_{sa}$ such that $c \le a \le \|c\| + \varepsilon$.

First assume that $p \le a$ for some a in A_{sa}. Then clearly $p \in \mathrm{her}_{A^{**}}(A)$. Thus $\forall\varepsilon > 0$, $\exists a' \in A_{sa}$ such that $p \le a' \le 1 + \varepsilon$. Therefore $p \le pa'p$, and hence $\alpha(p) \le 1 + \varepsilon$. Since ε is arbitrary, $\alpha(p) \le 1$.

Now assume that $\alpha(p) = 1$. We will prove $p \in \mathrm{her}_{A^{**}}(A)$. Let H be the Hilbert space of the universal representation of A, so that A^{**} is the von Neumann algebra generated by A in $B(H)$. Represent elements of A^{**} as 2×2 operator matrices relative to $H = pH \oplus (1-p)H$. Choose $\varepsilon > 0$ and a in A_+ such that $\|a\| < 1 + \varepsilon$ and $p \le pap$. Let $a = \left(\begin{smallmatrix} x & y \\ y^* & z \end{smallmatrix}\right)$. Since

$$\begin{pmatrix} x & y \\ y^* & z \end{pmatrix} \le \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1+\varepsilon \end{pmatrix}, \qquad \begin{pmatrix} 1+\varepsilon-x & -y \\ -y^* & 1+\varepsilon-z \end{pmatrix} \ge 0.$$

Therefore $\|y\| \le \|1+\varepsilon-x\|^{\frac{1}{2}}\|1+\varepsilon-z\|^{\frac{1}{2}} \le \varepsilon^{\frac{1}{2}}(1+\varepsilon)^{\frac{1}{2}}$, since $x \ge 1$ and $z \ge 0$. Since

$$\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \le \begin{pmatrix} x + \|y\| & y \\ y^* & z + \|y\| \end{pmatrix},$$

$p \le a + \|y\|$. Let $(e_i)_{i\in D}$ be an approximate identity of A. Then $\limsup \|(1-e_i) \times p(1-e_i)\| \le \limsup \|(1-e_i)a(1-e_i)\| + \|y\| \le \varepsilon^{\frac{1}{2}}(1+\varepsilon)^{\frac{1}{2}}$. Since ε is arbitrary, $\lim \|(1-e_i)p(1-e_i)\| = 0$. This implies that $p \in \mathrm{her}_{A^{**}}(A)$. ☐

We review some known facts about pairs of projections. A complete classification of these, up to unitary equivalence, was given by Dixmier [12] (see also [14], [17], [19]). If p and q are projections in $B(H)$ with ranges M and N, let $H_{11} = M \cap N$, $H_{10} = M \cap N^\perp$, $H_{01} = M^\perp \cap N$, $H_{00} = M^\perp \cap N^\perp$, and $H_0 = (H_{11} \oplus H_{10} \oplus H_{01} \oplus H_{00})^\perp$. A simple example of a pair of projections occurs when H is 2-dimensional and $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $q = \left(\begin{smallmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{smallmatrix}\right)$ for some θ in $(0, \frac{\pi}{2})$. In the most general example, $(H_0, p|_{H_0}, q|_{H_0})$ is a direct integral of such 2-dimensional examples, for various values of θ, and $\|p-q\|$ can be computed as follows. If H_{01} or H_{10} is nontrivial, then $\|p-q\| = 1$. Otherwise $\|p-q\| = \sin\theta$, where θ is the essential supremum of the angles occurring in the decomposition of H_0. For later use, we make a couple of other points.

(1) The usual concept of the angle between two projections (or subspaces) is the essential infimum of the angles occurring in the decomposition of H_0.
(2) Define $d_a(p,q) = \sin^{-1}(\|p-q\|)$. Then d_a is a metric on the set of projections, equivalent to the metric induced by the norm (see [9, Corollary 4]).

Theorem 2.2. *If p is a projection in A^{**}, then* $\operatorname{dist}(p, \mathrm{RC}) = [1 - \alpha(p)^{-1}]^{\frac{1}{2}}$.

Proof. (1) $\operatorname{dist}(p, \mathrm{RC}) \geq [1 - \alpha(p)^{-1}]^{\frac{1}{2}}$. For this we may assume $\operatorname{dist}(p, \mathrm{RC}) < 1$. Let q be in RC such that $\|p - q\| < 1$. Then $pqp \geq (\cos^2\theta)p$, where θ is as above, so that $\|p - q\| = \sin\theta$. Since q is relatively compact, there is a in A_{sa} such that $q \leq a \leq 1$. Thus $pap \geq (\cos^2\theta)p$, and

$$\cos^{-2}\theta \geq \alpha(p).$$

Therefore

$$\cos^2\theta \leq \alpha(p)^{-1},$$
$$\sin^2\theta \geq 1 - \alpha(p)^{-1},$$
$$\|p - q\| = \sin\theta \geq \left[1 - \alpha(p)^{-1}\right]^{\frac{1}{2}}.$$

Since q can be chosen so that $\|p - q\|$ approximates $\operatorname{dist}(p, \mathrm{RC})$, we conclude that $\operatorname{dist}(p, \mathrm{RC}) \geq [1 - \alpha(p)^{-1}]^{\frac{1}{2}}$.

(2) $\operatorname{dist}(p, \mathrm{RC}) \leq [1 - \alpha(p)^{-1}]^{\frac{1}{2}}$. For this we may assume $\alpha(p) < \infty$. Let a be in A_{sa} such that $p \leq pap$, let $\varepsilon > 0$, and let $q = E_{[\varepsilon,\infty)}(a)$ (q is a spectral projection of a). Then q is compact. Since $a \leq \|a\|q + \varepsilon(1 - q)$, $p \leq \|a\|pqp + \varepsilon p$. Therefore $pqp \geq \frac{1-\varepsilon}{\|a\|}p$. Let r be the range projection of qp. Then $r \leq q$ and hence $r \in \mathrm{RC}$. Since $rp = qp$, $prp = (rp)^*(rp) = (qp)^*(qp) = pqp$. Refer to the notation introduced above for the pair (p, r). If $\varepsilon < 1$, the initial projection of rp is p, and hence $H_{10} = 0$. Since r is the range projection of rp, $H_{01} = 0$. Therefore $\|p - r\| = \sin\theta$, where $\cos^2\theta \geq \frac{1-\varepsilon}{\|a\|}$. Thus $\operatorname{dist}(p, \mathrm{RC}) \leq [1 - \frac{1-\varepsilon}{\|a\|}]^{\frac{1}{2}}$. We can choose ε and a so that $\frac{1-\varepsilon}{\|a\|}$ approximates $\alpha(p)^{-1}$. □

Corollary 2.3. *If p is a projection in A^{**}, then $\alpha(p) = 1$ if and only if p is in the norm closure of* RC.

Remark. Clearly, if $\alpha(p) < \infty$ and p' is sufficiently close to p, then $\alpha(p') < \infty$. If one wants the best estimates (i.e., *how* close must p' be to p and what the best estimate for $\alpha(p')$ is), one should use the metric d_a. Thus $d_a(p, \mathrm{RC}) = \cos^{-1}(\alpha(p)^{-\frac{1}{2}})$; and if $d_a(p', p) + d_a(p, \mathrm{RC}) < \frac{\pi}{2}$, then $\alpha(p') < \infty$. It is easy to construct examples (see Section 3) where $d_a(p', p) + d_a(p, q) = \frac{\pi}{2}$, $q \in \mathrm{RC}$, and $\alpha(p') = \infty$.

Proposition 2.4. *Let h be a strongly upper semicontinuous element of A^{**}_+ such that the spectrum of h omits $(0, \varepsilon)$ for some $\varepsilon > 0$. Then $E_{(0,\infty)}(h)$ is compact.*

Proof. Proposition 2.44(b) of [7] asserts that $E_{(0,\infty)}(h)$ is closed under the hypothesis that h is weakly upper semicontinuous. The proof of the present result is almost identical. Alternatively, the present result can be deduced from the earlier one by adjoining an identity to A. □

Lemma 2.5. *Assume that p is a projection, $0 \leq a \leq 1$, and $pap \geq \varepsilon p$ for some $\varepsilon > 0$. Then $pa^{\frac{1}{2}}(pap)^{-1}a^{\frac{1}{2}}p \geq pap$, where the inverse is taken in $pA^{**}p$.*

Remark. Of course, this is an operator-theoretic lemma that has nothing to do with A.

Proof. Again, we represent elements of A^{**} as 2×2 operator matrices relative to $H = pH \oplus (1-p)H$. Write $a^{\frac{1}{2}} = \left(\begin{smallmatrix} x & y \\ y^* & z \end{smallmatrix}\right)$, so that $a = \left(\begin{smallmatrix} x^2+yy^* & * \\ * & * \end{smallmatrix}\right)$. Since $a^{\frac{1}{2}} \geq a$, $x \geq x^2 + yy^*$. Therefore, $x(x^2+yy^*)^{-1}x \geq x(x)^{-1}x = x \geq x^2 + yy^*$. This is the desired inequality. ☐

Theorem 2.6. *Let p be a projection in A^{**}.*

(a) *If p is open, then* $\operatorname{dist}(p, \mathrm{ORC}) = \operatorname{dist}(p, \mathrm{RC})$.
(b) *If p is closed, then* $\operatorname{dist}(p, \mathrm{CRC}) = \operatorname{dist}(p, \mathrm{RC})$. *Moreover, in this case, if $\exists a \in A_{sa}$ such that $p \leq pap$ and $\|a\| = \alpha(p)$, then $\exists q \in$* CRC *such that $\|p - q\| = \operatorname{dist}(p, \mathrm{CRC})$.*

Proof. The proofs of the two cases are similar. We start with a in A_{sa} such that $0 \leq a \leq 1$ and $pap \geq \varepsilon p$, $\varepsilon > 0$, and let q be the range projection of $a^{\frac{1}{2}}p$, where ε should approximate (or, for the last sentence of (b), be equal to) $\alpha(p)^{-1}$. In case (a), we also need the range projection of a to be in RC. This is accomplished by replacing the original a with $f_\delta(a)$, where

$$f_\delta(t) = \begin{cases} 0, & 0 \leq t \leq \delta, \\ t, & 2\delta \leq t \leq 1. \end{cases}$$

(This causes the original ε to be replaced by $\varepsilon - 2\delta$.)

The partial isometry in the polar decomposition of $a^{\frac{1}{2}}p$ is $u = a^{\frac{1}{2}}p(pap)^{-\frac{1}{2}}$. Thus $q = uu^* = a^{\frac{1}{2}}(pap)^{-1}a^{\frac{1}{2}}$. Lemma 2.5 implies that $pqp \geq pap \geq \varepsilon p$. Also $qpq = a^{\frac{1}{2}}(pap)^{-1}(pa^{\frac{1}{2}}p)^2(pap)^{-1}a^{\frac{1}{2}}$. Since pap and $pa^{\frac{1}{2}}p$ are invertible elements of $pA^{**}p$, this implies that $qpq \geq \delta_1(a^{\frac{1}{2}}pa^{\frac{1}{2}})$, for some $\delta_1 > 0$, and hence $qpq \geq \delta_2 q$ for some $\delta_2 > 0$. Thus the range projection of qp is q. Now the foregoing discussion and the proof of Theorem 2.2 imply that $\|p - q\| \leq (1-\varepsilon)^{\frac{1}{2}}$.

To complete the proof, we need only show that q is in ORC or CRC in the two cases. Note that q is the range projection of $(a^{\frac{1}{2}}p)(a^{\frac{1}{2}}p)^* = a^{\frac{1}{2}}pa^{\frac{1}{2}}$. In case (a), $a^{\frac{1}{2}}pa^{\frac{1}{2}}$ is strongly lower semicontinuous, and in case (b), $a^{\frac{1}{2}}pa^{\frac{1}{2}}$ is strongly upper semicontinuous. (This follows, for example, from Proposition 2.44(a) of [7].) In case (a), it follows from Proposition 2.44(a) of [7] that q is open. Since q is smaller than the range projection of a, which is in RC, q is in ORC. In case (b), Proposition 2.4 implies that q is compact. We need to know that $\sigma(a^{\frac{1}{2}}pa^{\frac{1}{2}})$ omits $(0, \varepsilon)$, and this follows from $\sigma(a^{\frac{1}{2}}pa^{\frac{1}{2}}) \cup \{0\} = \sigma(pap) \cup \{0\}$. ☐

Corollary 2.7. *If p is an open projection in A^{**}, then $\alpha(p) = 1$ if and only if p is in the norm closure of* ORC.

Remark. We also recover (in different language) a result of Akemann [4, Theorem II.5]: if p is closed, then $\alpha(p) = 1$ if and only if p is compact (see Section 1).

We now consider other interpretations of $\alpha(p)$. Some of these can be considered as methods of computing $\alpha(p)$.

Proposition 2.8. *Let p be a nonzero projection in A^{**}, let $(e_i)_{i \in D}$ be an approximate identity of A, and let ε_i be the least point in $\sigma(pe_ip)$, where the spectrum is computed in $pA^{**}p$. Then $\alpha(p)^{-1} = \lim \varepsilon_i$.*

Proof. Note that $\varepsilon_i \le \alpha(p)^{-1}$ so that $\limsup \varepsilon_i \le \alpha(p)^{-1}$. (We do not need to assume that (e_i) is increasing, though we do assume that $0 \le e_i \le 1$.) Assume that $0 \le a \le 1$ and that $pap \ge \varepsilon p$. For any $\delta > 0$, there is i_0 such that $\|a - e_i a e_i\| < \delta$ for $i \ge i_0$. Thus $\varepsilon p \le pap \le p(e_i a e_i + \delta)p \le p e_i^2 p + \delta p \le p e_i p + \delta p$. Therefore $\varepsilon - \delta \le \varepsilon_i$ for $i \ge i_0$, and $\liminf \varepsilon_i \ge \varepsilon$. Since ε can be chosen to approximate $\alpha(p)^{-1}$, $\liminf \varepsilon_i \ge \alpha(p)^{-1}$. □

Remark. It was pointed out in [8] (Remark 1 after Theorem 4) that if e is a strictly positive element of A, then $\alpha(p) < \infty$ if and only if $pep \ge \varepsilon p$ for some $\varepsilon > 0$.

Theorem 2.9. *Let p be a nonzero projection in A^{**}, and let $\overline{S}(p)$ be the weak* closure of $F(p) \cap S(A)$. Then $\alpha(p)^{-1} = \inf\{\|\varphi\| : \varphi \in \overline{S}(p)\}$.*

Remark.

(1) The infimum is actually a minimum.
(2) This result is most natural when p is closed, but it is valid generally.
(3) If $p = 1$, there is a well-known dichotomy: if A is unital, $\overline{S}(1) = S(A)$; and if A is nonunital, $\overline{S}(1) = Q(A)$. In our language, $\alpha(1) = 1$ or ∞ according to whether A is unital or not.

Proof. Assume that $a \in A_{sa}$ and $pap \ge p$. Then $\varphi(a) \ge 1$, $\forall \varphi \in F(p) \cap S(A)$. Therefore $\varphi(a) \ge 1$, $\forall \varphi \in \overline{S}(p)$. Thus $\|a\| \ge \|\varphi\|^{-1}$, $\forall \varphi \in \overline{S}(p)$. This implies that $\alpha(p)^{-1} \le \inf\{\|\varphi\| : \varphi \in \overline{S}(p)\}$.

To prove the reverse inequality, we may assume that $\inf\{\|\varphi\| : \varphi \in \overline{S}(p)\} > 0$. Choose ε such that $0 < \varepsilon < \inf\{\|\varphi\| : \varphi \in \overline{S}(p)\}$, and let $K = \{f \in A^* : f = f^* \text{ and } \|f\| \le \varepsilon\}$. Then K and $\overline{S}(p)$ are disjoint compact convex sets. By the separation theorem, we can find a in A_{sa} such that $\sup\{f(a) : f \in K\} < \inf\{\varphi(a) : \varphi \in \overline{S}(p)\}$. Since the supremum is $\varepsilon\|a\|$, we can normalize a so that $\|a\| = 1$, and then we find $pap \ge \varepsilon p$. This implies that $\alpha(p)^{-1} \ge \varepsilon$ and hence $\alpha(p)^{-1} \ge \inf\{\|\varphi\| : \varphi \in \overline{S}(p)\}$. □

Corollary 2.10. *We have that $\alpha(p) < \infty$ if and only if 0 is not in the weak* closure of $F(p) \cap S(A)$.*

If V is a partially ordered real normed linear space and $e \in V_+$, then e is an *order unit* of V if $\forall x \in V$, $\exists t \in \mathbb{R}_+$ such that $x \le te$. We will call e a *t-order unit* if $\|e\| = 1$ and $x \le t\|x\|e$, $\forall x \in V$. If V is a Banach space and the positive cone is closed, then every order unit of norm 1 is a t-order unit for t sufficiently large. The proof of this (presumably known) result is similar to an argument given in the next theorem. If p is a projection in A^{**}, then $pA_{sa}p$ is a partially ordered real normed linear space if regarded as a subspace of $pA_{sa}^{**}p$. If p is closed, then [6, Proposition 4.4] implies that $pA_{sa}p$ is a Banach space and its norm is the quotient norm from the natural map $A_{sa} \to pA_{sa}p$.

Theorem 2.11. *Let p be a projection in A^{**}.*

(a) *Then $\alpha(p) < \infty$ if and only if $pA_{sa}p$ has an order unit.*
(b) *If p is closed, then $\alpha(p) = \inf\{t : pA_{sa}p \text{ has a } t\text{-order unit}\}$. Also $pA_{sa}p$ has an $\alpha(p)$-order unit if and only if there is a in A_{sa} such that $pap \ge p$ and $\|a\| = \alpha(p)$.*

(c) *For general* $p, \alpha(p)$ *is the infimum of* t *such that there is an order unit* e *satisfying*

(i) $e = pa_1p$ *where* $\|a_1\| \leq 1$,

(ii) $pap \leq t\|a\|e, \forall a \in A_{sa}$.

Proof. (a) If $pap \geq p$, then clearly pap is an order unit for $pA_{sa}p$. Conversely, if e is an order unit, let $C = \{a \in A_{sa} : -e \leq pap \leq e\}$. Then C is closed, convex, and symmetric, and $A_{sa} = \bigcup_1^\infty nC$. A standard argument based on the Baire category theorem shows that nC contains the unit ball of A_{sa}, for some n. If (e_i) is an approximate identity of A, then $pe_ip \leq ne$, $\forall i$. Taking the strong limit, we see that $p \leq ne$. Therefore, $\alpha(p) < \infty$.

(c) If e and t satisfy (i) and (ii), then part of the argument just given shows that $p \leq te = p(ta_1)p$. Thus $\alpha(p) \leq t$. Therefore, $\alpha(p)$ is at most the infimum specified. On the other hand, if $pap \geq p$, then e and t satisfy (i) and (ii), where $t = \|a\|$ and $e = t^{-1}pap$. This implies the opposite inequality.

(b) If p is closed, the infima in (b) and (c) are the same, since the norm of $pA_{sa}p$ is the quotient norm under the map $a \mapsto pap$ (see [6]). The second sentence of (b) is deduced from Theorem 3.3 or Corollary 3.4 of [7]: $e = pap$, where $\|a\| = \|e\|$. □

Lemma 2.12. *Assume that* p *is a closed projection in* A^{**}, $a \in A_+$, *and* $pap \geq \varepsilon p$ *for some* $\varepsilon > 0$. *Then* $pa^{\frac{1}{2}}Aa^{\frac{1}{2}}p = pAp$.

Proof. Since $pap \geq \varepsilon p$, $\exists s \in A^{**}$ such that $p = sa^{\frac{1}{2}}p$. It follows that $pA^{**}p = pa^{\frac{1}{2}}(s^*A^{**}s)a^{\frac{1}{2}}p \subset pa^{\frac{1}{2}}A^{**}a^{\frac{1}{2}}p$. Now let $y = pa^{\frac{1}{2}}$, and define $\varphi : A \to pAp$ by $\varphi(b) = yby^*$. The result cited from [6] shows that $(pAp)^{**}$ can be identified with $pA^{**}p$ in such a way that φ^{**} becomes the map of A^{**} to $pA^{**}p$ given by $b \mapsto yby^*$. In general, if the second adjoint of a map between Banach spaces is surjective, then the original map is surjective. □

Theorem 2.13. *Let* p *be a closed projection in* A^{**}.

(a) *Then* $\alpha(p) < \infty$ *if and only if there are a compact projection* q *and a complete order isomorphism* $\theta : pAp \to qAq$.

(b) *Also,* $\alpha(p) = \inf\{\|\theta\|\|\theta^{-1}\| : q$ *and* θ *as above*$\}$.

(c) *There are* θ *and* q *as above such that* $\|\theta\|\|\theta^{-1}\| = \alpha(p)$ *if and only if there is* a *in* A_{sa} *such that* $pap \geq p$ *and* $\|a\| = \alpha(p)$.

Proof. First assume that θ and q are as in (a). Since q is compact, $q \in qAq$. Let $e = \theta^{-1}(q)$. If $b \in A_{sa}$, then $\theta(pbp) \leq \|\theta\|\|b\|q$. Therefore $pbp \leq \|\theta\|\|b\|e$. As in the proof of Theorem 2.11, we deduce that $p \leq \|\theta\|e$. By Corollary 3.4 of [7], we can write $\|\theta\|e = pap$ for a in A_{sa} such that $\|a\| = \|\theta\|\|e\| \leq \|\theta\|\|\theta^{-1}\|$.

Next assume that $\alpha(p) < \infty$, $a \in A$, $0 \leq a \leq 1$, and $pap \geq \varepsilon p$. Here ε approximates $\alpha(p)^{-1}$, and for (c), $\varepsilon = \alpha(p)^{-1}$. Let q be the range projection of $a^{\frac{1}{2}}p$. As in the proof of Theorem 2.6, we deduce that q is compact and $q = a^{\frac{1}{2}}(pap)^{-1}a^{\frac{1}{2}}$. Then

$$qAq = a^{\frac{1}{2}}(pap)^{-1}(a^{\frac{1}{2}}Aa^{\frac{1}{2}})(pap)^{-1}a^{\frac{1}{2}} = a^{\frac{1}{2}}(pap)^{-1}A(pap)^{-1}a^{\frac{1}{2}},$$

where the second equality uses Lemma 2.12. Let $x = a^{\frac{1}{2}}(pap)^{-1}$ and $y = pa^{\frac{1}{2}}$. Then $x \in qA^{**}p, y \in pA^{**}q, xy = q$, and $yx = p$. If we define θ and φ by

$\theta(b) = xbx^*$ and $\varphi(b) = yby^*$, then the above equation shows that θ maps pAp into qAq and it is obvious that φ maps qAq into pAp. It is now obvious that θ and φ are inverses of one another, and clearly both are completely positive. Now $\|\theta\| \leq \|x\|^2 = \|x^*x\| = \|(pap)^{-1}\| \leq \varepsilon^{-1}$, and $\|\theta^{-1}\| \leq \|y\|^2 \leq 1$.

The above arguments prove all three parts of the theorem. □

Remark. The first part of the proof used only the hypothesis that θ is an order isomorphism, not a *complete* order isomorphism. Therefore, the word "complete" could be omitted from the statement of the theorem.

3. Some examples and discussion

If $1 \leq s \leq t \leq \infty$, A is a C^*-algebra, p is a projection in A^{**}, and $(\alpha(p), \alpha(\overline{p})) = (s,t)$, we will say that p and A achieve (s,t), or, more briefly, that p achieves (s,t). The basic objective of this section is to show that every such pair can be achieved, but we want a little more. We want to consider various properties of projections and find which pairs can be achieved by projections satisfying one or more of these properties. The properties we will consider are open, closed, central, and regular, except that all discussion of regularity will be postponed to the next section (this does not cause much inefficiency). Of course there are many other properties which could be considered, and perhaps some of these would lead to deeper results. The gist of what we will show is that all pairs can be achieved with open projections, but the other properties are compatible only with very special pairs. If p is closed, obviously we must have $s = t$. The restrictions required for the other properties are not much deeper, but we will dignify them with numbers.

(3.1). If p is clopen (both open and closed), then either $\alpha(p) = \alpha(\overline{p}) = 1$ or $\alpha(p) = \alpha(\overline{p}) = \infty$.

Proof. Of course p is clopen if and only if $p \in M(A)$, the multiplier algebra of A. If $pap \geq p$ for a in A, then pap is in A also. From this we easily conclude that p is in A (look at the images in $M(A)/A$). □

(3.2). If p is a central projection in A^{**}, then either $\alpha(p) = \alpha(\overline{p}) = 1$ or $\alpha(p) = \alpha(\overline{p}) = \infty$.

Proof. If $\alpha(p) < \infty$, then there is a in A_+ such that $pap \geq p$. Since $pa = ap$, this clearly implies that $a \geq p$. Let

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 1, & x \geq 1. \end{cases}$$

Since $a \geq p$ and $ap = pa$, $1 \geq f(a) \geq p$. Therefore, $\alpha(\overline{p}) = 1$. □

In the examples $\mathcal{K}$ denotes the set of compact operators on a separable infinite-dimensional Hilbert space H, $\{e_1, e_2, \ldots\}$ is an orthonormal basis of H, and $v \times w$ denotes the rank 1 operator $x \mapsto (x,w)v$. In many cases we will take $A = c \otimes \mathcal{K}$. Then A can be regarded as the set of $\{x_n : 1 \leq n \leq \infty\}$ such that $x_n \in \mathcal{K}$ and $x_n \to x_\infty$ in norm, and A^{**} is the set of $\{h_n : 1 \leq n \leq \infty\}$ such that $h_n \in B(H)$ and $\{\|h_n\|\}$ is bounded. If $p = \{p_n\}$ is a projection in $(c \otimes \mathcal{K})^{**}$, then p is open if and only if $p_\infty \le h$ for every weak cluster point h of the sequence (p_n), and p is closed if and only if $p_\infty \ge h$ for every such h. This follows, for example, from the criterion for weak semicontinuity given in Sections 5.14 and 5.15 of [7].

(3.3). $(1,1)$.

It is trivial to achieve this pair with a clopen central projection. Just let A be any unital C^*-algebra and $p=1$.

(3.4). (∞,∞).

It is trivial to achieve this pair with a clopen central projection. Just let A be any nonunital C^*-algebra, and let $p=1$.

(3.5). $(s,s), 1<s<\infty$ (cf. [8, Remark 2, p. 276]).

For this we need two examples, one open and one closed. Let θ be in $(0,\frac{\pi}{2})$ such that $\sec^2\theta = s$. Let $A = c\otimes\mathcal{K}$ and $v_n = \cos\theta e_1 + \sin\theta e_{n+1}$. Define p and q by $p_n = q_n = v_n \times v_n, n<\infty$, $p_\infty = e_1\times e_1$, and $q_\infty = 0$. Then q is open, $p=\overline{q}$, and we claim that $\alpha(p)=\alpha(q)=s$. (Thus p and q both achieve the pair (s,s).) Define a in A by $a_n = a_\infty = s(e_1\times e_1)$. Then $pap\ge p$ and $qaq \ge q$ (actually, $qaq=q$). Thus $\alpha(q),\alpha(p)\le s$. If φ_n is the pure state of A given by $\varphi_n(a) = (a_n v_n, v_n)$, then $\varphi_n \in F(q)\cap S(A)\subset F(p)\cap S(A)$ and φ_n converges weak* to a functional of norm $\frac{1}{s}$. Thus $\alpha(q)$, $\alpha(p)\ge s$.

We now justify the remark after Corollary 2.3. Choose θ' such that $\theta<\theta'\le\frac{\pi}{2}$, and let $w_n = \cos\theta' e_1 + \sin\theta' e_{n+1}$. Define a closed projection p' by $p'_\infty = e_1\times e_1$, $p'_n = w_n\times w_n$. Then $\alpha(p') = \sec^2\theta'$, $d_a(p',RC) = \theta'$, $d_a(p',p) = \theta'-\theta$, and $d_a(p,RC)=\theta$. If $\theta'=\frac{\pi}{2}$, then $\alpha(p')=\infty$. We could equally well consider an open projection $q', (q')_\infty = 0$, $q'_n = p'_n$, and compare q' to q.

(3.6). (s,∞), $1<s<\infty$ (cf. [8, Remark 4, p. 276]).

Let A, θ, and v_n be as in the previous example. Let (m_n) be a sequence which includes each positive integer infinitely often. Define an open projection p in A^{**} by $p_\infty = 0$, $p_n = v_{m_n}\times v_{m_n}$, $n<\infty$. Then by essentially the same argument as above, $\alpha(p)=s$. Since $\{v_n\}$ is total in H, $(\overline{p})_\infty = 1$. Then it is obvious that $\alpha(\overline{p})=\infty$.

(3.7). $(1,\infty)$.

Akemann's Example IV.5 in [4] gives an open projection that achieves this pair, but we will give another example, somewhat similar in spirit, where $A=c\otimes\mathcal{K}$.

Lemma 3.1. *If $x>1>y>0$ and $u=(u_1,u_2)$, where $|u_1|^2 = \frac{x(1-y)}{x-y}$ and $|u_2|^2 = \frac{y(x-1)}{x-y}$, then $u\times u \le \left(\begin{smallmatrix} x & 0\\ 0 & y\end{smallmatrix}\right)$. (Here the Hilbert space is 2-dimensional.)*

Now let K be any one-to-one element of $\mathcal{K}_+$ such that $\|K\|>1$. If $\mathcal{V} = \{u\in H : \|u\|=1 \text{ and } u\times u\le K\}$, then the lemma implies that $\mathcal{V}$ is a total subset of H. Let (u_n) be a sequence which is dense in $\mathcal{V}$. Define an open projection p in A^{**} by $p_\infty = 0$, $p_n = u_n\times u_n$. Define a in A by $a_n = a_\infty = K$. Since $p\le a$, $\alpha(p)=1$. Since $(\overline{p})_\infty = 1$, $\alpha(\overline{p})=\infty$.

(3.8). $(1,t), 1<t<\infty$.

Let $A_0 = c\otimes\mathcal{K}$, and let p_0 be the projection in A_0^{**} called p in Example (3.7). Let $A_1 = A_0\otimes M_2$. For this example, A will be an extension of A_1 by $\mathbb{C}$. According to Busby [10], such an extension is determined by an element e' of $M(A_1)$ which maps onto a projection in $M(A_1)/A_1$. We will take e' to actually be a projection; namely,

$$e' = \begin{pmatrix} t^{-1} & [t^{-1}(1-t^{-1})]^{\frac{1}{2}} \\ [t^{-1}(1-t^{-1})]^{\frac{1}{2}} & 1-t^{-1} \end{pmatrix}.$$

Let e be the corresponding element of A. Thus $e^2=e=e^*$ and $ex=e'x$, $xe=xe'$ for x in A_1. Then $A^{**}\simeq A_1^{**}\oplus\mathbb{C}\simeq(A_0^{**}\otimes M_2)\oplus\mathbb{C}$.

Now let $p=\left(\begin{smallmatrix} p_0 & 0\\ 0 & 0\end{smallmatrix}\right)\oplus 0$. Clearly p is open. If a has the same meaning as in Example (3.7) (so that $a\in A_0$), then $p\le\left(\begin{smallmatrix} a & 0\\ 0 & 0\end{smallmatrix}\right)\oplus 0$, an element of A. Thus $\alpha(p)=1$. We claim that $\overline{p}=\left(\begin{smallmatrix} \overline{p}_0 & 0\\ 0 & 0\end{smallmatrix}\right)\oplus 1$. In fact, clearly $\overline{p}=\left(\begin{smallmatrix} \overline{p}_0 & 0\\ 0 & 0\end{smallmatrix}\right)\oplus r$, where r is 0 or 1. It is actually not important which is true. To show that $r=1$, we need only show $\nexists x\in A_1$ such that $(e-x)p=0$. This is equivalent to showing $\nexists x'\in A_0$ such that $(1-x')p_0=0$. This last follows from the fact that $\overline{p}_0$ is not compact in A_0^{**}.

Obviously, $\overline{p}\le\overline{p}(te)\overline{p}$. Therefore, $\alpha(\overline{p})\le t$. Since $\alpha(\overline{p}_0)=\infty$, there is a sequence (φ_n') in $F(\overline{p}_0)\cap S(A_0)$ such that $\varphi_n'\to 0$ in the weak* topology of A_0^*. Let (φ_n) be the corresponding sequence in $F(\overline{p})\cap S(A)$. Note that $A^*\simeq A_1^*\oplus\mathbb{C}$ and $\varphi_n\in A_1^*\oplus 0$. Since $\varphi_n'\to 0$, every weak* cluster point of (φ_n) has A_1^*-component 0. Since $\varphi_n(e)=t^{-1}$, $\forall n$, we conclude that $\varphi_n\to 0\oplus t^{-1}$ in the weak* topology of A^*. Therefore, $\alpha(\overline{p})\ge t$.

(3.9). (s,t), $1<s<t<\infty$.

Remark. If one is only interested in which of (1), (2), (3) (notation of Section 1) are satisfied by p and $\overline{p}$, then it is not necessary to consider this example, since (3.5) would suffice.

For this example, A is the same as in Example (3.8). In particular A_0, A_1, e', and e are the same. Let p_0 be the projection in A_0^{**} called p in Example (3.6), with the s of Example (3.6) replaced by s', where s' is a number in $(1,\infty)$ to be determined later. As in Example (3.8), we let $p=\left(\begin{smallmatrix} p_0 & 0\\ 0 & 0\end{smallmatrix}\right)\oplus 0$, an open projection in A^{**}, and $\overline{p}=\left(\begin{smallmatrix} \overline{p}_0 & 0\\ 0 & 0\end{smallmatrix}\right)\oplus 1$.

As in Example (3.8), we prove that $\alpha(\overline{p})=t$. It remains to calculate $\alpha(p)$. Since $\alpha(p_0)=s'$, there is a sequence (φ_n') in $F(p_0)\cap S(A_0)$ such that $\varphi_n'\to\varphi'$, where $\|\varphi'\|=(s')^{-1}$, in the weak* topology of A_0^*. Let (φ_n) be the corresponding sequence in $F(p)\cap S(A)$, and let φ be the element of A_1^* corresponding to φ'. Since $\varphi_n(e)=t^{-1}$, $\forall n$, we find that $\varphi_n\to\varphi\oplus(t^{-1}-(s')^{-1}t^{-1})$ in the weak* topology of A^* (see Example (3.8)). Since $\|\varphi\oplus(t^{-1}-(s')^{-1}t^{-1})\|=(s')^{-1}+t^{-1}-(s')^{-1}t^{-1}=(s')^{-1}+(1-(s')^{-1})t^{-1}$, $\alpha(p)^{-1}\le(s')^{-1}+(1-(s')^{-1})t^{-1}$. Let q be the projection in A_0 given by $q_n=q_\infty=e_1\times e_1$. Then $p_0qp_0=(s')^{-1}p_0$. Then we define an element a of A by

$$a = e+\begin{pmatrix} (1-t^{-1})q & -[t^{-1}(1-t^{-1})]^{\frac{1}{2}}q \\ -[t^{-1}(1-t^{-1})]^{\frac{1}{2}}q & -(1-t^{-1})q \end{pmatrix},$$

where the matrix is in A_1. Thus, relative to $A^{**} \simeq (A_0^{**} \otimes M_2) \oplus \mathbb{C}$, we have

$$a = \begin{pmatrix} q + t^{-1}(1-q) & [t^{-1}(1-t^{-1})^{\frac{1}{2}}](1-q) \\ [t^{-1}(1-t^{-1})]^{\frac{1}{2}}(1-q) & (1-t^{-1})(1-q) \end{pmatrix} \oplus 1.$$

Clearly, $\|a\| = 1$, and $pap = [(s')^{-1} + (1-(s')^{-1})t^{-1}]p$. Now, if we choose s' such that $(s')^{-1} + (1-(s')^{-1})t^{-1}) = s^{-1}$, we have that $\alpha(p) = s$.

4. Regularity, some variants, and relations with $\alpha(p)$

Before proceeding, the author would like to make a personal statement in the interest of full disclosure. In 1985 I was told that someone had done some work on variants of regularity. Specifically, I was told this mathematician's definition of k-regularity (given below); and I think I was told there was a special result on 2-regularity, but I was not told what this result was (it is likely similar to my Theorem 4.16, Corollary 4.17). Unfortunately, I was not interested enough then to ask this mathematician's name, and now (1990) the person who told me has forgotten the name. I made a strong effort to locate a name or paper without success. Except as noted above, all of my work is independent; in particular, all of my proofs are independent, but surely some of my results were obtained first by that anonymous inventor of k-regularity. Except for one comment in Example 4.15(b), I make no further reference to this unpleasant situation.

For p a projection in A^{**}, we have already defined $F(p)$, the norm-closed face of $Q(A)$ supported by p. There are many other convex subsets of A^* that can be defined in terms of p. Among these: $L(p) = \{f \in A^* : f(a) = f(ap), \forall a\}$, the left ideal generated by $F(p)$, $L_1(p) = \{f \in L(p) : \|f\| \leq 1\}$, $C(p) = \{f \in A^* : f \geq 0$ and $f(1-p) = 0\}$, the cone generated by $F(p)$, $V(p) = \{f \in A^* : f(a) = f(pap), \forall a\}$, the complex vector space generated by $F(p)$, $RV(p) = \{f \in V(p) : f = f^*\}$, the real vector space generated by $F(p)$, $V_1(p) = \{f \in V(p) : \|f\| \leq 1\}$, and $RV_1(p) = \{f \in RV(p) : \|f\| \leq 1\}$, the convex hull of $F(p) \cup (-F(p))$.

If p is closed, then all of the above sets are weak* closed; and if any of these sets is weak* closed, then p is closed. All of these facts were either proved by Effros in [13] or are easy consequences of results of [13]. The problem of relating the closure operation to these sets is more complicated. Effros showed that $L(p)^- = L(\overline{p})$, where $-$, when applied to a subset of A^*, always means weak* closure. We will use the following uninspired abbreviations:

(R_1) $L_1(p)^- = L_1(\overline{p})$,
(R_2) $F(p)^- = F(\overline{p})$,
(R_3) $C(p)^- = C(\overline{p})$,
(R_4) $V(p)^- = V(\overline{p})$,
(R_4') $RV(p)^- = RV(\overline{p})$,
(R_5) $V_1(p)^- = V_1(\overline{p})$,
(R_6) $RV_1(p)^- = RV_1(\overline{p})$,
$(R_7(K))$ $L_1(p)^- \supset K^{-1}L_1(\overline{p}),\ 1 < K < \infty$,
$(R_8(K))$ $F(p)^- \supset K^{-1}F(\overline{p}),\ 1 < K < \infty$,
$(R_9(K))$ $RV_1(p)^- \supset K^{-1}RV_1(\overline{p}),\ 1 < K < \infty$.

Then p is called *regular* (see Tomita [21]) if $\|a\overline{p}\| = \|ap\|$, $\forall a \in A$. Theorem 6.1 of [13] asserts that regularity is equivalent to each of $(R_1), (R_2), (R_3)$. Unfortunately, the proof tacitly assumed A to be unital in one place, and the theorem is not correct in the nonunital case. In general, regularity is equivalent to (R_1) and (R_2), and a correct proof of this is contained in [13], but (R_3) may be strictly weaker.

Each of the (R_i)'s is a variant of regularity. There are some deliberate omissions from the list. Aside from the one the reader has already noticed, we mention in passing a condition intermediate between $(R_8(K))$ and (R_3): every element of $C(\overline{p})$ is the weak* limit of a bounded net from $C(p)$. The reason for the omissions is not that we are trying to hide anything. We are simply trying to make a reasonable compromise between, on the one hand, presenting the minimum amount of material on regularity indicated by our interest in near relative compactness, and, on the other hand, attempting an exhaustive treatment of the variants of regularity. (We have not, in fact, done enough research for the latter course.)

The following implications are either obvious or were proved in [13]:

$$\begin{aligned} \text{regular} \Leftrightarrow (R_1) \Leftrightarrow (R_2) &\Rightarrow (R_3), (R_4), (R_4'), (R_6), \\ (R_4) &\Leftrightarrow (R_4'), \\ (R_1) &\Rightarrow \big(R_7(K)\big), \\ (R_2) &\Rightarrow \big(R_8(K)\big) \Rightarrow (R_3) \Rightarrow (R_4'), \\ (R_5) &\Rightarrow (R_6) \Rightarrow \big(R_9(K)\big) \Rightarrow (R_4'), \\ \big(R_8(K)\big) &\Rightarrow \big(R_9(K)\big). \end{aligned}$$

In particular, regularity implies all except (R_5), and all except $(R_7(K))$ imply (R_4). Therefore (R_4) is interesting, and we will say that p is *0-regular* if p satisfies (R_4). We say that p is *cone-regular* if it satisfies (R_3), *K-quasiregular* if it satisfies $(R_7(K))$, and *quasiregular* if K-quasiregular for some K. We believe that cone regularity and quasiregularity are the most interesting for near relative compactness, but we may have overlooked something. Finally, p is *k-regular* if

$$\begin{pmatrix} p & & 0 \\ & \ddots & \\ 0 & & p \end{pmatrix}$$

is regular in $(A \otimes M_k)^{**}$.

Before finally getting down to business, we need some more notation. A small amount of semicontinuity theory is used, and we follow the notation of [5]. Let $\tilde{A} = A + \mathbb{C}1$, where 1 is the identity of A^{**}. For $S \subset A_{sa}^{**}$, S^m is the set of (σ-strong) limits of bounded increasing nets from S, S_m is defined similarly with decreasing nets, and $-$, when applied to subsets of A^{**}, means norm closure. For example, $(\tilde{A}_{sa}^m)^-$ is the set of weakly lower semicontinuous elements of A^{**}, and $(A_+^m)^-$ is the set of positive strongly lower semicontinuous elements. Also, $M(A)$ is the multiplier algebra of A and $QM(A)$ is the space of quasimultipliers (both are subsets of A^{**}). In all the results of this section, A is an arbitrary C^*-algebra and p is a projection in A^{**}. The arguments presented below almost include a new proof of the equivalence of regularity, (R_1), and (R_2); but this is not our goal and we officially are assuming this equivalence.

A good way to deal with the (R_i)'s is to use the double polar theorem. For example, (R_1) is equivalent to the statement that $L_1(p)$ and $L_1(\overline{p})$ have the same polar in A. It is easy to compute the polars if one remembers that A^* is the predual of the W^*-algebra A^{**} and that the polar in A is just the intersection with A of the polar in A^{**}. The polar in A of $L(p)$ is $\{a \in A : ap = 0\}$. (Since always $L(p)^- = L(\overline{p})$, this tells us that $ap = 0 \Leftrightarrow a\overline{p} = 0$. This is often a good "working definition" of the closure of a projection.) The polar in A of $L_1(p)$ is $\{a \in A : \|ap\| \leq 1\}$. The polar in A_{sa} of $F(p)$ is $\{a \in A_{sa} : pap \leq p\}$. (The polar in A is $\{a \in A : \mathrm{Re}\, pap \leq p\}$.) The polar in A_{sa} of $C(p)$ is $\{a \in A_{sa} : pap \leq 0\}$. The polar in A of $V(p)$ is $\{a \in A : pap = 0\}$, and the polar in A_{sa} of $RV(p)$ is $\{a \in A_{sa} : pap = 0\}$. The polar in A of $V_1(p)$ is $\{a \in A : \|pap\| \leq 1\}$, and the polar in A_{sa} of $RV_1(p)$ is $\{a \in A_{sa} : \|pap\| \leq 1\}$.

The following is now obvious.

Proposition 4.1.

(a) *The projection p is K-quasiregular if and only if $\|a\overline{p}\| \leq K\|ap\|$, $\forall a \in A$.*
(b) *We have $(R_9(K)) \Leftrightarrow \|\overline{p}a\overline{p}\| \leq K\|pap\|$, $\forall a \in A_{sa}$.*
(c) *The projection p is 0-regular if and only if $pap = 0 \Rightarrow \overline{p}a\overline{p} = 0, \forall a \in A$.*
(d) *The projection p is cone-regular if and only if $pap \leq 0 \Rightarrow \overline{p}a\overline{p} \leq 0$, $\forall a \in A_{sa}$.*
(e) *We have $(R_5) \Leftrightarrow \|\overline{p}a\overline{p}\| = \|pap\|, \forall a \in A$.*

Throughout this section, $\sigma(php)$ means the spectrum of php relative to $pA^{**}p$.

Theorem 4.2.

(a) *The projection p is regular if and only if the top points in $\sigma(php)$ and $\sigma(\overline{p}h\overline{p})$ are the same, $\forall h \in (\tilde{A}^m_{sa})^-$.*
(b) *The projection p is K-quasiregular if and only if $\|\overline{p}h\overline{p}\| \leq K^2\|php\|$, $\forall h \in \overline{A^m_+}$.*

Proof. (a) Since $\|ap\|^2 = \|p(a^*a)p\|$, regularity is equivalent to the condition stated for all h in A_+. In general, if $h_i \nearrow h$ in a W^*-algebra, the top point in $\sigma(h_i)$ converges to the top point in $\sigma(h)$. This generalizes the condition to A^m_+. In general, the top point in $\sigma(p(h+\lambda)p)$ is $\lambda+$ top point in $\sigma(php)$. This generalizes the condition to $\{h \in A^{**}_{sa} : \exists\lambda \text{ with } h + \lambda \in A^m_+\} = \tilde{A}^m_{sa}$. In general, if $h_n \to h$ in norm, the top point in $\sigma(h_n)$ converges to the top point in $\sigma(h)$. This generalizes the condition to $(\tilde{A}^m_{sa})^-$.

The proof of (b) is similar, except that we leave out the step involving translation by λ. □

Corollary 4.3.

(a) *If p is regular, then $\|T\overline{p}\| = \|Tp\|$ whenever $T^*T \in (\tilde{A}^m_{sa})^-$, in particular whenever $T \in QM(A)$.*
(b) *If p is K-quasiregular, then $\|T\overline{p}\| \leq K\|Tp\|$ whenever $T^*T \in \overline{A^m_+}$, in particular whenever T is a right multiplier of A.*

(c) *If p is regular and h is in $QM(A)_{sa}$, in particular if h is in A_{sa} or $M(A)_{sa}$, then both extreme points of $\sigma(php)$ and $\sigma(\overline{p}h\overline{p})$ agree.*
(d) $(R_9(K^2)) \Rightarrow$ *K-quasiregular.*
(e) $(R_6) \Leftrightarrow$ *regular.*
(f) $(R_5) \Rightarrow$ *regular.*

Proof. For (a) and (b), we just have to quote [7, Proposition 4.1]. Condition (c) follows from the fact (see [5]) that $QM(A)_{sa} = \{h : h, -h \in (\tilde{A}^m_{sa})^-\}$. For (d), we use Proposition 4.1(b) to characterize $(R_9(K^2))$. By the proof of Theorem 4.2, $(R_7(K))$ is equivalent to the restriction of this condition from A_{sa} to A_+.

If we let $K \to 1^+$ in (d), we see that $(R_6) \Rightarrow$ regular. We already knew the converse. Thus (e) is proved and (f) follows. □

Pedersen [18] proved that if A is unital, then p is regular if and only if $a \geq p \Rightarrow a \geq \overline{p}, \forall a \in A_{sa}$. His arguments can be generalized to the following.

Theorem 4.4.

(a) *If there is a in A_{sa} such that $a \geq p$, then p is K-quasiregular if and only if $b \geq p \Rightarrow K^2 b \geq \overline{p}$, $\forall b \in A_{sa}$.*
(a′) *If there is a in A_{sa} such that $a \geq p$, then p is regular if and only if $b \geq p \Rightarrow b \geq \overline{p}$, $\forall b \in A_{sa}$.*
(b) *If general, p is K-quasiregular if and only if $h \geq p \Rightarrow K^2 h \geq \overline{p}, \forall h \in \tilde{A}_{sa}$ if and only if $h \geq p \Rightarrow K^2 h \geq \overline{p}$, $\forall h \in M(A)_{sa}$. If A is σ-unital, one can add $h \geq p \Rightarrow K^2 h \geq \overline{p}$, $\forall h \in QM(A)_{sa}$.*
(b′) *Same as (b) except omit "K" and "quasi-."*

Proof. (b) Assume that p is K-quasiregular and that $h \geq p$, where h is in $M(A)_{sa}$ or $QM(A)_{sa}$. Let $\varepsilon > 0$, and choose R a right multiplier of A such that R is invertible in A^{**} and $R^*R = (h+\varepsilon)^{-1}$. If A is σ-unital, the existence of R follows from [7, Proposition 4.8]. Otherwise, $h \in M(A)$ and we take $R = (h+\varepsilon)^{-\frac{1}{2}}$. Then $R^{-1}(R^*)^{-1} = h + \varepsilon \geq p$,

$$\begin{aligned} 1 &\geq RpR^*, \\ \|Rp\| &\leq 1, \\ \|R\overline{p}\| &\leq K, \quad \text{by Corollary 4.3(b)}, \\ R\overline{p}R^* &\leq K^2, \quad \text{and hence} \\ \overline{p} &\leq K^2 R^{-1}(R^*)^{-1} = K^2(h+\varepsilon). \end{aligned}$$

Since ε is arbitrary, $\overline{p} \leq K^2 h$.

Now assume that $h \geq p \Rightarrow K^2 h \geq \overline{p}$, $\forall h \in \tilde{A}_{sa}$. Assume that $x \in A$, $\|xp\| \leq 1$, and $\varepsilon > 0$. Then

$$\begin{aligned} \big\||x|p\big\| &\leq 1, \\ \big\|\big(|x|+\varepsilon\big)p\big\| &\leq 1+\varepsilon, \\ \big(|x|+\varepsilon\big)p\big(|x|+\varepsilon\big) &\leq (1+\varepsilon)^2, \quad \text{and hence} \\ p &\leq (1+\varepsilon)^2\big(|x|+\varepsilon\big)^{-2}. \end{aligned}$$

By hypothesis, $\overline{p} \le K^2(1+\varepsilon)^2(|x|+\varepsilon)^{-2}$. By reversing some of the above steps, we obtain $\|(|x|+\varepsilon)\overline{p}\| \le K(1+\varepsilon)$. Then taking limits as $\varepsilon \to 0^+$, we obtain $\|x\overline{p}\| = \||x|\overline{p}\| \le K$.

(a) Half of this follows from (b). Thus assume that $b \ge p \Rightarrow K^2 b \ge \overline{p}$, $\forall b \in A_{sa}$ and $\exists a \in A_{sa}$ such that $a \ge p$. Clearly, then, $\overline{p}$ satisfies property (2) of Section 1, and hence $\overline{p}$ is compact. Thus we can choose a in A_{sa} such that $\overline{p} \le a \le 1$. Now let (f_i) be any approximate identity of A, and let $e_i = a + (1-a)^{\frac{1}{2}} f_i (1-a)^{\frac{1}{2}}$. Then (e_i) is an approximate identity and $\overline{p} \le e_i \le 1$. Now suppose that h is in $\tilde{A}_{sa}$ and that $h \ge p$. Then $e_i h e_i \ge e_i p e_i = p$. Therefore, $K^2 e_i h e_i \ge \overline{p}$. Taking σ-strong limits in A^{**}, we see that $K^2 h \ge \overline{p}$. Then (b) implies that p is K-quasiregular.

(a′) and (b′) follow by letting $K \to 1^+$. □

Theorem 4.5. *If $\alpha(p) < \infty$, then p is cone-regular if and only if $pbp \ge p \Rightarrow \overline{p}b\overline{p} \ge \overline{p}$, $\forall b \in A_{sa}$.*

Proof. Assume that p is cone-regular and that $pbp \ge p$. Then

$$\begin{aligned} p(1-b)p &\le 0, \\ p(e_i-b)p &\le 0, \quad \forall i, \text{where } (e_i) \text{ is an approximate identity}, \\ \overline{p}(e_i-b)\overline{p} &\le 0, \quad \forall i, \text{by Proposition 4.1(d)}, \\ \overline{p}(1-b)\overline{p} &\le 0, \quad \text{by taking the } \sigma\text{-strong limit, and} \\ \overline{p}b\overline{p} &\ge \overline{p}. \end{aligned}$$

Next assume that $pap \ge p$ (possible since $\alpha(p) < \infty$), $pbp \ge p \Rightarrow \overline{p}b\overline{p} \ge \overline{p}$, $\forall b \in A_{sa}$, and $pxp \le 0$, $x \in A_{sa}$. Then

$$\begin{aligned} p(a-tx)p &\ge p, \quad \forall t > 0, \\ \overline{p}(a-tx)\overline{p} &\ge \overline{p}, \quad \forall t > 0, \\ \overline{p}x\overline{p} &\le t^{-1}\overline{p}(a-1)\overline{p}, \quad \forall t > 0, \text{and hence} \\ \overline{p}x\overline{p} &\le 0, \quad \text{by taking the limit as } t \to \infty. \end{aligned}$$

By Proposition 4.1(d), the above shows that p is cone-regular. □

Corollary 4.6. *If p is cone-regular, then $\alpha(p) = \alpha(\overline{p})$.*

Theorem 4.7. *Assume that p is cone-regular.*

(a) *If $\alpha(p) = 1$, then p is regular.*
(b) *If $\alpha(p) = s < \infty$, then p satisfies $(R_8(s))$. A fortiori p satisfies $(R_9(s))$ and p is $s^{\frac{1}{2}}$-quasiregular.*

Proof. (b) If $\varphi \in F(\overline{p})$, there is a net (φ_i) in $C(p)$ such that $\varphi_i \xrightarrow{w^*} \varphi$. Let $\psi_i = \frac{\varphi_i}{\|\varphi_i\|}$, an element of $F(p) \cap S(A)$. If $\|\varphi_i\| \to \infty$, then $\psi_i \xrightarrow{w^*} 0$, in contradiction to $\alpha(p) < \infty$ and Corollary 2.10. Thus, passing to a subnet if necessary, we may assume that $\|\varphi_i\| \to t < \infty$. Then $t^{-1}\varphi \in \overline{S}(p)$. By Theorem 2.9, $s \ge t$. By definition, this shows $(R_8(s))$.

Condition (a) follows from (b) if we let $s \to 1^+$. □

If A is nonunital, we can identify $\tilde{A}^{**}$ with $A^{**} \oplus \mathbb{C}$. Then any projection in A^{**} can also be regarded as an element of $\tilde{A}^{**}$. Of course, the next result is also true, trivially, if A is unital.

Theorem 4.8. *If p is a projection in A^{**}, then p is regular in $\tilde{A}^{**}$ if and only if p is regular in A^{**} and $\alpha(p)$, computed relative to A, is 1 or ∞.*

Proof. We first reduce to the case where p is closed, and hence regular, in A^{**}. Let p_1 and p_2 be the closures of p in A^{**} and $\tilde{A}^{**}$, respectively. It is easy to see that $p_1 \leq p_2$ (see [7, Proposition 3.54]). If p is regular in A^{**}, then by the weak* lower semicontinuity of norm, $[F(p) \cap S(A)]^- \supset F(p_1) \cap S(A)$. Since the weak* topologies of A^* and $\tilde{A}^*$ agree on $S(A)$, this shows that $F(p)$ and $F(p_1)$ have the same weak* closures in $\tilde{A}^*$. Thus p is regular in $\tilde{A}^{**}$ if and only if p_1 is. Also, $\alpha(p) = \alpha(p_1)$ by Corollary 4.6. Now assume that p is regular in $\tilde{A}^{**}$. Then the $\tilde{A}^*$-closure of $F(p) \cap S(A)$ includes $F(p_2) \cap S(\tilde{A})$, and hence it includes $F(p_1) \cap S(A)$. Again, since the two weak* topologies agree on $S(A)$, the A^*-closure of $F(p) \cap S(A)$ includes $F(p_1) \cap S(A)$, and hence p is regular in A^{**}.

From now on we assume that p closed in A^{**}, and we let $\overline{p}$ denote p_2. If $\alpha(p) = 1$, p is compact in A^{**}. This implies by results of Akemann [4] (see also [7, Definition-Lemma 2.47]) that p is closed in $\tilde{A}^{**}$, and hence regular in $\tilde{A}^{**}$. If $\alpha(p) > 1$, then p is not compact and, by the results cited above, p is not closed in $\tilde{A}^{**}$. Thus $\overline{p} = p \oplus 1$ in $A^{**} \oplus \mathbb{C}$. If p is regular in $\tilde{A}^{**}$, then $0 \oplus 1$ is in the weak* closure of $F(p) \cap S(A)$, where $\tilde{A}^*$ is identified with $A^* \oplus \mathbb{C}$. This implies that $0 \in \overline{S}(p)$, and by Corollary 2.10, $\alpha(p) = \infty$. If $\alpha(p) = \infty$, then by Corollary 2.10, 0 is in $\overline{S}(p)$. Since $\overline{S}(p)$ is convex, this shows that $\overline{S}(p) = F(p)$. Now $S(\tilde{A})$, with its weak* topology, can be identified with $Q(A)$, with its weak* topology. The map is $\varphi \leftrightarrow \varphi \oplus (1 - \|\varphi\|)$. Thus $\overline{S}(p) = F(p)$ implies that the $\tilde{A}^*$-closure of $F(p) \cap S(A)$ is $F(\overline{p}) \cap S(\tilde{A})$. This shows that p is regular in $\tilde{A}^{**}$. ☐

Theorem 4.9. *If p is K-quasiregular for $K < \sqrt{2}$, then p satisfies $(R_9(\frac{K^2}{2-K^2}))$.*

Proof. Use Proposition 4.1(b) to interpret $(R_9(\cdot))$. Assume that $a \in A_{sa}$, $\|pap\| = 1$, and $\|\overline{p}a\overline{p}\| = s$. By Theorem 3.3 or Corollary 3.4 of [7], there is b in A_{sa} such that $\|b\| = s$ and $\overline{p}b\overline{p} = \overline{p}a\overline{p}$. Therefore, also $pbp = pap$. Assume that the top point in $\sigma(\overline{p}a\overline{p})$ is s (otherwise replace a by $-a$ and b by $-b$). Since $b+s \in \overline{A_+^m}$, $\|\overline{p}(b+s)\overline{p}\| \leq K^2\|p(b+s)p\|$, by Theorem 4.2(b). Thus $2s \leq K^2(1+s)$, and hence $s \leq \frac{K^2}{2-K^2}$. ☐

Example 4.10. (a) First of all, we promised at the beginning of Section 3 to say what effect (ordinary) regularity has on $(\alpha(p), \alpha(\overline{p}))$. By Corollary 4.6, if p is regular, $\alpha(p) = \alpha(\overline{p})$. Thus we consider regularity only in connection with Examples (3.3), (3.4), and (3.5). By a result of Tomita (see [21, p. 25]), every central projection is regular. Thus our "examples" in Examples (3.3) and (3.4) are regular. In Example (3.5), we gave two examples with $\alpha(p) = \alpha(\overline{p}) = s$, $1 < s < \infty$. One example was closed, and hence regular. The other example was open but not regular (see Example 4.10(b) below). It is easy to modify this example and obtain a regular and even k-regular, $\forall k$, open projection with $\alpha(p) = \alpha(\overline{p}) = s$. Let A and v_n be as in Example (3.5), and define p by $p_\infty = 0$,

$p_{2n} = v_n \times v_n$, $p_{2n-1} = e_1 \times e_1$. As before, $\overline{p}$ differs from p only in that $(\overline{p})_\infty = e_1 \times e_1$. It is easy to check that p has the properties claimed. For regularity, we need that $\|a_\infty(\overline{p})_\infty\| \le \sup_n \|a_n p_n\|$, $\forall a \in A$. This follows from $\|a_\infty(e_1 \times e_1)\| \le \sup_n \|a_{2n-1}(e_1 \times e_1)\|$, which is true because $a_{2n-1} \to a_\infty$ in norm. The proof of k-regularity is similar.

(b) In this example p is cone-regular but not regular, and $\alpha(p) = s$, $1 < s < \infty$. Also this example shows that the estimates in Theorem 4.7(b) for the constants in $(R_i(\cdot))$ are sharp. This example is exactly the open example given in Example (3.5). If $pap \le 0$, a in A_{sa}, then $(a_n v_n, v_n) \le 0$. Since $a_n \to a_\infty$ in norm, $a_\infty \in \mathcal{K}$, and $v_n \overset{w}{\to} s^{-\frac{1}{2}} e_1$, we can take a limit and obtain $s^{-1}(a_\infty e_1, e_1) \le 0$. Therefore, $\overline{p}a\overline{p} \le 0$. By Proposition 4.1(d), p is cone-regular. If we define a in A by $a_n = a_\infty = e_1 \times e_1$, then $\|a\overline{p}\| = 1$. $\|ap\| = \sup|(e_1, v_n)| = s^{-\frac{1}{2}}$. This shows that p is at best $s^{\frac{1}{2}}$-quasiregular, precisely in accordance with Theorem 4.7(b). It follows a fortiori that the $(R_8(\cdot))$ and $(R_9(\cdot))$ estimates given in Theorem 4.7(b) are also sharp.

(c) In this example, p is cone-regular and open but not regular and not even quasiregular. By Theorem 4.7(b), $\alpha(p) = \infty$. Let $A = c \otimes \mathcal{K}$. Let $\mathcal{V}_m = \{u \in H : \|u\| = m^{-1}$ and $(u, e_k) = 0, \forall k > m\}$, $m = 1, 2, \ldots$. Let v_n be a sequence of unit vectors in H such that:

(i) $\forall n$, $\exists m$ such that $v_n = u_n + (1 - m^{-2})^{\frac{1}{2}} e_k$, where $u_n \in \mathcal{V}_m$ and $k > \max(m, n)$, and
(ii) $\{u_n\}$ contains a dense subset of each $\mathcal{V}_m$.

Define an open projection p in A^{**} by $p_\infty = 0$ and $p_n = v_n \times v_n$. For each u in $\mathcal{V}_m$, $u \times u$ is a weak cluster point of (p_n). Therefore, $(\overline{p})_\infty = 1$ (and, as always, $(\overline{p})_n = p_n$ for $n < \infty$). If $a \in A_{sa}$ and $pap \le 0$, then $(a_n v_n, v_n) \le 0$, $\forall n$. As in (b), it follows that $(a_\infty u, u) \le 0$, $\forall u \in \mathcal{V}_m$. Therefore, $a_\infty \le 0$ and $\overline{p}a\overline{p} \le 0$. Therefore, p is cone-regular. Now let $x = e_m \times e_m$. Then $xu = 0$ if $u \in \mathcal{V}_{m'}$, $m' < m$, $\|xe_k\| \to 0$ as $k \to \infty$, and $\|x\| = 1$. Thus $\limsup \|xv_n\| \le m^{-1}$. Now define a_N in A by

$$(a_N)_n = \begin{cases} 0, & n \le N, \\ x, & N+1 \le n \le \infty. \end{cases}$$

Then $\|a_N \overline{p}\| = 1$ and $\limsup_{N \to \infty} \|a_N p\| \le m^{-1}$. Since m is arbitrary, p is not quasiregular.

(d) In this example A is unital and p is $\sqrt{2}$-quasiregular but not 0-regular. This shows that the constant $\sqrt{2}$ in Theorem 4.9 is sharp. Let $A = c \otimes M_2$. Define an open projection p in A^{**} by $p_\infty = 0$, $p_{2k-1} = e_1 \times e_1$, and $p_{2k} = e_2 \times e_2$. Then $(\overline{p})_\infty = 1$. For a in A, $\|ap\| = \max(\sup_k \|a_{2k-1} e_1\|, \sup_k \|a_{2k} e_2\|)$. Since $a_n \to a_\infty$ in norm, $\|ap\| \ge \max(\|a_\infty e_1\|, \|a_\infty e_2\|)$. We conclude easily, for example, by looking at the Hilbert–Schmidt norm, that $\|a_\infty\| \le \sqrt{2}\|ap\|$. Therefore, $\|a\overline{p}\| \le \sqrt{2}\|ap\|$. If we define b in A_{sa} by $b_n = b_\infty = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then $pbp = 0$ and $\overline{p}b\overline{p} \ne 0$. Therefore, p is not 0-regular.

Theorem 4.11. *If $\alpha(p) = s$ and p is K-quasiregular with $K^2 < s/(s-1)$, then $\alpha(\overline{p}) \le s/[s - K^2(s-1)] < \infty$. In particular, if $s = 1$, then $\alpha(\overline{p}) = 1$.*

Proof. Choose $s_1 > s$ such that $K^2 < s_1/s_1 - 1$, and choose a in A_+ such that $pap \geq p$ and $\|a\| \leq s_1$. Then $p(1-a)p \leq 0$ and hence $p(s_1 - a)p \leq (s_1 - 1)p$. Since $s_1 - a \in \overline{A_+^m}$, Theorem 4.2(b) implies that $\overline{p}(s_1 - a)\overline{p} \leq K^2(s_1 - 1)\overline{p}$. Therefore, $\overline{p}a\overline{p} \geq [s_1 - K^2(s_1 - 1)]\overline{p}$. Since $s_1 - K^2(s_1 - 1) > 0$, this shows that $\alpha(\overline{p}) \leq s_1/s_1 - K^2(s_1 - 1)$. Now let $s_1 \to s^+$. □

Lemma 4.12. *Let H be a Hilbert space, let e_1 be a unit vector in H, and let Q be the projection with range $\{e_1\}^\perp$. Let $0 < t < 1$, $W_1 = \{u \in H : \|u\| = 1$ and $\|Qu\| \leq t\}$, and $W = \{u \in H : \|u\| \leq 1$ and $\|Qu\| \leq t\}$. Then W is a balanced convex set and is the closed convex hull of W_1.*

Proof (Sketch). W is weakly compact, and W_1 is the set of extreme points. □

Example 4.13. (a) In this example, $\alpha(p) = s$, $1 < s < \infty$, p is open and K-quasiregular with $K^2 = \frac{s}{s-1}$, and $\alpha(\overline{p}) = \infty$. This shows that the estimate on K in Theorem 4.11 is sharp. This also shows that K-quasiregularity does not imply cone regularity for $K > 1$.

Let $A = c \otimes \mathcal{K}$ and $t = (1 - s^{-1})^{\frac{1}{2}} = K^{-1}$, and let (v_n) be a dense sequence in W_1. Define an open projection p in A^{**} by $p_\infty = 0$ and $p_n = v_n \times v_n$. As in Example (3.6), we see that $(\overline{p})_\infty = 1$, so that $\alpha(\overline{p}) = \infty$. If a in A is defined by $a_n = a_\infty = e_1 \times e_1$, then $pap \geq s^{-1}p$. Therefore, $\alpha(p) \leq s$. If $b \in A$ and $\|bp\| \leq 1$, then $\|b_n v_n\| \leq 1$, $\forall n$. Therefore, $\|b_\infty u\| \leq 1$, $\forall u \in W_1$. By Lemma 4.12, $\|b_\infty u\| \leq 1$, $\forall u \in W$, and hence $\|b_\infty\| < t^{-1} = K$. We have shown that $\|b\overline{p}\| \leq K$, and thus p is K-quasiregular. Theorem 4.11 now shows that $\alpha(p) = s$.

(b) If we want a unital example where K-quasiregularity does not imply (cone) regularity, or better, if we want p to be K-quasiregular and not K'-quasiregular for any $K' < K$, we can use the same construction as in (a) for $A = c \otimes M_2$. Thus now the H of Lemma 4.12 is 2-dimensional and $\alpha(p) = \alpha(\overline{p}) = 1$. Since W contains the ball of radius t but no larger balls, the separation theorem shows that for any $t' > t$ we can find a linear functional h on H such that $|h(u)| \leq 1$, $\forall u \in W$, and $|h(u_0)| > 1$ for some u_0 with $\|u_0\| = t'$. Define a in A by $a_n = a_\infty = e_1 \times y$, where y in H is such that $h(\cdot) = (\cdot, y)$. Then $\|ap\| \leq 1$ and $\|a\overline{p}\| \geq (t')^{-1}$.

(c) This example shows that the estimate on $\alpha(\overline{p})$ in Theorem 4.11 is sharp when $K^2 < \frac{s}{s-1}$. The construction is similar to Example (3.9), but unfortunately it must be a bit more complicated if we want p to be open. Thus we assume given s and t such that $1 < s < t < \infty$, and we let s' be as in Example (3.9). Let $K^2 = \frac{s'}{s'-1}$. The reader can compute that $t = \frac{s}{s - K^2(s-1)}$. (If only s is given, then t can be chosen so that K takes any arbitrary value in $(1, (\frac{s}{s-1})^{\frac{1}{2}}$.)

Thus we perform the construction of (a) with s' instead of s. Let A_0 and p_0 be the algebra and projection produced by this, and let A_1, e', A, e, and q have the same meaning as in Example (3.9) (and Example (3.8)). (We now have $p_0 q p_0 \geq (s')^{-1} p_0$ instead of equality.)

For each n choose a unit vector $z_n = x_n \oplus y_n$ in $H \oplus H$ such that:

(i) $(x_n, e_1) = (y_n, e_1) = 0$,
(ii) $(x_n, v_n) = 0$,
(iii) $e' z_n = z_n$,
(iv) $z_n \xrightarrow{w} 0$ as $n \to \infty$.

Let p be the open projection in A_1^{**} defined by $p_\infty = 0$, and let $p_n = (v_n \oplus 0) \times (v_n \oplus 0) + z_n \times z_n$. As before, p is also regarded as an open projection in A^{**}, the closure, $\overline{p}$, of p in A_1^{**} is the same as p except that $(\overline{p})_\infty = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and the closure of p in A^{**} is $\overline{p} \oplus 1$.

Let a have the same meaning as in Example (3.9) (a is a specific element of norm 1 in A_+). Then using (i), (ii), (iii), we see, similarly to Example (3.9), that $pap \geq s^{-1}p$ and $(\overline{p} \oplus 1)e(\overline{p} \oplus 1) \geq t^{-1}(\overline{p} \oplus 1)$. Also, the proofs in Example (3.8) and Example (3.9) that $\alpha(p) \geq s$ and $\alpha(\overline{p} \oplus 1) \geq t$ still apply, since $p \geq \left(\begin{smallmatrix} p_0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \oplus 0$.

It remains to show that p is K-quasiregular. If $b \in A$, since the A_1^{**}-component of b is in $M(A_1)$, the calculations of (a) (see Corollary 4.3(b)) show that $\|b\overline{p}\| \leq (\frac{s'}{s'-1})^{\frac{1}{2}}\|bp\|$. (Note that to show this, we need only estimate $\|b_\infty(\overline{p})_\infty\|$, and we do not need to consider the z_n's.) If $b = \lambda e + x$, $x \in A_1$, then by (iii) and (iv) $\|bp\| \geq |\lambda|$. This concludes the sketch of the proof.

If we drop the openness requirement, we can get an easier example. Let $p = \left(\begin{smallmatrix} p_0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \oplus 1$. In this case p is abelian, as in many of our earlier examples.

The gist of what we have done so far is that, in general, cone regularity and quasiregularity are independent of one another, and that both have significant relations with near relative compactness. A special case of our results is that if p is either cone-regular or quasiregular, then condition (2) of Section 1 implies that p is relatively compact (briefly, $\alpha(p) = 1 \Rightarrow \alpha(\overline{p}) = 1$).

In Section 6, we will consider situations in which $\alpha(p_1 \vee p_2)$ can be bounded in terms of $\alpha(p_1)$ and $\alpha(p_2)$. For the question of regularity of $p_1 \vee p_2$, we will consider only the special case where $\overline{p}_1\overline{p}_2 = 0$, so that $p_1 \vee p_2 = p_1 + p_2$ and $(p_1 \vee p_2)^- = \overline{p}_1 + \overline{p}_2$. Also, we consider only the hypothesis that p_1 and p_2 are regular in the ordinary sense, except when generalizations are easy. The conclusions available even from these seemingly strong hypotheses are not strong. Of course, we do not expect to be able to prove that $p_1 + p_2$ is regular—otherwise there would be no purpose for the concept of k-regularity.

Proposition 4.14.

(a) *If p_i is K_i-quasiregular for $i = 1, 2$, and if $\overline{p}_1\overline{p}_2 = 0$, then $p_1 + p_2$ is $(K_1^2 + K_2^2)^{\frac{1}{2}}$-quasiregular.*

(b) *If p is 0-regular, then* $\mathrm{Diag}(p, \ldots, p)$ *is 0-regular in $(A \otimes M_k)^{**}$.*

Proof. (a) By [1], $\overline{p}_1 + \overline{p}_2$ is closed, so that $(p_1 + p_2)^- = \overline{p}_1 + \overline{p}_2$. If $\|a(p_1 + p_2)\| \leq 1$ for a in A, then $\|a\overline{p}_i\| \leq K_i$. Thus $a(\overline{p}_1 + \overline{p}_2)a^* \leq a\overline{p}_1a^* + a\overline{p}_2a^* \leq K_1^2 + K_2^2$, and $\|a(\overline{p}_1 + \overline{p}_2)\| \leq (K_1^2 + K_2^2)^{\frac{1}{2}}$.

(b) Since $\mathrm{Diag}(p, \ldots, p)^- = \mathrm{Diag}(\overline{p}, \ldots, \overline{p})$, this follows immediately from Proposition 4.1(c). □

Example 4.15. (a) Let $\pi : M_2 \to B(H)$ be a unital $*$-representation, so that π induces a faithful homomorphism from M_2 to the Calkin algebra, $B(H)/\mathcal{K}$. Let A be the extension of $\mathcal{K} \oplus \mathcal{K}$ by M_2 induced by $\pi \oplus \pi : M_2 \to B(H) \oplus B(H)$ (see [10]). Then A^{**} can be identified with $B(H) \oplus B(H) \oplus M_2$. Let $\{e_{ij} : i, j = 1, 2\}$ be a system of matrix units for M_2, and define $p_1 = \pi(e_{11}) \oplus 0 \oplus 0$, $p_2 = 0 \oplus \pi(e_{22}) \oplus 0$. It is easy to check that $\overline{p}_1 = \pi(e_{11}) \oplus 0 \oplus e_{11}$, $\overline{p}_2 = 0 \oplus \pi(e_{22}) \oplus e_{22}$, and that p_i is k-regular, $\forall k$, and open. If a_0 is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in M_2 and $a = \pi(a_0) \oplus \pi(a_0) \oplus a_0$, then

$a \in A$, $(p_1+p_2)a(p_1+p_2) = 0$, and $(\overline{p}_1+\overline{p}_2)a(\overline{p}_1+\overline{p}_2) \neq 0$. Thus p_1+p_2 is not even 0-regular. Thus p_1+p_2 cannot be better than $\sqrt{2}$-quasiregular by Theorem 4.9, so that Proposition 4.14(a) is sharp, at least in the special case $K_1 = K_2 = 1$. Note that A is unital.

(b) The fact that for every $k > 1$ there is a projection which is $(k-1)$-regular but not k-regular is surely due to the inventor of k-regularity. For completeness, we write down a natural example, but the proof that it is correct is left to the reader. Let $A = c \otimes M_k$, a unital algebra, and let (q_n) be a sequence dense in the set of rank $k-1$ projections in M_k. Define a $(k-1)$-regular open projection p in A^{**} by $p_\infty = 0$ and $p_n = q_n$. Then $(\overline{p})_\infty = 1$ and p is not k-regular.

Theorem 4.16. *If $\overline{p}_1\overline{p}_2 = 0$, then p_1+p_2 is regular if and only if*

(i) *p_1 and p_2 are regular, and*
(ii) *$\{f \in A^* : \|f\| \leq 1$ and $f(a) = f(p_1ap_2), \forall a\}$ is weak* dense in $\{f \in A^* : \|f\| \leq 1$ and $f(a) = f(\overline{p}_1a\overline{p}_2), \forall a\}$.*

Proof. We first assume (i) and (ii) and prove $(R_2), F(p_1+p_2)^- = F(\overline{p}_1+\overline{p}_2)$. Since $F(p_1+p_2)^-$ is convex, it is enough to show that it contains each pure state φ in $F(\overline{p}_1+\overline{p}_2)$. Let $\pi = \pi_\varphi$. Then $\varphi = (\pi(\cdot)v, v)$ for some unit vector v in H_π, $v = v_1+v_2$, where $v_i = \pi^{**}(\overline{p}_i)v$. If v_1 or v_2 is 0, then φ is in $F(\overline{p}_2)$ or $F(\overline{p}_1)$, and by (i) φ is in $F(p_1+p_2)^-$. Thus assume that $v_1, v_2 \neq 0$, and let $u_i = \frac{v_i}{\|v_i\|}$. If we define f in A^* by $f(\cdot) = (\pi(\cdot)u_2, u_1)$, then $\|f\| = 1$ since π is irreducible (use [15]), and clearly $f(\cdot) = f(\overline{p}_1 \cdot \overline{p}_2)$.

Consider A^{**} as a subset of $B(H)$, via the universal representation of A. (This is just a "bookkeeping" convenience.) By (ii), we can find a net (f_i) such that $\|f_i\| = 1$, $f_i(\cdot) = f_i(p_1 \cdot p_2)$, and $f_i \xrightarrow{w^*} f$. Write $f_i(\cdot) = (\cdot w_i^2, w_i^1)$, where w_i^j is a unit vector in p_jH, $j = 1, 2$. We obtain this representation of f_i from the polar decomposition (see [20]). Note that $|f_i| = (\cdot w_i^2, w_i^2), |f_i^*| = (\cdot w_i^1, w_i^1)$, $|f| = (\pi(\cdot)u_2, u_2)$, and $|f^*| = (\pi(\cdot)u_1, u_1)$. Since $f_i \xrightarrow{w^*} f$ and $\|f_i\| \to \|f\|$, it follows from [13] that $|f_i| \xrightarrow{w^*} |f|$ and $|f_i^*| \xrightarrow{w^*} |f^*|$. Now let $w_i = \|v_1\|w_i^1 + \|v_2\|w_i^2$ and $\varphi_i = (\cdot w_i, w_i)$. Then $\varphi_i \in F(p_1+p_2)$ and $\varphi_i \xrightarrow{w^*} \varphi$.

Next, assume that p_1+p_2 is regular. Every element f of $V(\overline{p}_1+\overline{p}_2)$ can be written uniquely as $\sum_{j,k=1}^2 f^{jk}$, where $f^{jk}(\cdot) = f^{jk}(\overline{p}_j \cdot \overline{p}_k)$. Also a bounded net (f_i) in $V(\overline{p}_1+\overline{p}_2)$ converges weak* to f if and only if $f_i^{jk} \xrightarrow{w^*} f^{jk}, \forall j, k$. Of course, $F(\overline{p}_1+\overline{p}_2) \subset V(\overline{p}_1+\overline{p}_2)$, so that the above applies. If $\varphi \in F(\overline{p}_1) \cap S(A)$, then by (R_2), there is a net (φ_i) in $F(p_1+p_2)$ such that $\varphi_i \xrightarrow{w^*} \varphi$. Then also $\varphi_i^{11} \xrightarrow{w^*} \varphi$. Since $\varphi_i^{11}(\cdot) = \varphi_i(\overline{p}_1 \cdot \overline{p}_1)$ and $\varphi_i \in F(p_1+p_2)$, $\varphi_i^{11} \in F(p_1)$. Therefore p_1 and, similarly, p_2 are regular.

Now assume that $\|f\| = 1$ and $f(\cdot) = f(\overline{p}_1 \cdot \overline{p}_2)$. Write $f(\cdot) = (\cdot u_2, u_1)$, where u_j is a unit vector in $\overline{p}_jH$. Let $v = 2^{-\frac{1}{2}}u_1 + 2^{-\frac{1}{2}}u_2$ and $\varphi = (\cdot v, v)$. Then $\varphi \in F(\overline{p}_1+\overline{p}_2) \cap S(A)$. By (R_2), there is a net (φ_i) in $F(p_1+p_2)$ such that $\varphi_i \xrightarrow{w^*} \varphi$. Therefore, $\varphi_i^{jk} \xrightarrow{w^*} \varphi^{jk}$. Then $2^{-1} = \|\varphi^{jj}\| \leq \liminf \|\varphi_i^{jj}\|$. Since $\|\varphi_i^{11}\| + \|\varphi_i^{22}\| = \|\varphi_i\| \leq 1$, it follows that $\|\varphi_i^{jj}\| \to 2^{-1}$. Now $\|\varphi_i^{12}\| \leq \|\varphi_i^{11}\|^{\frac{1}{2}}\|\varphi_i^{22}\|^{\frac{1}{2}}$, and $\varphi_i^{12} \xrightarrow{w^*}$

$\varphi^{12} = 2^{-1}f$. Therefore, $\|\varphi_i^{12}\| \to 2^{-1}$ and $\frac{\varphi_i^{12}}{\|\varphi_i^{12}\|} \to f$. Finally, since $\varphi_i \in F(p_1+p_2)$ and $\varphi_i^{12} = \varphi_i(\overline{p}_1 \cdot \overline{p}_2)$, $\varphi_i^{12} = \varphi_i^{12}(p_1 \cdot p_2)$. Thus (ii) is proved. □

Corollary 4.17. *We have 2-regular ⇔ (R_5). Explicitly, p is 2-regular if and only if $\{f \in A^* : \|f\| \le 1$ and $f(a) = f(pap), \forall a\}$ is weak* dense in $\{f \in A^* : \|f\| \le 1$ and $f(a) = f(\overline{p}a\overline{p}), \forall a\}$.*

Proof. By Corollary 4.3(f), $(R_5) \Rightarrow$ regular. □

Remark. The boundedness hypothesis on the net (f_i) in the proof of Theorem 4.16 is not really needed (to deduce $f_i^{jk} \to f^{jk}$), at least when A is σ-unital. A proof can be based on the Urysohn lemma (see [7, Lemma 3.31]).

5. Special results for open projections

Theorem 5.1. *Suppose that p and q are projections in A^{**} such that p is closed, q is open, $\alpha(p) < \infty$, and $\|p - q\| < 1$. Then q is compact. Thus q is actually in A.*

Proof. Since $\|p - q\| < 1$, there is $\varepsilon > 0$ such that $pqp \ge \varepsilon p$. Let B be the hereditary C^*-subalgebra of A supported by q (notation: $B = \text{her}(q)$), and let (e_i) be an approximate identity of B. Then $pe_ip \nearrow pqp$, with convergence in the σ-strong topology. Since p is closed, $\overline{S}(p)$ is a weak* compact subset of $F(p)$, and $\|\varphi\| \ge \alpha(p)^{-1}$ for φ in $\overline{S}(p)$. Therefore, $\forall\varphi \in \overline{S}(p)$, $\lim \varphi(e_i) = \lim \varphi(pe_ip) \ge \varepsilon\alpha(p)^{-1}$. By Dini's theorem, for i sufficiently large $\varphi(e_i) \ge 2^{-1}\varepsilon\alpha(p)^{-1}$, $\forall\varphi \in \overline{S}(p)$. Since $\overline{S}(p) \supset F(p) \cap S(A)$, we have shown that $pe_ip \ge \delta p$ for $i \ge i_0$, where $\delta = 2^{-1}\varepsilon\alpha(p)^{-1} > 0$.

Now let p_i be the range of projection of $e_i^{\frac{1}{2}}p$ for $i \ge i_0$. By the proof of Theorem 2.6(b), p_i is compact and $\|p_i - p\| < 1$. Also, clearly, $p_i \le q$. Since also $\|q - p\| < 1$, it follows that $p_i = q$ (see Lemma 7 of [9] and the subsequent remark).

Since q is both closed and open, $q \in M(A)$. It is easy to see that a projection in $M(A)$ is compact, as an element of A^{**}, if and only if it is in A. □

Example 5.2. There are a closed but not open projection p and an open but not closed projection q such that $\alpha(q) < \infty$ and $\|p - q\| < 1$. Let $A = c \otimes \mathcal{K}$. Let $v_n = 2^{-\frac{1}{2}}e_2 + 2^{-\frac{1}{2}}e_{n+2}$ and $w_n = 2^{-\frac{1}{2}}e_1 + 2^{-\frac{1}{2}}e_{n+3}$. Define p and q by $p_\infty = q_\infty = e_1 \times e_1$, $p_n = w_n \times w_n + e_{n+2} \times e_{n+2}$, and $q_n = e_1 \times e_1 + v_n \times v_n$. Then $\alpha(q) = 2$ and $\|p - q\| = 2^{-\frac{1}{2}}$. (Note that $d_a(q, RC) = \frac{\pi}{4}$, $d_a(p, q) = \frac{\pi}{4}$.)

We have already mentioned that if A is σ-unital and q is an open projection in A^{**}, then $\alpha(q) < \infty$ if and only if every closed subprojection of q is compact. This results from the combination of Theorems 2 and 4 of [8], and its proof there is indirect. It is shown that both conditions are equivalent to $M(A, B) = B$, where $B = \text{her}(q)$ and $M(A, B) = M(A) \cap (qA^{**}q)$. The proof of one half of this result in [8] is actually quite easy granted [7, Lemma 3.31], an operator-algebraic version of Urysohn's lemma. Nevertheless, we give a more direct proof of this half.

Lemma 5.3. *If A is σ-unital, p and q are projections in A^{**}, p is closed, q is open, $p \le q$, and $\alpha(q) < \infty$, then p is compact.*

Proof. By [7, Lemma 3.31], there is h in $M(A)_{sa}$ such that $p \le h \le q$. Choose a in A_{sa} such that $qaq \ge q$. Then $h^{\frac{1}{2}}ah^{\frac{1}{2}} = h^{\frac{1}{2}}qaqh^{\frac{1}{2}} \ge h^{\frac{1}{2}}qh^{\frac{1}{2}} = h \ge p$. Since $h^{\frac{1}{2}}ah^{\frac{1}{2}} \in A$, p satisfies property (2) of Section 1, and by Akemann [4] p is compact. □

Theorem 5.4. *If A is any C^*-algebra, p and q are projections in A^{**}, p is closed, q is open, $p \le q$, and $\alpha(q) < \infty$, then p is compact.*

Proof. The proof is by reduction to the separable case. Choose a in A_{sa} such that $qaq \ge q$. Let $q' = 1 - p$, $B = \text{her}(q)$, and $B' = \text{her}(q')$. Choose σ-unital hereditary C^*-subalgebras B_0 of B and B'_0 of B' such that $a \in \text{her}(B_0 \cup B'_0)$. (For a subset S of A, $\text{her}(S)$ denotes the smallest hereditary C^*-subalgebra including S.) Let e_0, e'_0 be strictly positive elements of B_0, B'_0, and let $A_0 = C^*(e_0, e'_0, a)$, a separable C^*-subalgebra of A. If p is not compact, there is a net $(\varphi_i)_{i \in D}$ in $F(p) \cap S(A)$ such that $\varphi_i \xrightarrow{w^*} \varphi$ and $\|\varphi\| < 1$. The construction of the desired separable C^*-subalgebra of A proceeds from here by recursion.

Step 1. Since A_0 is separable, we can choose $i_1, i_2, \dots$ such that $\varphi_{i_n}|_{A_0} \xrightarrow{w^*} \varphi|_{A_0}$. Since each $\|\varphi_i|B\| = 1$ (because $\varphi_i \in F(p) \cap S(A)$ and $p \le q$), we can find a countable subset E_1 of $\{b \in B : 0 \le b \le 1\}$ such that $\sup \varphi_{i_n}|_{E_1} = 1$ for each n. Then, by [7, Lemma 3.30], we can find open projections q_1, q'_1 such that $q_1 \le q$, $q'_1 \le q', q_1 q'_1 = q'_1 q_1$, $\text{her}(q_1)$ and $\text{her}(q'_1)$ are σ-unital, and $E_1 \cup \{e_0\} \subset \text{her}(q_1)$, $e'_0 \in \text{her}(q'_1)$. Let e_1, e'_1 be strictly positive elements of $\text{her}(q_1)$, $\text{her}(q'_1)$, and let $A_1 = C^*(A_0, E_1, e_1, e'_1)$. Note that since $A_0 \subset \text{her}(e_0, e'_0)$, $A_1 \subset \text{her}(q_1 \vee q'_1)$.

Step 2. We proceed in the same way as step 1, starting with A_1, e_1, e'_1 instead of A_0, e_0, e'_0. (The sequence (φ_{i_n}) constructed in step 2 might be disjoint from the one in step 1. They simply are both sequences, not *subsequences*, constructed from the elements of the original net (φ_i).)

The process is continued recursively, and we get increasing sequences $(q_n), (q'_n)$ of open projections and an increasing sequence (A_n) of separable subalgebras. Let $q_\infty = \lim q_n$, $q'_\infty = \lim q'_n$, and $A_\infty = C^*(\bigcup_n A_n)$. Then q_∞, q'_∞ are open, $\text{her}(q_\infty)$, $\text{her}(q'_\infty)$ are σ-unital, $q_\infty q'_\infty = q'_\infty q_\infty$, and A_∞ is separable. Since each q_n is the range projection of e_n, and $e_n \in A_\infty$, $q_\infty \in A_\infty^{**} \subset A^{**}$. Similarly, $q'_\infty \in A_\infty^{**}$. Since $A_n \subset \text{her}(q_n \vee q'_n)$, $q_\infty \vee q'_\infty$ is the identity of A_∞^{**}. Now q_∞ and q'_∞ are open as elements of A_∞^{**} (see [7, Proposition 2.14(a)]), and hence if $p_\infty = (q_\infty \vee q'_\infty) - q'_\infty$, then p_∞ is a closed projection in A_∞^{**}. Since each φ_i is supported by $p, \varphi_i|_{A_\infty}$ is supported by p_∞. Let $T = \{\varphi_i|_{A_\infty} : i \in D \text{ and } \|\varphi_i|_{A_\infty}\| = 1\}$. The set T contains all the φ_{i_n}'s constructed in all the steps. Thus $\varphi|_{A_\infty}$ is in the weak* closure of T, and $\|\varphi|_{A_\infty}\| < 1$. This shows that p_∞ is not compact in A_∞^{**}. Now $a \in A_\infty$ and $qaq \ge q$ implies that $q_\infty a q_\infty \ge q_\infty$. Thus all the hypotheses of Lemma 5.3 are satisfied by $A_\infty, p_\infty, q_\infty$ and we have a contradiction. Thus p is compact after all. □

Example 5.5. We give a commutative counterexample to the converse of Theorem 5.4. Of course, the example must be non-σ-unital. Let X be an ordered set with the order type of the first uncountable ordinal, endowed with the order topology. Then X is locally compact Hausdorff, and we let $A = C_0(X)$. Let U be the open set consisting of all isolated points of X (nonlimit ordinals), and let q be the corresponding open projection in A^{**}. Since U is cofinal in $X, \alpha(q) = \infty$. We claim that any closed subprojection p of q is compact. In fact, p corresponds to a closed subset F of X such that $F \subset U$. Note that $X \setminus U$ is a closed cofinal set. Any two closed cofinal subsets of X have a nonempty intersection (see [16]). Therefore F is not cofinal, and F and p are compact.

A result of Akemann (see [3, Theorem I.4]) states that if $a \in A$, p is a closed projection in A^{**}, and $\|ap\| \leq 1$, then for any $\varepsilon > 0$ there is an open projection q such that $q \geq p$ and $\|aq\| < 1+\varepsilon$. This is appealing from the point of view of noncommutative topology, and thus it is natural to consider similar questions. More discussion is given after the next theorem, but we have no particular applications in mind.

Theorem 5.6. *Assume that p is a closed projection in A^{**}, $a \in A$, and $0 < \varepsilon < 1$.*
- (a) *If $a^* = a$ and $pap \leq 0$, then there is an open projection q such that $q \geq p$ and $qaq \leq \varepsilon q$.*
- (b) *If $a^* = a$ and $pap \leq p$, then there is an open projection q such that $q \geq p$ and $qaq \leq (1+\varepsilon)q$.*
- (c) *There is an open projection q such that $q \geq p$ and $\|qaq\| < \|pap\| + \varepsilon$.*
- (d) *If $a^* = a$ and $pap \geq p$, then there is an open projection q such that $q \geq p$ and $qaq \geq (1-\varepsilon)q$ if and only if p is compact.*

Proof. Let L and R be the closed left and right ideals of A corresponding to p ($L = A\,\mathrm{her}(1-p), R = \mathrm{her}(1-p)A$). By a result of Combes (see [11, Proposition 6.2]), $L + R$ is closed, and $L + R = \{a \in A : pap = 0\}$. Let (e_i) be an approximate identity of $\mathrm{her}(1-p)$.

(c) It is known from [6] that $\|pap\| = \|a + L + R\|$ in $A/L + R$. Thus we can find $l \in L$ and $r \in R$ such that $\|a + l + r\| < \|pap\| + \frac{\varepsilon}{3}$. Then there is an i such that $\|l(1-e_i)\|$, $\|(1-e_i)r\| < \frac{\varepsilon}{3}$. If we let $q = E_{[0,\delta)}(e_i)$ for δ sufficiently small, then $\|lq\|$, $\|qr\| < \frac{\varepsilon}{3}$, and hence $\|qaq\| < \|pap\| + \varepsilon$.

(a) and (b) Now $a^* = a$ and we will, possibly unnecessarily, use [7, Corollary 3.4]. If I is a closed interval that contains $\sigma(pap) \cup \{0\}$, then there is b in A_{sa} such that $\sigma(b) \subset I$ and $pbp = pap$. In case (a), $I = [*, 0]$, and in case (b), $I = [*, 1]$. Then, similarly to case (c), $a = b + l + r$ where $b \leq 0$ or $b \leq 1$, and we need only choose q so that $\|q(l + r)q\| \leq \varepsilon$.

(d) If $qaq \geq (1-\varepsilon)q$, then $\alpha(q) < \infty$ and Theorem 5.4 implies that p is compact. If p is compact, then p is closed in $\tilde{A}^{**}$, and we can apply (a) to $1 - a$ and $\tilde{A}$. □

The inequalities considered in (a), (b), (c), and [3] correspond to different kinds of regularity, interpreted via polars as in Section 4. Thus (a) relates to (R_3), cone regularity; (b) relates to (R_2), regularity; (c) relates to (R_5) and special cases of (c) relate to (R_6) and (R_4); and [3] relates to (R_1), regularity. If, for example, p is a cone-regular projection and $pap \leq 0$, then $\overline{p}a\overline{p} \leq 0$ and (a) can be applied to $\overline{p}$.

Now Theorem 4.2 and Corollary 4.3 state that if p is regular, then the equality $\|xp\| = \|x\overline{p}\|$ is valid not only for x in A but also for x in $QM(A)$, in particular, for x in $\tilde{A}$. It might be hoped then that the following is true for a closed projection p in A^{**}:

(4) If $x \in \tilde{A}$ and $\|xp\| \leq 1$, then $\forall \varepsilon > 0$, there is an open projection q such that $q \geq p$ and $\|xq\| < 1 + \varepsilon$.

Theorem 5.7. *If p is a closed projection in A^{**}, then (4) is true for p if and only if p is regular in $\tilde{A}^{**}$.*

Proof. If p is regular in $\tilde{A}^{**}$, then $\|xp\| = \|x\overline{p}\|$, where $\overline{p}$ is the closure in $\tilde{A}^{**}$, and we can just apply [3] for $\tilde{A}$.

If p is not regular in $\tilde{A}^{**}$, then by Theorem 4.8, $1 < \alpha(p) < \infty$. Choose a in A_+ such that $pap \geq p$, and let $s = \|a\|$, so that $1 < s < \infty$. Then $p(s-a)p \leq (s-1)p$. If we apply (4) to $x = (s-1)^{-\frac{1}{2}}(s-a)^{\frac{1}{2}}$, we find an open projection q dominating p such that $q(s-a)q \leq (s-\frac{1}{2})q$. Thus $qaq \geq \frac{1}{2}q$, so that $\alpha(q) < \infty$, and p is compact by Theorem 5.4. Since $\alpha(p) > 1$, this is impossible, and hence (4) is false for p. □

The final result of this section is on the same subject as [8], which dealt with the noncommutative analogue of open relatively compact sets (except that the correct analogue turned out to be *nearly* relatively compact projections). Now we consider the noncommutative analogue of open sets with compact boundary.

Theorem 5.8. *Let A be a σ-unital C^*-algebra, let q be an open projection in A^{**}, and let $B = \mathrm{her}(q)$. The following are equivalent:*

(1) *$M(A,B)/B$ is σ-unital.*
(2) *There is a closed projection p such that $p \leq q$ and $\alpha(q-p) < \infty$.*
(3) *$M(A,B)/B$ is unital.*

Proof. $1 \Rightarrow 2$: Let h be a positive element of $M(A,B)$ such that the image of h is strictly positive in $M(A,B)/B$, and let e be a strictly positive element of A.

Claim. *We have that $q(e+h)q \geq \varepsilon q$ for some $\varepsilon > 0$.*

The proof of the claim is similar (and will refer) to the proof of [8, Theorem 4]. If false, we can find a sequence (φ_n) in $P(B)$ such that $\Sigma\varphi_n(e+h) < \infty$. We use this sequence as in [8] to construct a closed subprojection p' of q such that $p'Ap' \subset \mathcal{K}(p'H)$ and also $p'h^{\frac{1}{2}}M(A,B)h^{\frac{1}{2}}p' \subset \mathcal{K}(p'H)$. If $C = [h^{\frac{1}{2}}M(A,B)h^{\frac{1}{2}}]^-$, then $p'Cp' \subset \mathcal{K}$ and C is a hereditary C^*-subalgebra of $M(A,B)$ whose image in $M(A,B)/B$ is everything. Therefore $M(A,B) \subset C + B \subset C + A$. Thus $p'M(A,B)p' \subset \mathcal{K}$. But by [7, Lemma 3.31], there is x in $M(A,B)$ with $p' \leq x \leq q$. Then $p' = p'xp' \in \mathcal{K}(p'H)$, which is absurd, since p' is an infinite-rank projection on H. Thus the claim is proved.

Now let $p = E_{[\frac{\varepsilon}{2},\infty)}(h)$. Then

$$h \leq \|h\|p + \frac{\varepsilon}{2}(q-p).$$

Therefore

$$\begin{aligned} \varepsilon q &\leq q(e+h)q, \\ \varepsilon(q-p) &\leq (q-p)(e+h)(q-p) \leq (q-p)\Big[e + \|h\|p + \frac{\varepsilon}{2}(q-p)\Big](q-p), \\ \frac{\varepsilon}{2}(q-p) &\leq (q-p)e(q-p). \end{aligned}$$

Thus $\alpha(q-p) < \infty$.

(2) ⇒ (3): Choose h in $M(A)$ such that $p \le h \le q$ (see [7, Lemma 3.31]). Then $h \in M(A,B)$. If $x \in M(A,B)$, then $(1-h)xx^*(1-h)$ is in $M(A, \mathrm{her}(q-p))$. But by [8], $M(A, \mathrm{her}(q-p)) = \mathrm{her}(q-p)$. Thus $(1-h)xx^*(1-h)$ is in A, which implies that $(1-h)x \in A$. Similarly, $x(1-h) \in A$. This shows that the image of h is an identity for $M(A,B)/B$.

(3) ⇒ (1): This implication is trivial. □

6. $\alpha(p_1 \vee p_2)$

If p_1 and p_2 are closed projections with a positive angle, then $p_1 \vee p_2$ is closed, by [1], but if the angle is 0, then $p_1 \vee p_2$ may not be closed. The same applies to compactness, since p is compact in A^{**} if and only if it is closed in $\tilde{A}^{**}$. Therefore, it is natural to attempt to bound $\alpha(p_1 \vee p_2)$ in terms of $\alpha(p_1), \alpha(p_2)$, and the angle between p_1 and p_2.

At the cost of some redundancy, we first prove a special case which is considerably easier than the general case and is proved differently.

Theorem 6.1. *Assume that p_1 and p_2 are projections in A^{**} and that $p_1 p_2 = 0$.*

(a) *If p_1 and p_2 are closed and A is σ-unital, then $\alpha(p_1 + p_2) = \max(\alpha(p_1), \alpha(p_2))$.*
(b) *In general, $\alpha(p_1+p_2)^{-1} \ge \alpha(p_1)^{-1} + \alpha(p_2)^{-1} - 1$.*

Proof. (a) By [7, Lemma 3.31] and the continuous functional calculus, we can find h_1, h_2 in $M(A)_{sa}$ such that $p_j \le h_j \le 1$ and $h_1 h_2 = 0$. If $a_j \in A_{sa}$ and $p_j a_j p_j \ge p_j$, let $b_j = h_j a_j h_j$ and $b = b_1 + b_2$. Then $\|b\| \le \max(\|a_1\|, \|a_2\|)$, and $(p_1+p_2)b(p_1+p_2) \ge p_1 + p_2$. The result follows easily.

(b) This is vacuous unless $\alpha(p_1), \alpha(p_2) < \infty$. Therefore assume this and choose $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_j < \alpha(p_j)^{-1}$.

We use Proposition 2.8 for an approximate identity (e_i), where $\|e_i\| < 1$, $\forall i$. Let $p = p_1 + p_2$, and let pe_ip be represented by the operator matrix $\begin{pmatrix} a_i & b_i \\ b_i^* & c_i \end{pmatrix}$, relative to $pH = p_1 H \oplus p_2 H$. Since $\|a_i\| < 1$, the inequality $pe_ip \le p$ is equivalent to $b_i^*(1-a_i)^{-1} b_i \le 1 - c_i$. For i sufficiently large, $a_i \ge \varepsilon_1$ and $c_i \ge \varepsilon_2$. Then $(1-a_i)^{-1} \ge (1-\varepsilon_1)^{-1}$, and hence $\|b\|^2 \le (1-\varepsilon_1)(1-\varepsilon_2)$. Then $\begin{pmatrix} a_i & b_i \\ b_i^* & c_i \end{pmatrix} \ge \begin{pmatrix} \varepsilon_1 & b_i \\ b_i^* & \varepsilon_2 \end{pmatrix} \ge \begin{pmatrix} \varepsilon_1+\varepsilon_2-1 & 0 \\ 0 & \varepsilon_1+\varepsilon_2-1 \end{pmatrix}$, by an easy calculation, and hence $\alpha(p)^{-1} \ge \varepsilon_1 + \varepsilon_2 - 1$. Since ε_j can be taken arbitrarily close to $\alpha(p_j)^{-1}$, the result follows. □

Corollary 6.2. *If p_1 and p_2 are projections in A^{**} such that $p_1 p_2 = 0$, then we have the following.*

(a) *If $\alpha(p_1), \alpha(p_2) < \infty$ and $\alpha(p_1)^{-1} + \alpha(p_2)^{-1} > 1$, then $\alpha(p_1+p_2) < \infty$.*
(b) *If $\alpha(p_2) = 1$, then $\alpha(p_1+p_2) = \alpha(p_1)$.*

Example 6.3. (a) From Corollary 6.2(b) or otherwise, we see that $p_1 p_2 = 0$ and $\alpha(p_1) = \alpha(p_2) = 1$ imply $\alpha(p_1+p_2) = 1$. But it could be that $p_1, p_2 \in$ RC and $p_1 + p_2 \notin$ RC. Let $C^*(p,q)$ be the free C^*-algebra generated by two projections without an identity. (This C^*-algebra is described in Section 3 of [17].) Let $\pi : C^*(p,q) \to B(H)$ be a representation which induces a one-to-one map from $C^*(p,q)$ to $B(H)/\mathcal{K}$. Let A be the extension of $\mathcal{K} \oplus \mathcal{K}$ by $C^*(p,q)$

induced by $\pi \oplus \pi$ (see Example 4.15(a) and [10]). Then A^{**} can be identified with $B(H) \oplus B(H) \oplus C^*(p,q)^{**}$ so that any element x of $C^*(p,q)$ becomes $\pi(x) \oplus \pi(x) \oplus x$. Let $p_1 = \pi(p) \oplus 0 \oplus 0$ and $p_2 = 0 \oplus \pi(q) \oplus 0$. Since $p_1 \le p$ and $p_2 \le q$, p_1 and p_2 are relatively compact. We claim that $(p_1 + p_2)^- = \pi(p) \oplus \pi(q) \oplus 1$, a noncompact projection (since $C^*(p,q)$ is nonunital). To prove this, we just have to show that $(x+y)(p_1+p_2) = 0$, $x \in C^*(p,q)$, $y \in \mathcal{K} \oplus \mathcal{K}$, implies that $x = 0$. (It is then easy to compute $\{a \in A : a(p_1+p_2) = 0\}$, which equals $\{a \in A : a(p_1+p_2)^- = 0\}$.) If $(x+y)(\pi(p) \oplus \pi(q) \oplus 0) = 0$, then $\pi(x)\pi(p)$, $\pi(x)\pi(q) \in \mathcal{K}$. Therefore $\pi(x(p+q)) \in \mathcal{K}$, and hence $x(p+q) = 0$. Since $p+q$ is a strictly positive element of $C^*(p,q)$, $x = 0$.

(b) We give a simple example where $\alpha(p_1)^{-1} + \alpha(p_2)^{-1} = 1$ and $\alpha(p_1+p_2) = \infty$. Let $A = c \otimes \mathcal{K}$. Choose θ in $(0, \frac{\pi}{2})$, and let $v_n = \cos\theta e_1 + \sin\theta e_{n+1}$, $w_n = \sin\theta e_1 - \cos\theta e_{n+1}$. Define p_1 and p_2 by $(p_1)_\infty = (p_2)_\infty = 0$, $(p_1)_n = v_n \times v_n$, $(p_2)_n = w_n \times w_n$. Then $\alpha(p_1) = \cos^{-2}\theta$, $\alpha(p_2) = \sin^{-2}\theta$. Since $(p_1+p_2)_n \ge e_{n+1} \times e_{n+1}$, $\alpha(p_1+p_2) = \infty$.

Example 6.4. Before proceeding to a general result, we give a simple example to show that the hypothesis angle $(p_1,p_2) > 0$ is necessary. Let $A = c \otimes \mathcal{K}$ and $v_n = (1-n^{-1})^{\frac{1}{2}} e_1 + n^{-\frac{1}{2}} e_{n+1}$. Define projections p and q in A^{**} by $p_\infty = q_\infty = e_1 \times e_1$, $p_n = e_1 \times e_1$, $q_n = v_n \times v_n$. Then p and q are both compact, and $p \vee q$ is given by $(p \vee q)_\infty = e_1 \times e_1$ and $(p \vee q)_n = e_1 \times e_1 + e_{n+1} \times e_{n+1}$. Thus $p \vee q$ is closed and $\alpha(p \vee q) = \infty$.

If we consider instead p' and q', where $p'_n = p_n, q'_n = q_n$, and $p'_\infty = q'_\infty = 0$, then we obtain disjoint, open, relatively compact, and k-regular projections such that $\alpha(p' \vee q') = \infty$.

For the general case, we consider two situations.

I: angle$(p_1,p_2) = \theta$, $\alpha(p_j) = \sec^2\theta_j$, $0 < \theta \le \frac{\pi}{2}$, $0 \le \theta_j < \frac{\pi}{2}$, $\theta_1 + \theta_2 < \theta$. Then $\alpha(p_1 \vee p_2) < \infty$ and $\alpha(p_1 \vee p_2)^{-1} \ge \frac{S - \sqrt{T}}{2\sin^2\theta}$, where

$$\begin{aligned} S &= \cos^2\theta_1 + \cos^2\theta_2 - 2\cos^2\theta - 2\cos\theta\sin\theta_1\sin\theta_2, \qquad \text{and} \\ T &= (\cos^2\theta_1 + \cos^2\theta_2)^2 + 4\cos^2\theta\cos^2\theta_1\cos^2\theta_2 \\ &\quad - 4\cos\theta(\cos^2\theta_1 + \cos^2\theta_2)\sin\theta_1\sin\theta_2 \\ &\quad - 4(1+\cos^2\theta)(\cos^2\theta_1 + \cos^2\theta_2) + 8\cos\theta\sin\theta_1\sin\theta_2 \\ &\quad + 4\cos^2\theta + 4. \end{aligned}$$

If $\theta = \frac{\pi}{2}$, this formula is the same as Theorem 6.1(b); if $\alpha(p_2) = 1$, this gives $\alpha(p \vee p_2)^{-1} \ge \frac{\alpha(p_1)^{-1} - \cos^2\theta}{\sin^2\theta}$; and if $\alpha(p_1) = \alpha(p_2)$, this gives

$$\alpha(p_1 \vee p_2)^{-1} \ge \frac{1+\cos\theta}{\sin^2\theta}\big(2\alpha(p_j)^{-1} - 1 - \cos\theta\big).$$

This estimate and the hypothesis $\theta_1 + \theta_2 < \theta$ are sharp, even if we add the assumption that p_1 and p_2 are disjoint, open, and k-regular, $\forall k$, or if we add the assumption that p_1 are p_2 are closed.

II: p_1 and p_2 are closed *and* $p_1 \wedge p_2 = 0$, angle $(p_1,p_2) = \theta$, $\alpha(p_j) = \sec^2\theta_j$, $0 < \theta \le \frac{\pi}{2}$, $0 \le \theta_j < \frac{\pi}{2}$. Then $\alpha(p_1 \vee p_2) < \infty$, and we have the following.

(a) If $\cos\theta \le \frac{\sin\theta_1 \sin\theta_2}{1+\cos\theta_1\cos\theta_2}$, then $\alpha(p_1 \vee p_2)^{-1} \ge \frac{S-\sqrt{T}}{2\sin^2\theta}$, where

$$\begin{aligned} S &= \cos^2\theta_1 + \cos^2\theta_2 + 2\cos^2\theta\cos\theta_1\cos\theta_2, \qquad \text{and} \\ T &= (\cos^2\theta_1 + \cos^2\theta_2)^2 + 4\cos^2\theta_1\cos^2\theta_2(\cos^2\theta - \sin^2\theta) \\ &\quad + 4\cos\theta_1\cos\theta_2(\cos^2\theta_1 + \cos^2\theta_2)\cos^2\theta. \end{aligned}$$

(b) If $\cos\theta \ge \frac{\sin\theta_1 \sin\theta_2}{1+\cos\theta_1\cos\theta_2}$, then

$$\begin{aligned} \alpha(p_1 \vee p_2)^{-1} \ge\ & \cos^2\theta_1\cos^2\theta_2\sin^2\theta \\ & /\big(\cos^2\theta_1 + \cos^2\theta_2 - \cos^2\theta_1\cos^2\theta_2(1+\cos^2\theta) \\ & + 2\cos\theta\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\big). \end{aligned}$$

If $\theta = \frac{\pi}{2}$, this gives $\alpha(p_1 \vee p_2) = \max(\alpha(p_1), \alpha(p_2))$; if $\alpha(p_2) = 1$, this gives $\alpha(p_1 \vee p_2) \le \frac{\alpha(p_1) - \cos^2\theta}{\sin^2\theta}$; and if $\alpha(p_1) = \alpha(p_2)$, this gives

$$\alpha(p_1 \vee p_2) \le \begin{cases} \frac{1+\cos\theta}{1-\cos\theta}\alpha(p_j), & \cos\theta \le \frac{\alpha(p_j)-1}{\alpha(p_j)+1}, \\ \frac{2-\alpha(p_j)^{-1}(1+\cos\theta)}{1-\cos\theta}\alpha(p_j), & \cos\theta \ge \frac{\alpha(p_j)-1}{\alpha(p_j)+1}. \end{cases}$$

This estimate is sharp.

There are some preliminaries before the proof of the positive results. First, the angle between p_1 and p_2 is the same as the angle between p_1 and $p_2 - p_1 \wedge p_2$. Thus in both cases, we may assume that $p_1 \wedge p_2 = 0$. Then if $\varphi \in F(p_1 \vee p_2)$, we can write $\varphi = \sum_{j,k=1}^{2} \varphi^{jk}$, where $\varphi^{jk} \in \{f \in A^* : f(\cdot) = f(p_j \cdot p_k)\}$. We do this by considering A^{**} as a subalgebra of $B(H)$ via the universal representation of A. Then $\varphi = (\cdot v, v), v \in (p_1 \vee p_2)H$. Since angle $(p_1, p_2) > 0$, $(p_1 \vee p_2)H = p_1H + p_2H$, and $v = v_1 + v_2$, $v_j \in p_jH$. Then $\varphi^{11} = (\cdot v_1, v_1)$, $\varphi^{12} = (\cdot v_2, v_1)$, and so on. Note that $\|\varphi\| = \varphi(1) = \varphi^{11}(1) + 2\operatorname{Re}\varphi^{12}(1) + \varphi^{22}(1) = \|\varphi^{11}\| + 2\operatorname{Re}\varphi^{12}(1) + \|\varphi^{22}\|$ and $|\varphi^{12}(1)| \le \|\varphi^{11}\|^{\frac{1}{2}}\|\varphi^{22}\|^{\frac{1}{2}}\cos\theta$. It is important to know that the φ^{jk}'s are uniquely determined by $\varphi = \Sigma\varphi^{jk}$. To see this, note that $\varphi^{11} + \varphi^{21} \in L(p_1) = \{f \in A^* : f(\cdot) = f(\cdot p_1)\}$ and $\varphi^{12} + \varphi^{22} \in L(p_2)$. It is easy to see that $p_1 \wedge p_2 = 0$ implies that $L(p_1) \cap L(p_2) = \{0\}$. The reader can easily complete the proof that the four vector spaces in the decomposition are linearly independent. Finally, we will use a slightly different notation in the actual proof. Write $v_1 = su_1$, $v_2 = tu_2$, where $s, t \ge 0$ and $\|u_1\| = \|u_2\| = 1$. Then let $\psi^{11} = (\cdot u_1, u_1)$, $\psi^{12} = (\cdot u_2, u_1)$, and so on, so that $\varphi = s^2\psi^{11} + 2st\operatorname{Re}\psi^{12} + t^2\psi^{22}$. Note that the hypothesis angle $(p_1, p_2) = \theta$ implies that $|\psi^{12}(1)| \le \cos\theta$, and this implies that $s^2 + t^2 \le (1-\cos\theta)^{-1}$.

One more remark may be helpful. In the proof below, we first show that $\alpha(p_1 \vee p_2)^{-1}$ is at least the solution to a certain minimum problem for a function of several real variables. We then sketch the solution of this minimum problem. In the examples where we show that our bounds are sharp, we use the minimum problem itself rather than the explicit formula. Thus the reader may not wish to verify that our solution of the minimum problem is correct.

Theorem 6.5. *The upper bounds given for $\alpha(p_1 \vee p_2)$ in I and II above are valid under the hypotheses stated.*

Proof. We use Theorem 2.9. Thus let $\varphi_i \in F(p_1 \vee p_2) \cap S(A)$, and assume that $\varphi_i \xrightarrow{w^*} \varphi$. Of course, in case I, φ may not be in $F(p_1 \vee p_2)$. Using the notation above and passing to a subnet, we may assume that $s_i \to s, t_i \to t, \operatorname{Re}\psi_i^{12}(1) \to x$, and $\psi_i^{jk} \xrightarrow{w^*} \psi^{jk}$. Let $y = \operatorname{Re}\psi^{12}(1)$ (y need not equal x, since $1 \notin A$). Also, let $\delta_j = \psi^{jj}(1)$. Clearly, $s^2+2stx+t^2 = 1$, $|x| \le \cos\theta$, and $\delta_j \ge \cos^2\theta_j$. Also, by the lower semicontinuity of norm $\|(s')^2\psi^{11} + 2s't'\operatorname{Re}\psi^{12} + (t')^2\psi^{22}\| \le \liminf \|(s')^2\psi_i^{11} + 2s't'\operatorname{Re}\psi_i^{12} + (t')^2\psi_i^{22}\|$; that is, $\delta_1(s')^2 + 2ys't' + \delta_2(t')^2 \le (s')^2 + 2xs't' + (t')^2$, $\forall s', t' \in \mathbb{R}$. Thus $\left(\begin{smallmatrix} 1-\delta_1 & x-y \\ x-y & 1-\delta_2 \end{smallmatrix}\right) \ge 0$, and $|x-y| \le (1-\delta_1)^{\frac12}(1-\delta_2)^{\frac12} \le \sin\theta_1 \sin\theta_2$. Since $\varphi = s^2\psi^{11} + 2st\operatorname{Re}\psi^{12} + t^2\psi^{22}$, we find that $\|\varphi\|$ is at least the minimum of $\cos^2\theta_1 s^2 + 2yst + \cos^2\theta_2 t^2$ subject to $s^2+2xst+t^2 = 1$, $|x| \le \cos\theta$, $|x-y| \le \sin\theta_1\sin\theta_2$, and $s,t \ge 0$.

For case I, we compute this minimum and show that it is the formula given. Note that $\theta_1+\theta_2 < \theta$ implies that $\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 > \cos\theta$. Thus $y \ge -\cos\theta - \sin\theta_1\sin\theta_2 > -\cos\theta_1\cos\theta_2$. Thus the minimum *is* positive. One can see without computation that, at the minimum, $x = -\cos\theta$ and $y = -\cos\theta - \sin\theta_1\sin\theta_2$. (It is obvious that $y = x - \sin\theta_1\sin\theta_2$. To see that $x = -\cos\theta$, note that if x and y are decreased by the same amount, for fixed s,t, both quadratics change by the same amount, and thus the smaller quadratic changes by the larger percentage.) Once x and y are known, it is a matter of routine calculus (Lagrange multipliers) to calculate the minimum. This is left to the reader.

In case II, $\varphi \in F(p_1 \vee p_2)$ and $\psi^{jk}(\cdot) = \psi^{jk}(p_j \cdot p_k)$. Thus, using the same notation as for case I, we find that $\|\varphi\| = \delta_1 s^2 + 2yst + \delta_2 t^2$ and $y \le \delta_1^{\frac12}\delta_2^{\frac12}\cos\theta$. Thus now $\|\varphi\|$ is at least the minimum of $\delta_1 s^2 + 2yst + \delta_2 t^2$ subject to $s^2+2xst+t^2 = 1$, $|x| \le \cos\theta$, $|y| \le \delta_1^{\frac12}\delta_2^{\frac12}\cos\theta$, $|x-y| \le (1-\delta_1)^{\frac12}(1-\delta_2)^{\frac12}$, $\cos^2\theta_j \le \delta_j \le 1$, and $s,t \ge 0$. (Unlike case I, it is not yet obvious that $\delta_j = \cos^2\theta_j$ at the minimum.)

We can see by reasoning similar to that of case I that, for fixed δ_1, δ_2, the minimum occurs at $y = -\delta_1^{\frac12}\delta_2^{\frac12}\cos\theta$ and

$$x = \begin{cases} y + (1-\delta_1)^{\frac12}(1-\delta_2)^{\frac12}, & (1-\delta_1)^{\frac12}(1-\delta_2^{\frac12}) \le \cos\theta(1+\delta_1^{\frac12}\delta_2^{\frac12}), \\ \cos\theta, & (1-\delta_1)^{\frac12}(1-\delta_2)^{\frac12} \ge \cos\theta(1+\delta_1^{\frac12}\delta_2^{\frac12}). \end{cases}$$

We then substitute these values of x and y and prove that the minimum in (s,t) is a monotone increasing function of δ_1 and δ_2 (so that the minimum occurs for the smallest values of δ_1, δ_2). The easiest way to see the monotonicity is to perform a change of variable. Replace s by $\delta_1^{-\frac12}s$ and t by $\delta_2^{-\frac12}t$. The rest of the calculation is left to the reader. □

Remark. It follows from the formulas in both I and II that $\alpha(p_1) = \alpha(p_2) = 1$ implies that $\alpha(p_1 \vee p_2) = 1$, but this is nothing new. It follows from $p_1 \vee p_2 \le K(\theta)(p_1+p_2)$, where, of course, $K(\theta) \to \infty$ as $\theta \to 0$.

Corollary 6.6. *If p_1 and p_2 are closed projections such that $p_1 \wedge p_2 = 0$ and at least one of $\alpha(p_1), \alpha(p_2)$ is finite, then angle $(p_1, p_2) > 0$. Thus if both of $\alpha(p_1), \alpha(p_2)$ are finite, then $\alpha(p_1 \vee p_2)$ is finite. In particular, if p_1 and p_2 are compact and $p_1 \wedge p_2 = 0$, then $p_1 \vee p_2$ is compact.*

Proof. Assume angle $(p_1, p_2) = 0$. Then there are unit vectors v_n in p_1H and w_n in p_2H such that $\|v_n - w_n\| \to 0$. Thus there are states φ_n in $F(p_1)$ and ψ_n in $F(p_2)$ such that $\|\varphi_n - \psi_n\| \to 0$. Assume that $\alpha(p_1) < \infty$, and let φ be a weak* cluster point of (φ_n). Then $\varphi \neq 0$ and φ is also a weak* cluster point of (ψ_n). Thus $F(p_1) \cap F(p_2) \neq \{0\}$, which is a contradiction. □

Example 6.7. We show the sharpness claimed in I and II. Let $A = c \otimes \mathcal{K}$.

I. Let θ, θ_1, and θ_2 be as above, except that now we allow the possibility that $\theta_1 + \theta_2 = \theta$. Choose x, y, s, t in $\mathbb{R}$ such that $s^2 + 2xst + t^2 = 1$, $|x| \leq \cos\theta$, $|x - y| \leq \sin\theta_1 \sin\theta_2$, $s, t \geq 0$, and $\cos^2\theta_1 s^2 + 2yst + \cos^2\theta_2 t^2$ is minimized subject to the above. Of course, we know that $x = -\cos\theta$ and $y = -\cos\theta - \sin\theta_1\sin\theta_2$, and we could calculate s, t. If $\theta_1 + \theta_2 = \theta$, it is easily seen that this minimum value is 0, and hence the example in this case will have $\alpha(p_1 \vee p_2) = \infty (0 \in \overline{S}(p_1 \vee p_2))$.

Choose vectors u^1, u^2 in H such that $\|u^j\| = \cos\theta_j$ and $(u^1, u^2) = y$. The proof of Theorem 6.5 showed that $|y| \leq \cos\theta_1 \cos\theta_2$ (actually an equality), and therefore this is possible. For each n choose vectors w_n^1, w_n^2 in H such that $\|w_n^j\| = \sin\theta_j$, $(w_n^j, u^k) = 0$, $(w_n^1, w_n^2) = x - y$, and $w_n^j \stackrel{w}{\to} 0$ as $n \to \infty$. This is clearly possible. Let $u_n^j = u^j + w_n^j$, and let $v_n = su_n^1 + tu_n^2$. Then $\|u_n^j\| = 1$, $(u_n^1, u_n^2) = x$ and $u_j^n \stackrel{w}{\to} u^j$ as $n \to \infty$. It follows that $\|v_n\| = 1$. Let r be the projection in $B(H)$ whose range is $\operatorname{span}(u^1, u^2)$.

Define closed projections p^1, p^2 in A^{**} by $p_\infty^1 = p_\infty^2 = r$, $p_n^j = u_n^j \times u_n^j$. Define a in A_{sa} by $a_n = a_\infty = r$. Then $\|a\| = 1$ and $p^j a p^j \geq \cos^2\theta_j p^j$. Therefore $\alpha(p^j)^{-1} \geq \cos^2\theta_j$. Since $\operatorname{angle}(p^1, p^2) = \operatorname{angle}(p^1 - p^1 \wedge p^2, p^2 - p^1 \wedge p^2)$, and since $|(u_n^1, u_n^2)| \leq \cos\theta$, $\operatorname{angle}(p^1, p^2) \geq \theta$. Let φ_n in $S(A)$ be defined by $\varphi_n(a) = (a_n v_n, v_n)$. Clearly $\varphi_n \in F(p^1 \vee p^2)$ and $\varphi_n \stackrel{w^*}{\to} \varphi$, where $\varphi(a) = (a_\infty v, v)$, $v = su^1 + tu^2$. Therefore $\|\varphi\| = \|v\|^2 = \cos^2\theta_1 s^2 + 2yst + \cos^2\theta_2 t^2$. This shows that $\alpha(p^1 \vee p^2)$ is at least the value specified in I. (Of course, by Theorem 6.5, the inequalities for $\alpha(p^j), \alpha(p^1 \vee p^2)$, and $\operatorname{angle}(p^1, p^2)$ are actually equalities.)

Now we show how to modify the above to obtain disjoint, open, k-regular projections q^1, q^2. Let $q_\infty^j = 0$,

$$q_n^1 = \begin{cases} p_m^1, & n = 3m, \\ \frac{u^1}{\|u^1\|} \times \frac{u^1}{\|u^1\|}, & n = 3m+1, \\ 0, & n = 3m+2, \end{cases}$$

and

$$q_n^2 = \begin{cases} p_m^2, & n = 3m, \\ 0, & n = 3m+1, \\ \frac{u^2}{\|u^2\|} \times \frac{u^2}{\|u^2\|}, & n = 3m+2. \end{cases}$$

The reader can easily verify that q^1, q^2 have the required properties. (The closures of q^j have $(\overline{q}^j)_\infty = \frac{u^j}{\|u^j\|} \times \frac{u^j}{\|u^j\|}$.)

II. The construction is very similar. Of course now we have hardly any restrictions on $\theta, \theta_1, \theta_2$. Choose $x, y, s, t, \delta_1, \delta_2$ in $\mathbb{R}$ such that $s^2 + 2xst + t^2 = 1$, $|x| \leq \cos\theta$, $|y| \leq \delta_1^{\frac{1}{2}} \delta_2^{\frac{1}{2}} \cos\theta$, $|x - y| \leq (1 - \delta_1)^{\frac{1}{2}} (1 - \delta_2)^{\frac{1}{2}}$, $\cos^2\theta_j \leq \delta_j \leq 1$,

$s, t \geq 0$, and $\delta_1 s^2 + 2yst + \delta_2 t^2$ is minimized subject to the above. Of course we know that $\delta_j = \cos^2\theta_j$, $y = -\delta_1^{\frac{1}{2}}\delta_2^{\frac{1}{2}}\cos\theta$, the formula for x was given in the proof of Theorem 6.5, and we could calculate s, t.

The definitions of u^j, w_n^j, u_n^j, and v_n are by the same formulas used in part I, except that now $\|u^j\| = \delta_j^{\frac{1}{2}}$ and $\|w_n^j\| = (1-\delta_j)^{\frac{1}{2}}$.

Disjoint closed projections p^1, p^2 in A^{**} are defined by $p_\infty^j = \frac{u^j}{\|u^j}\| \times \frac{u^j}{\|u^j\|}$ and $p_n^j = u_n^j \times u_n^j$. It is easy to see that $\mathrm{angle}(p^1, p^2) \geq \theta$. Everything else is the same as in part I.

7. Attainment of extreme values

If p is a projection in A^{**}, we say that $\alpha(p)$ is attained if $\alpha(p) < \infty$ and there is a in A_{sa} such that $\|a\| = \alpha(p)$ and $pap \geq p$. We say that $\mathrm{dist}(p, \mathrm{RC})$ is attained if $\mathrm{dist}(p, \mathrm{RC}) < 1$ and there is q in RC such that $\|p-q\| = \mathrm{dist}(p, \mathrm{RC})$. We define attainment similarly for $\mathrm{dist}(p, \mathrm{ORC})$ and $\mathrm{dist}(p, \mathrm{CRC})$.

Proposition 7.1. *Let p be a projection in A^{**}.*

(a) *If $\alpha(p) = 1$, then $\alpha(p)$ is attained if and only if* $\mathrm{dist}(p, \mathrm{RC})$ *is attained if and only if $p \in \mathrm{RC}$.*

(b) *If p is closed, then $\alpha(p)$ is attained if and only if* $\mathrm{dist}(p, \mathrm{RC})$ *is attained if and only if* $\mathrm{dist}(p, \mathrm{CRC})$ *is attained.*

(c) *If p is open, then* $\mathrm{dist}(p, \mathrm{RC})$ *is attained if and only if* $\mathrm{dist}(p, \mathrm{ORC})$ *is attained.*

(d) *In general, if* $\mathrm{dist}(p, \mathrm{RC})$ *is attained, then $\alpha(p)$ is attained.*

Proof. (d) By the proof of Theorem 2.2, if q is a projection in RC such that $\|p-q\| = \mathrm{dist}(p, \mathrm{RC})$, then $pqp \geq \alpha(p)^{-1}p$. Let a_1 be in A_{sa} such that $q \leq a_1 \leq 1$, and $a = \alpha(p)a_1$. Then $\|a\| = \alpha(p)$ and $pap \geq p$.

(a) The second equivalence is obvious, since $\mathrm{dist}(p, \mathrm{RC}) = 0$. In view of (d), we need just assume that $\alpha(p)$ is attained and prove p is in RC. If a is in A_{sa}, $\|a\| \leq 1$, and $pap \geq p$, the proof of Theorem 2.1 shows that $ap = pa$. Therefore $p \leq a_+ \leq 1$, and p is in RC.

(b) Since the two distances are the same, $\mathrm{dist}(p, \mathrm{CRC})$ attained implies $\mathrm{dist}(p, \mathrm{RC})$ attained. Theorem 2.6(b) and Proposition 7.1(d) complete the proof.

(c) Again the two distances are the same. Thus assume that $\mathrm{dist}(p, \mathrm{RC})$ is attained. Choose q in RC such that $pqp \geq \alpha(p)^{-1}p$, as in the proof of (d), and choose a in A_{sa} such that $q \leq a \leq 1$. Choose a continuous function $f : \mathbb{R} \to [0,1]$ such that $f(1) = 1$ and $f = 0$ on $(-\infty, \frac{1}{2}]$, and let $b = f(a)$. Then the range projection of b is in RC, and $pbp \geq \alpha(p)^{-1}p$ since $b \geq q$. By the proof of Theorem 2.6(a), the range projection of $b^{\frac{1}{2}}p$ is in ORC and it attains $\mathrm{dist}(p, \mathrm{ORC})$. □

Example 7.2. (a) An open projection p such that $1 < \alpha(p) < \infty$ and $\alpha(p)$ is not attained. Let $A = \{a \in c \otimes \mathcal{K} : a_\infty \text{ is diagonal}\}$. Then $A^{**} = \{h \in (c \otimes \mathcal{K})^{**} : h_\infty$ is diagonal$\}$, where in both cases diagonality is with respect to our usual fixed orthonormal bases of H. Let v_0 be a unit vector in H with all coordinates nonzero. Let $\{f_1, f_2, \ldots\}$ be an orthonormal basis for $\{v_0\}^\perp$, and let $v_n = 2^{-\frac{1}{2}}v_0 + 2^{-\frac{1}{2}}f_n$.

Define p by $p_\infty = 0$ and $p_n = v_n \times v_n$. Define x' in A_{sa} by $x'_n = x'_\infty = \sum_1^k e_i \times e_i$. Then $p_n x'_n p_n = \varepsilon_n p_n$, where $\lim_{n\to\infty} \varepsilon_n = \frac{1}{2}\sum_1^k |(v_0, e_i)|^2$. For any $\delta > 0$, we can modify x'_n for finitely many values of n to obtain x in A_{sa} such that $\|x\| = 1$ and $pxp \geq (2^{-1}\sum_1^k |(v_0, e_i)|^2 - \delta)p$. Since k can be arbitrarily large and δ arbitrarily small, $\alpha(p)^{-1} \geq 2^{-1}$.

We claim there is no x in A_{sa} such that $\|x\| \leq 1$ and $pxp \geq 2^{-1}p$. If such x existed, $p_n x_\infty p_n \geq (2^{-1} - \delta_n)p_n$, where $\delta_n \to 0$ as $n \to \infty$. Since $v_n \xrightarrow{w} 2^{-\frac{1}{2}} v_0$, this implies that $2^{-1}(x_\infty v_0, v_0) \geq 2^{-1}$. This is impossible for x_∞ compact and diagonal. It follows from the proved claim that $\alpha(p) = 2$.

(b) An open projection p such that $\alpha(p)$ is attained and $\operatorname{dist}(p, \mathrm{RC})$ is not attained. Let $A = c \otimes \mathcal{K}$. Let $a_0 = \operatorname{Diag}(1, d_2, d_3, \ldots)$, where $0 < d_n < \frac{1}{2}$, $d_n \to 0$, and $\sum_2^\infty d_n = \infty$. For $n \geq 2$, let $v_n = (\frac{2^{-1} - d_n}{1 - d_n})^{\frac{1}{2}} e_1 + (\frac{2^{-1}}{1 - d_n})^{\frac{1}{2}} e_n$. Then $(a_0 v_n, v_n) = \frac{1}{2}$ and $\|v_n\| = 1$. Define p so that $p_\infty = 0$ and the sequence (p_n) includes each $v_k \times v_k$ infinitely often. Define a in A_{sa} by $a_\infty = a_n = a_0$. Then $\|a\| = 1$ and $pap \geq \frac{1}{2}p$. We claim that there is no q in RC such that $pqp \geq \frac{1}{2}p$. If there is, there is c in A_{sa} such that $q \leq c \leq 1$. Then $q \leq E_{\{1\}}(c)$. It is easy to see from the holomorphic functional calculus that $E_{\{1\}}(c)$ is majorized by a projection in A. Changing notation, we assume that $q \in A$. For any k, we can choose $n_1 < n_2 < \cdots$ such that $p_{n_i} = v_k \times v_k$. Then $(q_{n_i} v_k, v_k) \geq \frac{1}{2}$ and $q_n \to q_\infty$ imply $(q_\infty v_k, v_k) \geq \frac{1}{2}$. Let $r = q_\infty \vee (e_1 \times e_1)$. Thus r is a finite-rank projection, $r = e_1 \times e_1 + r'$ for a projection r', and $(r v_k, v_k) \geq \frac{1}{2}$, $\forall k$. Now $(r v_k, v_k) = \frac{2^{-1} - d_k}{1 - d_k} + \frac{2^{-1}}{1 - d_k}(r' e_k, e_k) \geq \frac{1}{2}$ implies that $(r' e_k, e_k) \geq d_k$. But r' finite rank implies that $\sum_1^\infty (r' e_k, e_k) < \infty$, in contradiction to the choice of (d_k). Again, we can conclude a posteriori that $\alpha(p) = 2$.

(c) A closed projection p such that $1 < \alpha(p) < \infty$ and $\alpha(p)$ is not attained. Let A_0 be the C^*-algebra called A in (a), and let $A_1 = A_0 \otimes M_2$. Let e' be the projection in $M(A_1)$ given by $\left(\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{smallmatrix}\right)$; and as in Example (3.8) and Example (3.9), let A be the extension of A_1 by $\mathbb{C}$ induced by e', and let e be the corresponding projection in A. Let v'_0 be a unit vector in H with all coordinates nonzero, and let $v_0 = v'_0 \oplus 0$ in $H \oplus H$. Choose an orthonormal sequence $f_1, f_2, \ldots$ in $H \oplus H$ such that $(f_n, v_0) = 0$ and $e' f_n = 0$, $\forall n$. Let $v_n = 2^{-\frac{1}{2}} v_0 + 2^{-\frac{1}{2}} f_n$, and define a closed projection p' in A_1^{**} by $p'_n = v_n \times v_n$ and $p'_\infty = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Identify A^{**} with $A_1^{**} \oplus \mathbb{C}$ and define $p = p' \oplus 1$, so that p is a closed projection in A^{**}. For each m, let $Q_m = \sum_1^m e_k \times e_k$, a diagonal projection in $\mathcal{K}$, and define a_m in A_1 by $(a_m)_n = (a_m)_\infty = \left(\begin{smallmatrix} \frac{1}{2}Q_m & -\frac{1}{2}Q_m \\ -\frac{1}{2}Q_m & \frac{1}{2}Q_m \end{smallmatrix}\right)$. Then define $b_m = e + a_m$, an element of A. Note that for $1 \leq n \leq \infty$, the nth component of $e' + a_m$ is $\left(\begin{smallmatrix} Q_m + \frac{1}{2}(1 - Q_m) & \frac{1}{2}(1 - Q_m) \\ \frac{1}{2}(1 - Q_m) & Q_m + \frac{1}{2}(1 - Q_m) \end{smallmatrix}\right)$, a projection.

We claim that $p b_m p \geq \varepsilon_m p$, where $\varepsilon_m \to \frac{1}{2}$ as $m \to \infty$. Thus $\alpha(p)^{-1} \geq \frac{1}{2}$. To prove this, it is enough to consider the A_1^{**}-components. It is obvious that $p'_\infty (e' + a_m)_\infty p'_\infty \geq \frac{1}{2} p'_\infty$. For n finite, $p'_n (e' + a_m)_n p'_n = \varepsilon_{nm} p'_n$, where $\varepsilon_{nm} = \frac{1}{2}((e' + a_m)_n v_0, v_0) + \operatorname{Re}((e' + a_m)_n v_0, f_n) + \frac{1}{2}((e' + a_m)_n f_n, f_n)$. For all n,

$\|(e' + a_m)_n v_0 - v_0\| = 2^{-\frac{1}{2}}\|(1 - Q_m)v_0\| = \delta_m$, where $\delta_m \to 0$. Thus $\varepsilon_{nm} \geq \frac{1}{2}(1 - \delta_m) - (\delta_m + |(v_0, f_n)|) \geq \frac{1}{2} - \frac{3}{2}\delta_m \ \forall n$.

Now let φ_n be the state given by $\varphi_n(x) = (x_n v_n, v_n)$. Since $v_n \xrightarrow{w} 2^{-\frac{1}{2}}v_0, \varphi_n$ converges on A_1 to $\frac{1}{2}\varphi$, where φ is the state on A_1 defined by $\varphi(x) = (x_\infty v_0, v_0)$. Also, since $e' f_n = 0$, $\varphi_n(e) = (e' v_n, v_n) = \frac{1}{2}(e' v_0, v_0) = \frac{1}{4}$. Thus φ_n converges in the weak* topology of A^* to $\frac{1}{2}\varphi \oplus 0$, where A^* is identified, as usual, with $A_1^* \oplus \mathbb{C}$. This shows not only that $\alpha(p)^{-1} \leq \frac{1}{2}$ but also that if $a \in A_{sa}$ and $pap \geq \frac{1}{2}p$, then $\varphi(a) \geq 1$. If $\alpha(p)$ were attained, then there would be such a with $\|a\| = 1$. If the A_1^{**}-component of a has ∞-component $\left(\begin{smallmatrix} r & * \\ * & * \end{smallmatrix}\right)$, then $r \leq 1$ and $r = \frac{1}{2}\lambda + K$, where $\lambda \leq 1 (a = \lambda e + a_1, a_1 \in A_1)$ and K is a diagonal compact operator. Since $\varphi(a) = 1$, $(rv_0', v_0') = 1$; and this is impossible since r is diagonal, all components of v_0' are nonzero, and $r \neq 1$. Thus $\alpha(p)$ is not attained.

8. Majorization, $\alpha(p)$, and semicontinuity

If $h \in A_{sa}^{**}$ and $h \leq a$ for some a in A_{sa}, then we expect that some of the spectral projections of h will be nearly relatively compact. This is trivially proved and has a couple of complements, one related to semicontinuity.

Proposition 8.1. *If $h \leq a$, where $h \in A_{sa}^{**}$ and $a \in A_{sa}$, then $\alpha(E_{[\varepsilon,\infty)}(h)) \leq \frac{\|a\|}{\varepsilon}$, $0 < \varepsilon \leq \|h_+\|$.*

Proof. If $p = E_{[\varepsilon,\infty)}(h)$, then $pap \geq php \geq \varepsilon p$. □

Corollary 8.2. *If, in addition, $h \geq 0$, then $\alpha(E_{[\varepsilon,\infty)}(h)) = 1$.*

Proof. Now we have $p \leq \varepsilon^{-1}h \leq \varepsilon^{-1}a$. □

Corollary 8.3. *If $h \in (A_{sa})_m^-$ and $h_+ \neq 0$, then $\alpha(E_{[\varepsilon,\infty)}(h)) \leq \frac{\|h_+\|}{\varepsilon}$ for $0 < \varepsilon \leq \|h_+\|$. Also $E_{\{\|h_+\|\}}(h)$ is compact. Similarly, if $h \in \overline{A_{sa}^m}$ and $h_- \neq 0$, then $\alpha(E_{(-\infty,-\varepsilon]}(h)) \leq \frac{\|h_-\|}{\varepsilon}$ for $0 < \varepsilon \leq \|h_-\|$ and $E_{\{-\|h_-\|\}}(h)$ is compact.*

Proof. Since h is strongly upper semicontinuous, [7, Corollary 3.16] implies that there is a in A_{sa} such that $h \leq a \leq \|h_+\|$. Thus the inequality follows from Proposition 8.1. Since $\|h_+\| - h$ is positive and strongly lower semicontinuous, [7, Proposition 2.44(a)] implies that its range projection is open; in other words, $E_{\{h_+\|\}}(h)$ is closed. Also $\alpha(E_{\{\|h_+\|\}}(h)) = 1$, by the case $\varepsilon = \|h_+\|$, and hence this projection is compact. The second part follows from the first applied to $-h$. □

Remark. Proposition 2.4 and the compactness assertion of Corollary 8.3 are closely analogous to Proposition 2.44 of [7]. (Before going on, we should remind the reader that in [7] we disclaimed originality for Proposition 2.44 and much of the rest of Section 2.D.) If p is a projection in A^{**}, then p is open if and only if lower semicontinuous (in any sense) (see [5]), closed if and only if weakly or middle upper semicontinuous (see [5]), and compact if and only if strongly upper semicontinuous (see [7, Definition-Lemma 2.47]). With the help of Proposition 2.4 and Corollary 8.3, we can now state a symmetrical result containing this last. If $h \in A_{sa}^{**}$ and $\sigma(h)$ has at most two elements, then h is weakly or middle lower semicontinuous if and only if q-lower semicontinuous, and h is strongly lower semicontinuous if and only if strongly q-lower semicontinuous. Of course, the same is true for upper semicontinuity. We also mention that there are at least two other ways of proving the compactness assertion in Corollary 8.3. The other proofs would not mention $\alpha(p)$.

Corollary 8.4. *Assume that $h \in A_+^{**}$, that h is strongly upper semicontinuous, and that h is q-upper semicontinuous. Then h is strongly q-upper semicontinuous.*

Proof. Let $p = E_{[\varepsilon,\infty)}(h)$ for ε in $(0, \|h\|]$. By [7, Corollary 3.22], $h \leq a$ for some a in A_{sa}. Then Corollary 8.2 implies that $\alpha(p) = 1$. By the definition of q-upper semicontinuous, p is closed. Therefore p is compact. Then by definition, h is strongly q-upper semicontinuous. □

Example 8.5. This example will show that the positivity assumption in Corollary 8.4 is necessary and that the estimate for $\alpha(E_{[\varepsilon,\infty)}(h))$ in Corollary 8.3 is sharp. Also, $\sigma(h)$ has only three elements. Choose λ_1, λ_2 in $\mathbb{R}$ such that $\lambda_1 > 2\lambda_2 > 0$. Let $A = c \otimes \mathcal{K}$ and $v_n = 2^{-\frac{1}{2}}e_1 + 2^{-\frac{1}{2}}e_{n+1}$. Define p in A^{**} by $p_\infty = e_1 \times e_1$ and $p_n = v_n \times v_n$. Also define p_0 by $(p_0)_\infty = e_1 \times e_1$ and $(p_0)_n = 0$. Then p_0 is a compact projection, p is a closed projection, $\alpha(p) = 2$, and $p_0 \leq p$. Let h in A_{sa}^{**} be determined by $\sigma(h) = \{\lambda_1, \lambda_2, \lambda_3\}$, $E_{\{\lambda_1\}}(h) = p_0$, and $E_{\{\lambda_1,\lambda_2\}}(h) = p$, where λ_3 is a negative number to be determined. Clearly, h is q-upper semicontinuous, and since p is not compact, h is not strongly q-upper semicontinuous. The following criterion for determining that h is strongly upper semicontinuous is given in [7, Section 5.13 and Remark (i)]. Choose a sequence (k_m) in $\mathcal{K}$ such that $k_m \searrow h_\infty$. Then we require that for each m and each $\varepsilon > 0$, there is N such that $k_m \geq h_n - \varepsilon$ for $n \geq N$. We can take $k_m = \lambda_1 e_1 \times e_1 + \lambda_3 \sum_2^m e_k \times e_k$. Then if $n \geq m$, we need look only at $\operatorname{span}(e_1, e_{n+1})$ to check the inequality $k_m \geq h_n - \varepsilon$. It is sufficient that

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \geq \lambda_2 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \lambda_3 \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The reader can easily check that this is true for $|\lambda_3|$ sufficiently large.

If $\varepsilon = \lambda_2$, then the inequality in Corollary 8.3 states that $\alpha(p) \leq \frac{\lambda_1}{\lambda_2}$. Since $\frac{\lambda_1}{\lambda_2}$ can be close to 2, the estimate in Corollary 8.3 cannot be improved. By slightly modifying this example, we can show that the inequality in Corollary 8.3 is not valid under the weaker hypothesis of Proposition 8.1. Let h' in A_{sa}^{**} be determined by $\sigma(h') = \{\lambda_2, \lambda_3\}$ and $E_{\{\lambda_2\}}(h') = p$. Then h' is q-upper semicontinuous, and h' satisfies all the hypothesis of Corollary 8.3 except that it is not strongly upper semicontinuous.

9. Concluding remarks

(1) The reader has probably noticed that in many of our examples p is abelian, in the (usual) sense that the W^*-algebra $pA^{**}p$ is abelian. In a few examples $\overline{p}$ is also abelian. We have not systematically tried to determine which phenomena can be exhibited with abelian projections. We merely were making a reasonable effort to keep our examples simple. It might be interesting to know the consequences of the hypothesis p is abelian or the hypothesis $\overline{p}$ is abelian.

(2) The idea of looking at $\operatorname{dist}(p, \mathrm{CRC})$ for general projections p was an afterthought. Of course, if p is closed, $\operatorname{dist}(p, \mathrm{CRC}) = \operatorname{dist}(p, \mathrm{RC})$, and if p is open but not closed, $\operatorname{dist}(p, \mathrm{CRC}) = 1$ by Theorem 5.1. For p neither open nor closed, all we have done is to look at the most obvious example.

Let $A = c \otimes \mathcal{K}$ and $v_n = 2^{-\frac{1}{2}} e_1 + 2^{-\frac{1}{2}} e_{n+1}$. Consider p_0 in A^{**} given by $(p_0)_\infty = 0$ and $(p_0)_n = v_n \times v_n$ and $p(u)$ given by $p(u)_n = v_n \times v_n$ and $p(u)_\infty = u \times u$, where u is a unit vector. Then p_0 is an open projection, and $p(u)$ is closed if and only if $u = \lambda e_1$. If q is a projection such that $\|p(u) - q\| < 1$, then q_n has rank 1, $1 \le n \le \infty$. It is then easy to see that q is compact if and only if $q_n \to q_\infty$ in norm.

Of course $\alpha(p(u)) = \alpha(p_0) = 2$, and $\operatorname{dist}(p(u), \mathrm{RC}) = \operatorname{dist}(p_0, \mathrm{RC}) = 2^{-\frac{1}{2}}$, for any u. And $(p_0, \mathrm{CRC}) = 1$ by Theorem 5.1. The determination of $\operatorname{dist}(p(u), \mathrm{CRC})$ reduces to an elementary problem. Clearly, for q compact, $\|p(u) - q\| \ge \|p(u)_\infty - q_\infty\|$, $\limsup_n \|p(u)_n - q_\infty\|$.

If the $\limsup$ is L, we can modify the q_n's so that $\|p(u)_n - q_n\| \le L + \varepsilon$, $\forall n$. (Actually, the ε is unnecessary.) Also, if $q_\infty = w \times w$, $\|w\| = 1$, then $\limsup d_a(p(u)_n, q_\infty) = \cos^{-1} |2^{-\frac{1}{2}}(e_1, w)|$, since $v_n \xrightarrow{w} 2^{-\frac{1}{2}} e_1$. Therefore $d_a(p(u), \mathrm{CRC}) = \cos^{-1} \sup\{\min(|(u, w)|, 2^{-\frac{1}{2}}|(e_1, w)|) : \|w\| = 1\}$. (Recall that $\operatorname{dist}(\cdot, \mathrm{CRC}) = \sin d_a(\cdot, \mathrm{CRC})$.) Assume, as we may, that $(u, e_1) \ge 0$. Then we can solve this maximin problem as follows. If $(u, e_1) \ge 2^{-\frac{1}{2}}$, let $w = e_1$. If $(u, e_1) < 2^{-\frac{1}{2}}$, choose w of the form $se_1 + tu$, $s, t \ge 0$ so that $(u, w) = 2^{-\frac{1}{2}}(e_1, w)$.

From the above we see that $\operatorname{dist}(p(u), \mathrm{CRC}) < 1$, $\forall u$, and $\operatorname{dist}(p(u), \mathrm{CRC}) = \operatorname{dist}(p(u), \mathrm{RC})$ if and only if $|(u, e_1)| \ge 2^{-\frac{1}{2}}$. The largest value of $\operatorname{dist}(p(u), \mathrm{CRC})$, $(\frac{2}{3})^{\frac{1}{2}}$, occurs when $(u, e_1) = 0$. The closure of $p(u)$ is given by $\overline{(p(u))_\infty} = (u \times u) \vee (e_1 \times e_1)$, a rank 2 projection except when $u = \lambda e_1$. It is easy to see that $\alpha(\overline{p(u)}) = 2$. Thus if $|(u, e_1)| < 2^{-\frac{1}{2}}$, we have $p_0 \le p(u) \le \overline{p(u)}$ and $\operatorname{dist}(p_0, \mathrm{CRC}) > \operatorname{dist}(p(u), \mathrm{CRC}) > \operatorname{dist}(\overline{p(u)}, \mathrm{CRC})$.

Let φ_n be the pure state given by $\varphi_n(a) = (a_n v_n, v_n)$. Then $\varphi_n \xrightarrow{w^*} \frac{1}{2}\varphi$, where $\varphi(a) = (a_\infty e_1, e_1)$. If $(u, e_1) = 0$, then the support projection of φ is orthogonal to $p(u)$, just as it is orthogonal to p_0.

It would seem that the study of $\operatorname{dist}(p, \mathrm{CRC})$, for general p, is more complicated than the study of $\operatorname{dist}(p, \mathrm{RC})$. It would be interesting to know whether there is any natural hypothesis on p (other than that p be closed) which, together with $\alpha(p) < \infty$, implies that $\operatorname{dist}(p, \mathrm{CRC}) < 1$.

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