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Boolean Algebras in Visser Algebras

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Abstract We generalize the double negation construction of Boolean algebras in Heyting algebras to a double negation construction of the same in Visser algebras (also known as basic algebras). This result allows us to generalize Glivenko's theorem from intuitionistic propositional logic and Heyting algebras to Visser's basic propositional logic and Visser algebras.

1 Introduction

Basic propositional calculus (BPC), which was introduced by Albert Visser in [8], captures a sublogic of intuitionistic propositional calculus (IPC) which corresponds to modal logic K4 in essentially the same way that IPC corresponds to modal logic S4. In Ardeshir [3] and Ardeshir and Ruitenburg [4] we introduce Visser algebras (where we named them basic algebras), which correspond to BPC in the same way that Heyting algebras correspond to IPC and that Boolean algebras correspond to classical propositional calculus (CPC). In Section 2 we present axiomatizations and some elementary properties of both BPC and Visser algebras.

The double negation construction of Boolean algebras from Heyting algebras is well known (see Balbes and Dwinger [6, Theorem IX.5.3] or Johnstone [7, p. 10]). It is natural to consider how closely one can repeat this construction over Visser algebras. Surprisingly the end result still works, although in details we use several new ideas.

Glivenko's well-established theorem (see [6, Section VIII.4 plus Theorem IX.5.3] or van Dalen [5, end of Section 5.2]) also goes through, but with an interesting reformulation. Given propositional formula ψ, define

$$\xi(\psi) \ := \ ((\top \to \psi) \to \psi) \to (\top \to \psi).$$

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Formulas $\xi(\psi)$ are of interest in their own right, see [4, p. 323]. Over IPC, formulas ψ and $\xi(\psi)$ are equivalent. So, in particular, IPC proves $\neg\xi(\bot)$. With Theorem 5.7 we show that for all (sequent) theories $\Gamma \supseteq$ BPC we have

$$\Gamma \text{ proves } \varphi \to \xi(\bot) \quad \text{if and only if } \ \Gamma + \text{CPC proves } \varphi \to \xi(\bot).$$

So if $\Gamma \supseteq$ IPC, then Γ proves $\neg\varphi$ if and only if Γ + CPC proves $\neg\varphi$ (Glivenko's theorem).

2 Basic Logic Axioms and Algebras

This section covers a quick overview of BPC and Visser algebras.

We choose an axiomatization of BPC by using sequents. We briefly recall some relevant points from [4]. Our formulas are built from propositional variables by using $\top$, $\bot$, $\varphi\wedge\psi$, $\varphi\vee\psi$, and $\varphi \to \psi$. Negation $\neg\varphi$ and bi-implication $\varphi \leftrightarrow \psi$ are defined in the usual way by $\varphi \to \bot$ and $(\varphi \to \psi) \wedge (\psi \to \varphi)$, respectively. Symbols $\top$ and $\bot$ are both atoms and nullary connectives. The foundation of our sequent theories is a collection of closure rules which, on its own, generates a bounded distributive prelattice with preorder $\Rightarrow$, on the collection of formulas. In each rule below, a single horizontal line means that if the sequents above the line hold, then so do the ones below the line. A rule with multiple conclusions is an abbreviation for several rules with single conclusions. A double line means the same as a single line but in both directions, so it is really an abbreviation for two (possibly abbreviated) rules. The absence of a line means that the conclusion holds without premises. So sequents are identifiable as special rules. Here they are:

$$\varphi \Rightarrow \varphi \quad \text{reflexivity,}$$

$$\frac{\varphi \Rightarrow \psi \quad \psi \Rightarrow \theta}{\varphi \Rightarrow \theta} \quad \text{transitivity,}$$

$$\varphi \Rightarrow \top \quad \bot \Rightarrow \varphi \quad \text{top and bottom,}$$

$$\frac{\varphi \Rightarrow \psi \quad \varphi \Rightarrow \theta}{\overline{\varphi \Rightarrow \psi \wedge \theta}} \quad \frac{\psi \Rightarrow \varphi \quad \theta \Rightarrow \varphi}{\overline{\psi \vee \theta \Rightarrow \varphi}} \quad \text{meet and join,}$$

$$\varphi \wedge (\psi \vee \theta) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \theta) \quad \text{distributivity, and}$$

$$\frac{\varphi \wedge \psi \Rightarrow \theta}{\varphi \Rightarrow \psi \to \theta} \quad \text{implication introduction.}$$

If in the implication introduction rule we replace the single horizontal line by a double line, we essentially add modus ponens to the system and so get IPC. In the absence of modus ponens we need to add the "formalized" versions of some of the rules of $\Rightarrow$:

$$(\varphi \to \psi) \wedge (\psi \to \theta) \Rightarrow \varphi \to \theta \quad \text{formal transitivity,}$$

$$(\varphi \to \psi) \wedge (\varphi \to \theta) \Rightarrow \varphi \to (\psi \wedge \theta) \quad \text{formal meet,} \qquad \text{and}$$

$$(\psi \to \varphi) \wedge (\theta \to \varphi) \Rightarrow (\psi \vee \theta) \to \varphi \quad \text{formal join.}$$

This completes the axiomatization of BPC. It is possible to define theories over BPC in terms of adding rules. For the purposes of this paper we restrict ourselves to theories that are obtained by only adding sequents (or rules without premises). Examples are IPC, which is axiomatizable by adding the schema $(\top \to \varphi) \Rightarrow \varphi$, and CPC, which is axiomatizable by adding the schema $((\varphi \to \bot) \to \bot) \Rightarrow \varphi$.

A Visser algebra $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1)$ (called a basic algebra in Alizadeh and Ardeshir [1], [2] and in [3], [4]) is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ with an arrow satisfying the schemas

$(a \to b \wedge c) = (a \to b) \wedge (a \to c)$,
$(b \vee c \to a) = (b \to a) \wedge (c \to a)$,
$(a \to a) = 1$,
$a \leq (1 \to a)$,
$(a \to b) \wedge (b \to c) \leq (a \to c)$ (transitivity),

where $\leq$ is the usual order relation implied by the lattice. This completes the axiomatization of a Visser algebra. A Heyting algebra is a Visser algebra satisfying the extra schema $(1 \to a) = a$. Visser algebras need not be Heyting algebras, but they always satisfy

$a \wedge b \leq c$ implies $a \leq b \to c$.

Further such properties can be found in [1], [4], or [2].

Visser algebra morphisms preserve sequent theories. For example, the image of a Heyting algebra under a Visser algebra morphism is again a Heyting algebra. The same applies to Boolean algebras.

3 Boolean Algebras

For the purposes of this paper we introduce notations $\Box a$ for $1 \to a$, and x^a for $x \to a$. So $\Box\Box a = 1 \to (1 \to a)$, and $x^{aaa} = ((x \to a) \to a) \to a$. For all terms $t(x)$ built from the defining functions of $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1)$ and the elements of A, and for all $x \in A$, we have $x \wedge t(x) = x \wedge t(1)$ (simple substitution). For example, $x \wedge (x \wedge y)^a = x \wedge (1 \wedge y)^a = x \wedge y^a$. Positive and negative occurrences in formulas and terms are defined in the usual way. If x is only positive in $t(x)$, then $x \leq y$ implies $t(x) \leq t(y)$. For example, $x^{aa} \leq \Box a \to a$. If x is only negative in $t(x)$, then $x \leq y$ implies $t(y) \leq t(x)$. For example, $\Box a \leq x^a$.

An element a is called *Heyting* if $\Box a = a$. A Visser algebra is a Heyting algebra exactly when all its elements are Heyting. Since $a \leq \Box a$ for all a, we have that 1 is always Heyting, but 0 need not be Heyting.

Proposition 3.1 *Let a be an element of a Visser algebra $\mathfrak{A}$. Then*

1. $(x \wedge y)^a \leq x \to y^a \leq (x \wedge y)^{\Box a}$,
2. $(x \wedge y)^{aa} = x^{aa} \wedge y^{aa}$.

Proof Item 1: First, $(x \wedge y)^a \wedge x = y^a \wedge x \leq y^a$, so $(x \wedge y)^a \leq x \to y^a$. Second, $x \wedge y \leq x$ implies $x \to y^a = x \to (y \to a) \leq (x \wedge y) \to (y \to a) = (x \wedge y) \to (1 \to a) = (x \wedge y)^{\Box a}$.

Item 2: Direction $(x \wedge y)^{aa} \leq x^{aa} \wedge y^{aa}$ is immediate from the positive positions of x and y. For the other direction, with $(x \to y^a) \wedge y^{aa} \leq x^a$ and item 1 we get $x^{aa} \wedge y^{aa} \wedge (x \wedge y)^a \leq x^{aa} \wedge y^{aa} \wedge (x \to y^a) \leq x^{aa} \wedge x^a = \Box a \wedge x^a = \Box a$. So $x^{aa} \wedge y^{aa} \leq (x \wedge y)^a \to \Box a$. From the positive position of x we get $x^{aa} \leq \Box a \to a$. Thus, with transitivity, $x^{aa} \wedge y^{aa} \leq (x \wedge y)^{aa}$. □

Let $a \in A$. An element x is called *a-regular* if $x^{aa} = x$. Let $R^a(\mathfrak{A})$ be the set of a-regular elements of $\mathfrak{A}$. Clearly we have $\{x^a : x \in A\} \supseteq \{x^{aa} : x \in A\} \supseteq \{x^{aaa} : x \in A\} \supseteq \cdots \supseteq R^a(\mathfrak{A})$. Since x is positive in x^{aa} and $0^{aa} = \Box a$ and $1^{aa} = \Box a \to a$, we also have $R^a(\mathfrak{A}) \subseteq [\Box a, \Box a \to a]$. The set $R^a(\mathfrak{A})$ inherits a partial order from $\mathfrak{A}$.

Proposition 3.2 *Let a be an element of a Visser algebra $\mathfrak{A}$. Then*

1. $x \in R^a(\mathfrak{A})$ *implies* $x^a \in R^a(\mathfrak{A})$,

2. $\Box a \in R^a(\mathfrak{A})$ *(this is [4, Proposition 2.12]),*
3. $\Box a \to a \in R^a(\mathfrak{A})$,
4. $x, y \in R^a(\mathfrak{A})$ *implies* $x \wedge y \in R^a(\mathfrak{A})$.

Proof Item 1: This is immediate, since $x^{aa} = x$ implies $x^{aaa} = x^a$.

Item 2: By positivity of $\Box a$ we have $(\Box a \to a) \to a \leq \Box a \to a$, so with simple substitution $(\Box a \to a) \to a \leq \Box a$. Thus $(\Box a)^{aa} = \Box a$.

Item 3: This is immediate from items 1 and 2.

Item 4: This is Proposition 3.1.2. $\Box$

So $R^a(\mathfrak{A})$ inherits top and bottom from the interval $[\Box a, \Box a \to a]$ and is closed under $\wedge$. We show below that closure under $x \mapsto x^a$ essentially means closure under (relative) complement.

Given $a \in A$, define $x \vee_a y = (x \vee y)^{aa}$.

Proposition 3.3 *Let a be an element of a Visser algebra* $\mathfrak{A}$. *Then*

1. $x, y \in R^a(\mathfrak{A})$ *implies* $x \vee_a y \in R^a(\mathfrak{A})$,
2. $x, y \in R^a(\mathfrak{A})$ *implies* $x \vee y \leq x \vee_a y$,
3. $z \in R^a(\mathfrak{A})$ *plus* $x \vee y \leq z$ *imply* $x \vee_a y \leq z$,
4. $x \in R^a(\mathfrak{A})$ *implies* $x \wedge (y \vee_a z) = (x \wedge y) \vee_a (x \wedge z)$.

Proof Item 1: $x \vee_a y = (x^a \wedge y^a)^a$. Apply Propositions 3.2.1 and 3.2.4.

Item 2: By positivity, $x \vee y = x^{aa} \vee y^{aa} \leq (x \vee y)^{aa} = x \vee_a y$.

Item 3: $x \vee y \leq z$ implies $x \vee_a y \leq z^{aa} = z$.

Item 4: By Proposition 3.1.2, $x \wedge (y \vee_a z) = x^{aa} \wedge (y \vee z)^{aa} = (x \wedge (y \vee z))^{aa} = ((x \wedge y) \vee (x \wedge z))^{aa} = (x \wedge y) \vee_a (x \wedge z)$. $\Box$

Given $a \in A$, define $x \to_a y = x^a \vee_a y$. Let $\mathfrak{R}^a(\mathfrak{A})$ be structure $(R^a(\mathfrak{A}), \wedge, \vee_a, \to_a, \Box a, \Box a \to a)$. By Propositions 3.2 and 3.3.1, this structure is well defined.

Theorem 3.4 *Let a be an element of a Visser algebra* $\mathfrak{A}$. *Then* $\mathfrak{R}^a(\mathfrak{A})$ *is a Boolean algebra.*

Proof By Propositions 3.2.2, 3.2.3, 3.2.4, and 3.3, $(R^a(\mathfrak{A}), \wedge, \vee_a, \Box a, \Box a \to a)$ is a bounded distributive lattice. So it suffices to show that $x \mapsto x^a$ gives a (relative) Boolean complement.

For all x we have $x \wedge x^a = x \wedge \Box a$. In case $x \in R^a(\mathfrak{A})$ this means that $x \wedge x^a = \Box a$, and so x and x^a are relatively disjoint.

Suppose that x and y are such that both $x \leq y$ and $x^a \leq y$ hold. Then $y^a \leq x^a \leq y$, so $y^a \leq \Box a$ and so $y^a = \Box a$. So also $y^{aa} = \Box a \to a$. So if $x \leq y$ plus $x^a \leq y$ plus $y \in R^a(\mathfrak{A})$, then $y = \Box a \to a$ is the largest element of $\mathfrak{R}^a(\mathfrak{A})$. $\Box$

4 Boolean Elements and Morphisms

We have a further characterization of the elements of $R^a(\mathfrak{A})$ which allows us to find an idempotent Visser algebra morphism from the "subalgebra" of $\mathfrak{A}$ on the interval $[a, 1]$, onto $\mathfrak{R}^a(\mathfrak{A})$.

Proposition 4.1 *Let a be an element of a Visser algebra* $\mathfrak{A}$. *Then*

1. $x \wedge x^{aa} = x \wedge (\Box a \to a)$ *(so* $x \leq \Box a \to a$ *if and only if* $x \leq x^{aa}$*),*
2. $\Box a \leq x$ *implies* $x^a \leq x^{aaa}$,

3. $x^{aaa} = x^a \wedge (\Box a \to a)$,
4. $x^{aaaa} = x^{aa}$,
5. $R^a(\mathfrak{A}) = \{x^{aa} \;:\; x \in A\}$.

Proof Item 1: This is immediate by simple substitution.

Item 2: $\Box a \leq x$ implies $x^a \leq \Box a \to a$. Apply item 1.

Item 3: By simple substitution of x^a we have $x^a \wedge x^{aaa} = x^a \wedge (\Box a \to a)$. So $x^{aaa} \geq x^a \wedge (\Box a \to a)$. For the other direction, inequality $x \wedge (\Box a \to a) \leq x^{aa}$ of item 1 implies $x^{aaa} \leq (x \wedge (\Box a \to a))^a$. From positivity of x^a we get $x^{aaa} \leq \Box a \to a$. Thus, with simple substitution, $x^{aaa} \leq (\Box a \to a) \wedge (x \wedge (\Box a \to a))^a = (\Box a \to a) \wedge x^a$.

Item 4: For all x we have $x^{aa} \leq \Box a \to a$. Apply item 3 with x replaced by x^a.

Item 5: This is immediate from item 4. □

Proposition 4.1.3 may be viewed as the natural generalization of Brouwer's triple negation theorem. If a is Heyting, then it yields $x^{aaa} = x^a$ for all x, and so $R^a(\mathfrak{A}) = \{x^a \;:\; a \in A\}$.

Now we have the tools to present an idempotent Visser algebra morphism from the subinterval $[a, 1]$ of $\mathfrak{A}$ onto $\mathfrak{R}^a(\mathfrak{A})$.

First some facts about Visser algebras on intervals. Let $a, b \in A$ be with $a \leq b$. We construct a Visser algebra $\mathfrak{I}^{[a,b]}(\mathfrak{A})$ on the interval $[a, b]$ as follows. Define $x \to_I y = (x \to y) \wedge b$. Define $\mathfrak{I}^{[a,b]}(\mathfrak{A}) = ([a, b], \wedge, \vee, \to_I, a, b)$. Clearly $\mathfrak{I}^{[a,b]}(\mathfrak{A})$ is well defined. The map $\pi_{[a,b]} : x \mapsto (x \wedge b) \vee a = (x \vee a) \wedge b$ is a well-defined map from A onto $[a, b]$. If $b = 1$, then $x \to_I y = x \to y$, so $\mathfrak{I}^{[a,1]}(\mathfrak{A})$ is clearly a Visser algebra and is a subalgebra of $\mathfrak{A}$ except for the bottom element.

Proposition 4.2 *Let $a \leq b$ be elements of a Visser algebra $\mathfrak{A}$. Then $\mathfrak{I}^{[a,b]}(\mathfrak{A})$ is a Visser algebra, and $\pi_{[a,b]}$ is an idempotent bounded distributive lattice morphism from $\mathfrak{A}$ onto $\mathfrak{I}^{[a,b]}(\mathfrak{A})$.*

Proof The bounded distributive lattice properties are well known. One easily verifies the defining Visser algebra properties of Section 2 for arrow $x \to_I y$. □

Map $\pi_{[a,b]}$ does not need to respect arrows even when $\mathfrak{A}$ is a Heyting algebra and $b = 1$, since $\pi_{[a,1]}(x \to y) = (x \to y) \vee a$ and $\pi_{[a,1]}(x) \to_I \pi_{[a,1]}(y) = x \to (y \vee a)$ need not be the same.

Finally the morphism of primary interest: let $a \in A$. Define the map $\gamma_a : A \to R^a(\mathfrak{A})$ by $\gamma_a(x) = x^{aa}$. By Proposition 4.1.5, map γ_a is well defined. We are primarily interested in γ_a with restriction to subdomain $[a, 1]$.

Proposition 4.3 *Let a and b be elements of a Visser algebra $\mathfrak{A}$. Then*

1. $(x^{aa} \vee y^{aa})^{aa} = (x \vee y)^{aa}$,
2. $(x \to (b \vee y))^a \leq ((x \to b) \vee y)^a$,
3. $(x \to (a \vee y))^a = ((x \to a) \vee y)^a$.

Proof Item 1: With Propositions 3.1.2 and 4.1.4 we have $(x^{aa} \vee y^{aa})^{aa} = (x^{aaa} \wedge y^{aaa})^a = (x^a \wedge y^a)^{aaa} = (x \vee y)^{aaaa} = (x \vee y)^{aa}$.

Item 2: This immediately follows from $(x \to b) \vee y \leq x \to (b \vee y)$.

Item 3: By item 2 we need only to show one direction. Since $(a \vee y)^a = y^a$ we have $(x \to (a \vee y))^a \geq (x \to (a \vee y))^a \wedge y^a = ((x \to (a \vee y)) \wedge ((a \vee y) \to a))^a \wedge y^a \geq (x^a)^a \wedge y^a = ((x \to a) \vee y)^a$. □

Theorem 4.4 *Let a be an element of a Visser algebra $\mathfrak{A}$. Then γ_a is an idempotent Visser algebra morphism from $\mathfrak{F}^{[a,1]}(\mathfrak{A})$ onto $\mathfrak{R}^a(\mathfrak{A})$.*

Proof Preservation of top 1, bottom a, and conjunction are easy. Surjectivity and idempotency of γ_a follow from Propositions 4.1.4 and 4.1.5. Equation $\gamma_a(x \vee y) = \gamma_a(x) \vee_a \gamma_a(y)$ is Proposition 4.3.1. Finally, let $y \in [a,1]$. Then $a \leq y$, so with Proposition 4.3.3 we have $(x \to y)^a = (x^a \vee y)^a$. Combined with Proposition 4.3.1 we then have $\gamma_a(x \to y) = (x \to y)^{aa} = (x^a \vee y)^{aa} = (x^{aaa} \vee y^{aa})^{aa} = \gamma_a(x) \to_a \gamma_a(y)$. □

Map $\gamma_a : \mathfrak{A} \to \mathfrak{R}^a(\mathfrak{A})$ is an idempotent onto bounded lattice morphism with $\gamma_a(x \to y) = (x \to y)^{aa} \leq (x \to (a \vee y))^{aa} = (x^a \vee y)^{aa} = \gamma_a(x) \to_a \gamma_a(y)$. In general these two expressions are not equal, even when $\mathfrak{A}$ is a Heyting algebra.

5 Glivenko Theorems

Let $\mathfrak{A}$ be a bounded distributive lattice with a binary function $x \to y$ satisfying $x \to y = 1$ for all $x, y \in A$. Then $\mathfrak{A}$ is clearly a Visser algebra. All Visser algebras satisfying $\Box 0 = 1$ can so be obtained from bounded distributive lattices. They belong to the very interesting collection of Visser algebras that satisfy the principle of excluded middle $x \vee x^0 = 1$, a collection which was essentially introduced in [3] (see also [4, Proposition 5.11]). So the principle of excluded middle is not sufficient to yield just Boolean algebras. Therefore the following is not completely self-evident.

Proposition 5.1 *Let $\mathfrak{A}$ be a Visser algebra satisfying the schema of double negation elimination $x^{00} \leq x$. Then $\mathfrak{A}$ is a Boolean algebra.*

Proof Clearly $\Box 0 = 0^{00} \leq 0$, so $\Box 0 = 0$. Let $x \in A$. Then $\Box x \wedge x^0 \leq \Box 0 = 0$, so $\Box x \leq x^{00} \leq x$. So $\mathfrak{A}$ is a Heyting algebra satisfying double negation elimination and thus is a Boolean algebra. □

The Glivenko theorems we describe below involve inverse images of $\Box 0$ and $\Box 0 \to 0$ under the Visser algebra morphism $\gamma_0 : \mathfrak{A} \to \mathfrak{R}^0(\mathfrak{A})$ of Section 4 (note that $\mathfrak{F}^{[0,1]}(\mathfrak{A}) = \mathfrak{A}$). We use the following defined term in the description of these inverse images.

For every element a of a Visser algebra, define $\xi(a) = (\Box a \to a) \to \Box a$.

Proposition 5.2 *Let a be an element of a Visser algebra $\mathfrak{A}$. Then*

1. $\xi(a) \wedge (\Box a \to a) = \Box a$,
2. $\Box \xi(a) = \xi(a)$ *(this is [4, Proposition 2.11]),*
3. $x \to \xi(a) = 1$ *if and only if* $x \leq \xi(a)$,
4. $\xi(a) \to a = \Box a \to a$.

Proof Item 1: With simple substitution, $\xi(a) \wedge (\Box a \to a) = \Box\Box a \wedge (\Box a \to a) \leq \Box a$.

Item 2: By item 1 we have $1 = \xi(a) \wedge (\Box a \to a) \to \Box a$. So $\Box \xi(a) \leq (\Box a \to a) \to \xi(a) = (\Box a \to a) \to \xi(a) \wedge (\Box a \to a) = (\Box a \to a) \to \Box a = \xi(a)$.

Item 3: From right to left is immediate. For the converse, suppose $x \to \xi(a) = 1$. Then with item 2 we have $x = x \wedge (x \to \xi(a)) = x \wedge \Box \xi(a) \leq \xi(a)$.

Item 4: Obviously $\xi(a) \to a \leq \Box a \to a$. Conversely, with item 1 and simple substitution we have $(\Box a \to a) \wedge (\xi(a) \to a) = (\Box a \to a) \wedge (\xi(a) \wedge (\Box a \to a) \to a) = (\Box a \to a) \wedge (\Box a \to a)$. □

Proposition 5.3 *Let a be an element of a Visser algebra $\mathfrak{A}$. Then*
$$\Box a \to a \le x^a \text{ if and only if } x^{aa} = \Box a \text{ if and only if } x \le \xi(a).$$

Proof With Propositions 3.2.2 and 4.1.3 we have that $\Box a \to a \le x^a$ implies $x^{aa} \le (\Box a)^{aa} = \Box a$ (and so $x^{aa} = \Box a$) implies $\Box a \to a \le x^{aaa} \le x^a$. So the first two statements are equivalent. By Proposition 5.2.4 we have that $x \le \xi(a)$ implies $\Box a \to a \le x \to a$. So the third statement implies the first. For the converse, suppose the first statement. Then $x \wedge (\Box a \to a) \le x \wedge x^a \le \Box a$. So $x \le (\Box a \to a) \to \Box a = \xi(a)$. □

So the inverse image of $\Box a$ under γ_a is the principal ideal $[0, \xi(a)]$.

Theorem 5.4 *Let a be an element of a Visser algebra $\mathfrak{A}$, and let $\gamma_a(x) = x^{aa}$ be the idempotent bounded distributive lattice morphism from $\mathfrak{A}$ onto $\mathfrak{R}^a(\mathfrak{A})$. Then $\gamma_a^{-1}(\Box a) = \{x \in A : x^{\xi(a)} = 1\}$ and $\gamma_a^{-1}(\Box a \to a) = \{x \in A : x^{a\xi(a)} = 1\}$.*

Proof With Propositions 5.3 and 5.2.3 we have $\gamma_a(x) = \Box a$ if and only if $x^{\xi(a)} = 1$. Similarly, $\gamma_a(x) = \Box a \to a$ if and only if $x^{aa} = \Box a \to a$ if and only if (use Propositions 3.2.2 and 4.1.4) $x^{aaa} = \Box a$ if and only if (see Proposition 5.3) $x^a \le \xi(a)$ if and only if (see Proposition 5.2.3) $x^{a\xi(a)} = 1$. □

Fix a propositional language $\mathcal{L}$. With its presentation in [3] (see also [4, Proposition 2.4]), the Lindenbaum algebra of basic propositional logic BPC is isomorphic in the natural way with the free Visser algebra on the set of propositional letters of $\mathcal{L}$. Sequent theories $\Gamma \supseteq$ BPC correspond to adding equations between (equivalence classes of) formulas of $\mathcal{L}$. Examples are intuitionistic propositional logic $\Gamma =$ IPC, which is axiomatizable by the schema $\top \to \varphi \Rightarrow \varphi$, and classical propositional logic $\Gamma =$ CPC, which is axiomatizable by the schema $(\varphi \to \bot) \to \bot \Rightarrow \varphi$, also written as $\neg\neg\varphi \Rightarrow \varphi$. Write $\mathfrak{A}_\Gamma$ for the Lindenbaum Visser algebra of Γ, with elements $[\varphi]_\Gamma = \{\psi \in \mathcal{L} : \Gamma \vdash \psi \Leftrightarrow \varphi\}$. Given sequent theories $\Gamma \subseteq \Delta$, the map $\pi^\Gamma_\Delta : [\varphi]_\Gamma \mapsto [\varphi]_\Delta$ is a Visser algebra morphism from $\mathfrak{A}_\Gamma$ onto $\mathfrak{A}_\Delta$. A Visser algebra morphism $\mu : \mathfrak{A} \to \mathfrak{B}$ induces a congruence on $\mathfrak{A}$ in the usual way by $x \sim y$ exactly when $\mu(x) = \mu(y)$. If $\mathfrak{A} = \mathfrak{A}_\Gamma$ for some sequent theory Γ, then $\Delta(\mu) = \{\varphi \Rightarrow \psi : [\varphi]_\Gamma \sim [\varphi \wedge \psi]_\Gamma\}$ is the unique sequent theory containing Γ such that $\mathfrak{A}_\Gamma/(\sim) \cong \mathfrak{A}_{\Delta(\mu)}$ by the usual isomorphism $[[\varphi]_\Gamma]_\sim \mapsto [\varphi]_{\Delta(\mu)}$. We call $\Delta(\mu)$ the congruence theory implied by μ. Given sequent theories $\Gamma \subseteq \Delta \subseteq \Delta(\mu)$, the map $\nu([\varphi]_\Delta) = \mu([\varphi]_\Gamma)$ is the unique function (and Visser algebra morphism) that makes the following diagram commute:

$$\begin{tikzcd}
\mathfrak{A}_\Gamma \arrow[r, "\mu"'] \arrow[d, "\pi^\Gamma_\Delta"'] & \mathfrak{B} \\
\mathfrak{A}_\Delta \arrow[ur, "\exists!\ \nu"'] &
\end{tikzcd}$$

where ν is an isomorphism exactly when $\Delta = \Delta(\mu)$.

Given an element a of Visser algebra $\mathfrak{A}$, we have $R^a(\mathfrak{A}) \subseteq A$. So each function μ from $\mathfrak{A}$ uniquely determines a restricted function μ_a from $\mathfrak{R}^a(\mathfrak{A})$. Let $\mu : \mathfrak{A} \to \mathfrak{B}$ be a Visser algebra morphism. Then the following diagram commutes, with $\mu_a(x^{aa}) = \mu(x)^{\mu(a)\mu(a)}$:

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\gamma_a"'] \arrow[d, "\mu"'] & \mathfrak{R}^a(\mathfrak{A}) \arrow[d, "\mu_a"] \\
\mathfrak{B} \arrow[r, "\gamma_{\mu(a)}"] & \mathfrak{R}^{\mu(a)}(\mathfrak{B})
\end{tikzcd}$$

The following is not immediately self-evident since the idempotent onto maps γ_a and $\gamma_{\mu(a)}$ need not be Visser algebra morphisms.

Proposition 5.5 *Let a be element of a Visser algebra $\mathfrak{A}$, and let $\mu : \mathfrak{A} \to \mathfrak{B}$ be a Visser algebra morphism. Then μ_a is a Visser algebra morphism.*

Proof This is essentially immediate from the definition of the Boolean algebra in terms of the defining functions of the original Visser algebra. For example, $\mu_a(x \vee_a y) = \mu(x \vee_a y) = \mu((x \vee y)^{aa}) = (\mu(x) \vee \mu(y))^{\mu(a)\mu(a)} = \mu_a(x) \vee_{\mu(a)} \mu_a(y)$. $\square$

So map μ_a is also a Boolean algebra morphism.

Proposition 5.6 *Let Γ be a sequent theory. Then the congruence theory implied by $\gamma_0 : \mathfrak{A}_\Gamma \to \mathfrak{R}^0(\mathfrak{A}_\Gamma)$ equals* $\Gamma + \mathrm{CPC}$.

Proof By Proposition 4.1.4 we have $\gamma_0([\varphi]_\Gamma) = \gamma_0([\neg\neg\varphi]_\Gamma)$. So $\Gamma \cup \mathrm{CPC} \subseteq \Delta(\gamma_0)$. Consider the following diagram:

$$\begin{tikzcd}
\mathfrak{A}_\Gamma \arrow[r, "\gamma_0"'] \arrow[d, "\pi"'] & \mathfrak{R}^0(\mathfrak{A}_\Gamma) \arrow[d, "\pi_0"] \\
\mathfrak{A}_{\Gamma+\mathrm{CPC}} \arrow[r, "\gamma_0=\mathrm{id}"] \arrow[ur, "\exists !\ \nu"'] & \mathfrak{R}^0(\mathfrak{A}_{\Gamma+\mathrm{CPC}})
\end{tikzcd}$$

where π is short for $\pi^\Gamma_{\Gamma+\mathrm{CPC}}$. The bottom γ_0 is clearly an identity between Boolean algebras. The outer square and the top left triangle both commute. An easy diagram chase plus π onto gives that the lower right triangle also commutes. So $\pi_0\nu = 1$. Since γ_0 is onto, ν is also onto. With $\nu = \nu\pi_0\nu$ this gives $\nu\pi_0 = 1$. Thus ν is a Visser algebra isomorphism, and $\Gamma + \mathrm{CPC} = \Delta(\gamma_0)$, $\square$

This is essentially all we need to generalize the Glivenko theorems from IPC to BPC. We employ the following notation for formulas and sequent theories over BPC.

We write $\Gamma \vdash \varphi$ as short for $\Gamma \vdash (\top \Rightarrow \varphi)$. This agrees with default practice over IPC, where, with modus ponens, $\varphi \Rightarrow \psi$ and $\top \Rightarrow \varphi \to \psi$ are provably equivalent. So intuitionistic theories can ignore sets of sequents in favor of sets of formulas, by simply dropping the $(\top \Rightarrow)$-part.

Define $\xi(\varphi)$ as short for $((\top \to \varphi) \to \varphi) \to (\top \to \varphi)$. This is in agreement with the function ξ over Visser algebras of the form $\mathfrak{A}_\Gamma$, since $\xi([\varphi]_\Gamma) = [\xi(\varphi)]_\Gamma$.

Theorem 5.7 *Let Γ be a sequent theory over* BPC. *Then for all formulas φ we have*

1. $\Gamma \vdash \varphi \to \xi(\bot)$ *if and only if* $\Gamma + \mathrm{CPC} \vdash \varphi \to \bot$,

2. $\Gamma \vdash (\varphi \to \bot) \to \xi(\bot)$ *if and only if* $\Gamma + \mathrm{CPC} \vdash \varphi$.

Proof Item 1: $\Gamma \vdash \varphi \to \xi(\bot)$ if and only if $[\varphi]_\Gamma^{\xi(0)} = 1$ in $\mathfrak{A}_\Gamma$ if and only if (see Theorem 5.4) $[\varphi]_\Gamma^{00} = 0$ in $\mathfrak{R}^0(\mathfrak{A}_\Gamma)$ if and only if (see Proposition 5.6) $[\varphi]_{\Gamma+\mathrm{CPC}} = 0$ in $\mathfrak{A}_{\Gamma+\mathrm{CPC}}$ if and only if $\Gamma + \mathrm{CPC} \vdash \varphi \to \bot$.

Item 2: By item 1 we have $\Gamma \vdash (\varphi \to \bot) \to \xi(\bot)$ if and only if $\Gamma + \mathrm{CPC} \vdash (\varphi \to \bot) \to \bot$. Apply double negation elimination over CPC. $\square$

Since IPC $\vdash ((\top \to \varphi) \Leftrightarrow \varphi)$, we have IPC $\vdash (\xi(\varphi) \Leftrightarrow \varphi)$. In particular, IPC $\vdash \neg\xi(\bot)$. So over IPC, Theorem 5.7 reduces to the following well-known theorem.

Theorem 5.8 (Glivenko) *Let* Γ *be a theory over* IPC. *Then for all formulas* φ *we have*

1. $\Gamma \vdash \neg\varphi$ *if and only if* $\Gamma + \mathrm{CPC} \vdash \neg\varphi$,
2. $\Gamma \vdash \neg\neg\varphi$ *if and only if* $\Gamma + \mathrm{CPC} \vdash \varphi$.

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