

Vapnik–Chervonenkis Density on Indiscernible Sequences, Stability, and the Maximum Property

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Abstract This paper presents some finite combinatorics of set systems with applications to model theory, particularly the study of dependent theories. There are two main results. First, we give a way of producing lower bounds on VC_{ind} -density and use it to compute the exact VC_{ind} -density of polynomial inequalities and a variety of geometric set families. The main technical tool used is the notion of a maximum set system, which we juxtapose to indiscernibles. In the second part of the paper we give a maximum set system analogue to Shelah’s characterization of stability using indiscernible sequences.

1 Introduction

In the recent past, there have been a number of papers relating various measures of the combinatorial structure of NIP theories to one another (see Aschenbrenner, Dolich, Haskell, MacPherson, and Starchenko [1], Guingona and Hill [5], and Kaplan, Onshuus, and Usvyatsov [7]). One fact which emerged from this is the close relation of dp-rank and VC-density restricted to indiscernible sequences. Guingona and Hill have used the term VC_{ind} -density to describe VC-density restricted to indiscernibles. At the end of their paper [5], Guingona and Hill ask if there is a useful characterization of when a formula has VC_{ind} -density equal to VC-density. We offer such a characterization below (see Corollary 3.8) and use it to compute the exact VC_{ind} -density of certain formulas.

A separate goal of this paper is to show how maximum set systems can in many cases be used as more accessible surrogates for indiscernible sequences. To this end we, translate Shelah’s well-known characterization of stability in terms of indiscernible sets to a version involving maximum set systems.

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2 Notation

Let there be a fixed complete theory T , with a large saturated model $\mathfrak{M} = \langle M, \dots \rangle$. All sets and models, unless otherwise stated, are assumed to be elementarily embedded in the model \mathfrak{M} . We write formulas in partitioned form $\varphi(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = \langle x_1, \dots, x_l \rangle$, and $\mathbf{y} = \langle y_1, \dots, y_k \rangle$. We use $\mathcal{P}(X)$ to denote the power set of X .

For $A \subseteq M^{|\mathbf{y}|}$ and $\mathbf{b} \in M^{|\mathbf{x}|}$,

$$\varphi(\mathbf{b}, A) := \{\mathbf{a} \in A : \models \varphi(\mathbf{b}, \mathbf{a})\}.$$

We use $S_\varphi(A)$ for $A \subseteq M^{|\mathbf{y}|}$ to denote the set of φ -types over A , where a φ -type over A is a maximal consistent set of the form $\{\pm\varphi(\mathbf{x}, \mathbf{a}) : \mathbf{a} \in M^{|\mathbf{y}|}\}$. We let $S_\varphi(A)|_B = \{tp_\varphi(\mathbf{b}/A) : \mathbf{b} \in B\}$ when $B \subseteq M^{|\mathbf{x}|}$. For an individual type $p \in S_\varphi(A)$, we often identify p and the set of its positive parameter instances $\{\mathbf{a} \in A : \varphi(\mathbf{x}, \mathbf{a}) \in p\}$ without further comment. Similarly, we sometimes identify $S_\varphi(A)|_B$ and $\{\varphi(\mathbf{b}, A) : \mathbf{b} \in B\}$, as in the following definition.

Definition 2.1 Let $\mathcal{C} \subseteq \mathcal{P}(M^{|\mathbf{x}|})$. We will say that $\varphi(\mathbf{x}, \mathbf{y})$ *traces* \mathcal{C} if for some $A \subseteq M^{|\mathbf{x}|}$, $\mathcal{C} \subseteq S_\varphi(A)$.

We now give some purely combinatorial definitions. For the rest of this section suppose that X is a set and that $\mathcal{C} \subseteq \mathcal{P}(X)$. For $A \subseteq X$, let $\mathcal{C}|^A = \{C \cap A : C \in \mathcal{C}\}$. Say that \mathcal{C} *shatters* A if $\mathcal{C}|^A = \mathcal{P}(A)$.

Definition 2.2 The *Vapnik–Chervonenkis (VC) dimension* of \mathcal{C} , denoted $\text{VC}(\mathcal{C})$, is $|A|$, where $A \subseteq X$ is of maximum finite cardinality such that \mathcal{C} shatters A .

If the VC-dimension of \mathcal{C} does not exist, we write $\text{VC}(\mathcal{C}) = \infty$. The VC-dimension was first considered in Vapnik and Chervonenkis [16] and was introduced into model theory by Laskowski [11]. The following notion of a maximum VC family was defined by Welzl [17].

Definition 2.3 Say that \mathcal{C} is *d-maximum* for $d \in \omega$ if, for any finite $A \subseteq X$, $|\mathcal{C}|^A| = \binom{|A|}{\leq d}$.

Here $\binom{n}{\leq k}$ is shorthand for $\sum_{i=0}^k \binom{n}{i}$ if $k < n$ and 2^n otherwise.

Lemma 2.4 (Sauer’s lemma [13], Shelah [14]) *If \mathcal{C} has $\text{VC}(\mathcal{C}) = d$, then for any finite $A \subseteq X$,*

$$|\mathcal{C}|^A| \leq \binom{|A|}{\leq d}.$$

Thus a d -maximum set system is “extremal” among set systems of VC-dimension d . These set systems are highly structured and well understood (see Floyd [4] and Kuzmin and Warmuth [10]). There are several examples that arise from natural algebraic situations. In fact it is conjectured that all d -maximum set systems arise from (or embed naturally in) arrangements of half-spaces, either in a Euclidean or hyperbolic space (see Rubinstein and Rubinstein [12]).

It is easy to see that if \mathcal{C} is d -maximum on X , and $A \subseteq X$ has $|A| = d + 1$, then

$$\mathcal{C}|^A = \mathcal{P}(A) \setminus \{C\}$$

for some $C \subseteq A$. Floyd [4] calls such a C the *forbidden label* of \mathcal{C} on A .

Let $[X]^m := \{A \subseteq X : |A| = m\}$. For a fixed d -maximum $\mathcal{C} \subseteq \mathcal{P}(X)$, associate with each $A \in [X]^{d+1}$ the forbidden label $C_A \subseteq A$, where $\mathcal{C}|^A = \mathcal{P}(A) \setminus \{C_A\}$.

Floyd proves the following.

Proposition 2.5 ([4, Lemma 3.15]) *On a finite domain X , any d -maximum set system \mathcal{C} is characterized by its forbidden labels, in the sense that $\forall B \subseteq X$,*

$$B \in \mathcal{C} \iff \forall A \in [X]^{d+1} (B \cap A) \neq C_A.$$

Proof Left-to-right is obvious. For right-to-left, let B satisfy the given conditions. Then $\mathcal{C} \cup \{B\}$ shatters no sets not shattered by \mathcal{C} . By Sauer’s lemma, B must already be in \mathcal{C} . □

We now define the notion of a forbidden code, which is essentially the “form” of a forbidden label when an ordering is present.

Let $\mathcal{L}_{\mathcal{C}}(X) = \{C_A : A \in [X]^{d+1}\}$ denote the forbidden labels of \mathcal{C} on X . Let $<$ be a fixed but arbitrary linear order on X . For each $C_A \in \mathcal{L}_{\mathcal{C}}(X)$, let $\overline{C_A} = \langle t_0, \dots, t_d \rangle$, where each $t_i \in \{0, 1\}$, and $t_i = 1$ if and only if the i th element of A is in C_A . Define $\overline{\mathcal{L}_{\mathcal{C}}(X)} = \{\overline{C_A} : C_A \in \mathcal{L}_{\mathcal{C}}(X)\}$.

We will refer to $\overline{\mathcal{L}_{\mathcal{C}}(X)}$ as the set of *forbidden codes* on X for \mathcal{C} , with respect to $<$. When a maximum set system has a unique forbidden code, that code determines everything about the system at the level of finite traces. Technically, we say that the system is *finitely characterized* by the code.

Definition 2.6 The set system $\mathcal{C} \subseteq \mathcal{P}(X)$ is *finitely characterized* by $\eta \in 2^{d+1}$ if for any finite $X_0 \subseteq X$, and $A \subseteq X_0$, the following are equivalent:

1. $A \in \mathcal{C}|^{X_0}$,
2. there are not elements $a_0 < \dots < a_d$ in X_0 such that $a_i \in A \iff \eta(i) = 1$.

There is a natural way in which forbidden codes can serve as combinatorial invariants for finite unions of points and $<$ -convex subsets in X . To see this, suppose that $(X, <)$ is a dense and complete linear order and that $B \subseteq X$ is a finite union of convex subsets. Let d be the number of boundary points of B . We can imagine that B is defined by some $L_{<} = \{<\}$ formula $\psi(x, c_1, \dots, c_d)$ with $c_1 < \dots < c_d \in X$. Define $\mathcal{F}(B) = \{\psi(X, c'_1, \dots, c'_d) : c'_1 < \dots < c'_d \in X\}$. Intuitively, the elements of $\mathcal{F}(B) \subseteq \mathcal{P}(X)$ are the “homeomorphic images” of B in $(X, <)$. In Johnson [8] we show that $\mathcal{F}(B)$ is finitely characterized by some $\eta \in 2^{d+1}$.

Define the *genus* of B , denoted $\mathbb{G}(B)$, to be the $\eta \in 2^{d+1}$ that finitely characterizes $\mathcal{F}(B)$. Equivalently, define $\mathbb{G}(B)$ to be any $\eta \in 2^{d+1}$ such that there are no $a_0 < \dots < a_d$ in X such that $a_i \in B$ if and only if $\eta(i) = 1$, for all $i = 0, \dots, d$.

In [8] we show that such an η exists and is unique, as well as the further fact that for any $\eta \in 2^{<\omega}$, there is some $B \subseteq X$ such that $\mathbb{G}(B) = \eta$. Simple rules for computing genus are given in Table 1.

To give an example of applying the table, suppose that $X = \mathbb{R}$ and that $<$ is the usual ordering. Then the genus of the point $\{0\}$ is $\langle 11 \rangle$, and the genus of the interval $(0, 1)$ is $\langle 101 \rangle$. Conversely, to consider all subsets of \mathbb{R} with genus $\langle 11 \rangle$, let \mathcal{C} be all the singletons. Similarly, the collection of all subsets of genus $\langle 101 \rangle$ is exactly the set of all infinite convex subsets which are not coinital or cofinal.

The assumption that $(X, <)$ is complete was made to give a clear presentation of the genus concept and is sufficient for this paper. One can, however, define the genus

Table 1 A key for assigning forbidden codes to unions of convex sets.

Code	Translation
$\langle 1 \dots \rangle$	do nothing
$\langle 0 \dots \rangle$	$(-\infty, \dots$
$\langle \dots 0, 0 \dots \rangle$	remove point
$\langle \dots 0, 1 \dots \rangle$	end interval
$\langle \dots 1, 0 \dots \rangle$	begin interval
$\langle \dots 1, 1 \dots \rangle$	add point
$\langle \dots 0 \rangle$	$\dots, \infty)$
$\langle \dots 1 \rangle$	do nothing

of $B \subseteq X$ on other orders by considering the shortest $\eta \in 2^{<\omega}$ which B does not induce, sidestepping the issue of boundary points.

The link between genus and forbidden codes is given in the following theorem.

Theorem 2.7 *Suppose that $(X, <)$ is a complete dense linear order without endpoints. If $\eta \in 2^{d+1}$ and $\mathcal{C} = \{C \subseteq X : \mathbb{G}(C) = \eta\}$, then \mathcal{C} is d -maximum on X and $\mathfrak{L}_{\mathcal{C}}(X) = \{\eta\}$.*

Proof We provide a sketch of the proof that \mathcal{C} is d -maximum, which is very similar to the well-known proof that unions of intervals are maximum.

Let $X_0 \subseteq X$ be finite, and let $\mathcal{C} = \{C \subseteq X : \mathbb{G}(C) = \eta\}$. Let $a := \max X_0$, and let $X_0^a = X_0 \setminus \{a\}$. Define $\mathcal{C}^a = \{C \in \mathcal{C}|^{X_0^a} : C \cup \{a\} \in \mathcal{C}|^{X_0} \ \& \ C \in \mathcal{C}|^{X_0}\}$. By induction on $|X_0|$ and d , \mathcal{C}^a is $(d - 1)$ -maximum and $\mathcal{C}|^{X_0^a}$ is d -maximum. Then $|\mathcal{C}|^{X_0} = |\mathcal{C}^a| + |\mathcal{C}|^{X_0^a}$, and, by Pascal’s identity, $|\mathcal{C}|^{X_0} = \binom{|X_0|}{\leq d}$. \square

3 Results

3.1 VCm and VC_{ind}-density We now apply the above to achieve our results. Recall the following definitions.

Definition 3.1 A formula $\varphi(\mathbf{x}, \mathbf{y})$ has *VC-density* at most r for $r \in \mathbb{R}$ if there is $K \in \omega$ such that for every finite $A \subseteq M^{|\mathbf{y}|}$, $|S_\varphi(A)| < K \cdot |A|^r$. We denote this by $\text{VCd}(\varphi) \leq r$.

Definition 3.2 A formula $\varphi(\mathbf{x}, \mathbf{y})$ has *VC_{ind}-density* at most r for $r \in \mathbb{R}$ if there is $K \in \omega$ such that for every finite and indiscernible sequence $\bar{\mathbf{b}} = \langle \mathbf{b}_i : i < N \rangle \in M^{|\mathbf{y}| \cdot N}$, $|S_\varphi(\text{range}(\bar{\mathbf{b}}))| < K \cdot N^r$. We denote this by $\text{VCd}_{\text{ind}}(\varphi) \leq r$.

The study of VC-density has emerged several times in model theory. See [1] for historical remarks.

There has been some study of the fact that frequently $\text{VCd}(\varphi)$ is bounded by a simple (and uniform) function of $|\mathbf{x}|$ (see [1] and Johnson and Laskowski [9]). When this is true, it justifies the heuristic practice of “parameter counting” to guess the complexity of set systems. Guingona and Hill showed that in a dp -minimal theory, $\text{VCd}_{\text{ind}}(\varphi) \leq |\mathbf{x}|$. Thus there is interest in bounding $\text{VCd}(\varphi)$ by a function of $\text{VCd}_{\text{ind}}(\varphi)$ (obviously $\text{VCd}(\varphi) \geq \text{VCd}_{\text{ind}}(\varphi)$). This may not be possible in general, but we now show a practicable route to achieving it for a given formula.

Definition 3.3 For a set $A \subseteq M^{|\mathbf{y}|}$, we denote the traces of $\varphi(\mathbf{x}, \mathbf{y})$ on A by

$$\text{Tr}(\varphi, A) = \mathcal{P}(S_\varphi(A)).$$

We refer to the traces of φ on sets of cardinality κ by

$$\text{Tr}_\kappa(\varphi) = \bigcup_{A \in [M^{|y|}]^\kappa} \text{Tr}(\varphi, A).$$

The following is easy, but interesting, because the collection of d -maximum set systems would seem, a priori, to be very diverse. It also informs Definition 3.5.

Lemma 3.4 *Let κ be infinite. For each $d \in \omega$, there is some $\mathcal{C} \in \text{Tr}_\kappa(\varphi)$ such that \mathcal{C} is d -maximum if and only if for each $n \in \omega$, $n \geq d$, there is $\mathcal{C}_n \in \text{Tr}_n(\varphi)$ such that \mathcal{C}_n is d -maximum.*

Proof For the right-to-left direction, it is easily seen that the property of being d -maximum is elementary. Apply the compactness theorem and the saturation of \mathfrak{M} . For left-to-right, note that if \mathcal{C} is d -maximum and $X' \subseteq X$ with $|X'| = n$, $n \geq d$, then $\mathcal{C}|^{X'}$ is d -maximum. \square

As a consequence of Lemma 3.4, if κ is infinite, then there exists $\mathcal{C} \in \text{Tr}_\kappa(\varphi)$ which is d -maximum if and only if there is $\mathcal{C}' \in \text{Tr}_{\aleph_0}(\varphi)$ such that \mathcal{C}' is d -maximum.

Definition 3.5 The VC-maximum dimension of φ is defined by

$$\text{VCm}(\varphi) := \max\{d \in \omega : \exists \mathcal{C} \in \text{Tr}_{\aleph_0}(\varphi) \text{ s.t. } \mathcal{C} \text{ is } d\text{-maximum}\}.$$

Note that the definition of VC-maximum dimension does not use model theory.

If $\langle \mathbf{a}_i \rangle_{i \in I}$ is a sequence of indiscernibles and $(I, <)$ is a complete and dense linear order without endpoints, for $B \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$, define $\mathbb{G}(B)$ to be the genus of $\{i \in I : \mathbf{a}_i \in B\} \subseteq I$. With these assumptions, for $m \in \omega$, make the definition

$$P_m(B) = \{\rho \in 2^m : \exists i_0 < \dots < i_{m-1} \in I : \mathbf{a}_{i_j} \in B \iff \rho(j) = 1\}.$$

If $\mu \in 2^k$, $\eta \in 2^l$, and $l \leq k$, write $\eta \trianglelefteq \mu$ if η is a subsequence of μ , meaning that for some order-preserving embedding $f : l \rightarrow k$ (where k and l are regarded as ordinals) $\forall i \in l, \mu(f(i)) = \eta(i)$.

We observe that

$$\forall \rho \in 2^m, \quad \rho \in P_m(B) \iff \mathbb{G}(B) \trianglelefteq \rho. \tag{1}$$

Assume in Lemma 3.6 that the formula $\varphi(\mathbf{x}, \mathbf{y})$ is NIP.

Lemma 3.6 (Transfer lemma) *Let $\langle \mathbf{a}_i \rangle_{i \in I}$ be a sequence of indiscernibles, where $(I, <)$ is a complete and dense linear order without endpoints. Assume that $B \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$ is defined by $\varphi(\langle \mathbf{a}_i \rangle_{i \in I}, \mathbf{c})$, $A \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$, and $A' \subseteq A$ can be traced as $A' = A \cap B'$, where $B' \subseteq \langle \mathbf{a}_i \rangle_{i \in I}$ is such that $\mathbb{G}(B') = \mathbb{G}(B)$. Then, there exists $\mathbf{c}' \in M^{|y|}$ such that $\varphi(A, \mathbf{c}') = A'$.*

Proof First, consider the case in which A is finite.

Let $m = |A|$, and suppose that $\mathbf{a}_{i_0} < \dots < \mathbf{a}_{i_{m-1}}$ is an enumeration of A . Since $\mathbb{G}(B') = \mathbb{G}(B)$, we have $P_m(B) = P_m(B')$ by (1). Let $\rho \in P_m(B')$ be such that for each $j = 0, \dots, m-1$, $\mathbf{a}_{i_j} \in A' \iff \rho(j) = 1$. Then $\rho \in P_m(B)$, and for some $\mathbf{a}_{k_0} < \dots < \mathbf{a}_{k_{m-1}}$ in $\langle \mathbf{a}_i \rangle_{i \in I}$,

$$\mathfrak{M} \models \bigwedge_{j=0}^{m-1} \varphi(\mathbf{a}_{k_j}, \mathbf{c})^{\rho(j)}$$

and thus

$$\mathfrak{M} \models \exists \mathbf{y} \bigwedge_{j=0}^{m-1} \varphi(\mathbf{a}_{k_j}, \mathbf{y})^{\rho(j)}.$$

Then, by indiscernibility,

$$\mathfrak{M} \models \exists \mathbf{y} \bigwedge_{j=0}^{m-1} \varphi(\mathbf{a}_{i_j}, \mathbf{y})^{\rho(j)},$$

and the witnessing \mathbf{c}' is the desired parameter.

The case in which A is infinite now follows by compactness and the saturation of \mathfrak{M} . □

Theorem 3.7 For any $\varphi(\mathbf{x}, \mathbf{y})$, $\text{VCm}(\varphi) = \text{VCd}_{\text{ind}}(\varphi)$.

Proof First, note that we always have $\text{VCm}(\varphi) \geq 0$ and $\text{VCd}_{\text{ind}}(\varphi) \geq 0$.

Now suppose that $\text{VCd}_{\text{ind}}(\varphi) \geq d$, for some positive $d \in \omega$. Let $0 < \epsilon < 1/2$. By compactness, Ramsey’s theorem, and saturation of the monster model, there is some indiscernible sequence $\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$ such that $|S_\varphi(A)| \geq |A|^{d-\epsilon}$ for arbitrarily large finite $A \subseteq \langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$.

Claim $\exists B \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$ with $lg(\mathbb{G}(B)) = d + 1$.

First, we argue that there is $B \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$ with $lg(\mathbb{G}(B)) \geq d + 1$. Suppose, to the contrary, that $\forall B \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$, we have $lg(\mathbb{G}(B)) \leq d$. Let $A \subseteq \langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$ be finite, and consider an arbitrary $B \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$. There are $2^{d+1} - 1$ possibilities for $\mathbb{G}(B)$. For any choice of $\mathbb{G}(B)$, by Theorem 2.7,

$$|\{C \cap A : C \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}), \mathbb{G}(C) = \mathbb{G}(B)\}| \leq \binom{|A|}{\leq d-1}.$$

These two facts imply that $|S_\varphi(A)| \leq (2^{d+1} - 1) \cdot \binom{|A|}{\leq d-1} = O(|A|^{d-1})$. Because this holds for any finite $A \subseteq \langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$, the hypothesis on $\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$ is violated.

We now assume that $B \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$ and that $lg(\mathbb{G}(B)) \geq d + 1$. Without loss of generality, φ is NIP, because otherwise we have $\text{VCm}(\varphi) = \text{VCd}_{\text{ind}}(\varphi) = \infty$. Under these assumptions, $\mathbb{G}(B) = n \geq d + 1$ for some $n \in \omega$. Inducting on n , it follows by compactness and saturation of the monster that there is some $B' \in S_\varphi(\langle \mathbf{a}_i \rangle_{i \in \mathbb{R}})$, with $lg(\mathbb{G}(B')) = d + 1$. This proves the claim.

Now take B as in the claim. It follows from the transfer lemma that on any finite $A \subseteq \langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}$,

$$\{B' \cap A : B' \subseteq \langle \mathbf{a}_i \rangle_{i \in \mathbb{R}}, \mathbb{G}(B') = \mathbb{G}(B)\} \in \text{Tr}(\varphi, A).$$

This implies, by Theorem 2.7, that φ traces arbitrarily large d -maximum set systems, and, by Lemma 3.4, $\text{Tr}_{\aleph_0}(\varphi)$ contains a d -maximum set system. Thus $\text{VCm}(\varphi) \geq d$.

To show the other direction, suppose that $\text{VCm}(\varphi) \geq d$. By compactness, saturation, and Ramsey’s theorem (or, alternatively, by the Erdős–Rado theorem), there is an infinite indiscernible sequence $A = \langle \mathbf{a}_i \rangle_{i \in \omega}$ on which φ traces a d -maximum set system. It follows from the definition of d -maximum that $S_\varphi(A)$ witnesses that $\text{VCd}_{\text{ind}}(\varphi) \geq d$. □

It should be noted that Guingona and Hill prove that $\text{VCd}_{\text{ind}}(\varphi)$ is equal to several other invariants, among which $\text{VCm}(\varphi)$ may obviously be included.

An immediate corollary is the following.

Corollary 3.8 *For any formula $\varphi(\mathbf{x}, \mathbf{y})$, $\text{VCd}(\varphi) = \text{VCd}_{\text{ind}}(\varphi)$ if and only if φ traces an infinite d -maximum set system, where $d = \text{VCd}(\varphi)$.*

This condition is easier to use in practice than a direct appeal to a nonconstructive combinatorial principle such as the Ramsey or Erdős–Rado theorem. We give an algebraic example in the theory of real closed fields (RCF). Though we make efforts to be precise in the following statements, we are just considering a definable family that results from a polynomial inequality where the coefficients form the parameter set.

Without loss of generality we work in \mathbb{R} . Let \mathbf{y} be a finite sequence of variables. For a given $m \in \omega$, let Y_m denote the set of monomials which occur in the general polynomial of degree m with variables \mathbf{y} . Consider a family of polynomials of the form

$$p(c_1, \dots, c_d, \mathbf{y}) = u_0(\mathbf{y}) + c_1 u_1(\mathbf{y}) + \dots + c_d u_d(\mathbf{y}),$$

where for each $i = 0, \dots, d$, $u_i(\mathbf{y}) \in Y_m$ and for each $i = 1, \dots, d$, $c_i \in \mathbb{R}$. Define $\mathcal{C} = \{\text{pos}(p(\mathbf{c}, \mathbf{y})) : \mathbf{c} \in \mathbb{R}^d\}$, where $\text{pos}(p(\mathbf{y})) = \{\mathbf{a} \in \mathbb{R}^{|\mathbf{y}|} : p(\mathbf{a}) \geq 0\}$.

Such a \mathcal{C} can clearly be traced by some $\varphi(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}, \mathbf{y}) \in L_{\text{ring}}$, in $\mathbb{R} \models \text{RCF}$. It is known (see Floyd [4, Section 8.1]) that for such a φ , we have $\text{VCm}(\varphi) \geq d$. As it is well known that $\text{VCd}(\varphi) = d$, we have $\text{VCd}_{\text{ind}}(\varphi) = \text{VCd}(\varphi)$ for polynomial inequalities φ .

Floyd’s result is based on an application of Dudley’s theorem (see [3, Theorem 4.2.1]), which can apply to somewhat more general situations (see Ben-David and Litman [2]).

Many familiar geometric families, such as circles, ellipses, positive half-spaces, hyperbolas, and so on, therefore have VC_{ind} -density equal to VC-density. The above approach notably does not apply to geometric families which are not polynomially definable (in the above sense) such as axis-parallel rectangles, or convex d -gons.

3.2 Stability Here we show how to characterize the stability of φ using the maximum systems in $\text{Tr}(\varphi, A)$. (For a review of the relevant notions from stability theory, see Hodges [6].)

Recall that the *ladder dimension* of a formula $\varphi(\mathbf{x}, \mathbf{y})$ is defined by writing $\text{LD}(\varphi) \geq n$ if and only if there are $\mathbf{a}_0, \dots, \mathbf{a}_{n-1}$ in $M^{|\mathbf{x}|}$ and $\mathbf{b}_0, \dots, \mathbf{b}_{n-1}$ in $M^{|\mathbf{y}|}$ such that $\mathfrak{M} \models \varphi(\mathbf{a}_i, \mathbf{b}_j) \iff i < j$. Finite ladder dimension is equivalent to stability for formulas. The VC-dimension can be thought of as a generalization of ladder dimension, and in general $\text{coVC}(\varphi) \leq \text{LD}(\varphi)$, where $\text{coVC}(\varphi)$ denotes the VC-dimension of $S_\varphi(M^{|\mathbf{y}|})$ conceived as a set family.

For a set X and $\mathcal{C} \subseteq \mathcal{P}(X)$, define a graph $\mathcal{G}_{\mathcal{C}} = (V, E)$ where $V = \mathcal{C}$ and where $E(C_1, C_2)$ holds if and only if $|C_1 \Delta C_2| = 1$. For $C_1, C_2 \in \mathcal{C}$, define $\text{dist}_h(C_1, C_2)$ to be the Hamming distance $|C_1 \Delta C_2|$, and $\text{dist}_{\mathcal{G}_{\mathcal{C}}}(C_1, C_2)$ to be the graph distance in $\mathcal{G}_{\mathcal{C}}$, with $\text{dist}_{\mathcal{G}_{\mathcal{C}}}(C_1, C_2) = \infty$ if C_1 and C_2 belong to different components.

The following was proved by Kuzmin and Warmuth [10].

Lemma 3.9 ([10, Lemma 14]) *Let X be a finite set. Suppose that $\mathcal{C} \subseteq \mathcal{P}(X)$ is d -maximum and that $C_1, C_2 \in \mathcal{C}$. Then*

$$\text{dist}_h(C_1, C_2) = \text{dist}_{\mathcal{G}_{\mathcal{C}}}(C_1, C_2).$$

In particular, $\mathcal{G}_{\mathcal{C}}$ is connected.

The equivalence of dist_h and $\text{dist}_{\mathcal{G}_{\mathcal{C}}}$ is clearly still true when X is infinite, though the graph $\mathcal{G}_{\mathcal{C}}$ will not be connected in general. In fact, in many natural maximum set systems (e.g., open intervals on a densely ordered set), $\mathcal{G}_{\mathcal{C}}$ is totally disconnected.

Theorem 3.10 For $\varphi(\mathbf{x}, \mathbf{y})$ a formula, φ is stable with $\text{LD}(\varphi) \leq n$ if and only if for every $A \subseteq M^{|\mathbf{y}|}$ and every $\mathcal{C} \in \text{Tr}(\varphi, A)$ which is d -maximum for some $d \in \omega$, for any $C_1, C_2 \in \mathcal{C}$, $|C_1 \setminus C_2| \leq n$.

Proof First, suppose that $\varphi(\mathbf{x}, \mathbf{y})$ is a stable formula with $\text{LD}(\varphi) \leq n$, and $A \subseteq M^{|\mathbf{y}|}$.

Let $C_1, C_2 \in \mathcal{C}$, where $\mathcal{C} \in \text{Tr}(\varphi, A)$ is d -maximum for some $d \in \omega$. We will show that $|C_1 \setminus C_2| \leq n$. Note that it suffices to consider the case in which A is finite. Thus we may assume, by Lemma 3.9, that $\mathcal{G}_{\mathcal{C}}$ is connected.

Let $\{a_1, \dots, a_k\} \subseteq C_1 \setminus C_2$ be a set of distinct elements. By Lemma 3.9, after possibly reordering, there are B_k, B_{k-1}, \dots, B_1 in \mathcal{C} , on the path from C_1 to C_2 in $\mathcal{G}_{\mathcal{C}}$, such that $a_i \in B_j$ iff $i < j$. Thus $k \leq n$, and consequently $|C_1 \setminus C_2| \leq n$.

Conversely, suppose that $\text{LD}(\varphi) > n$. Let $B = \{\mathbf{b}_0, \dots, \mathbf{b}_n\} \subseteq M^{|\mathbf{x}|}$, and define $A = \{\mathbf{a}_0, \dots, \mathbf{a}_n\} \subseteq M^{|\mathbf{y}|}$ such that $\mathfrak{M} \models \varphi(\mathbf{b}_i, \mathbf{a}_j) \iff i < j$. Put $\mathcal{C} = S_{\varphi}(A)|_B$. Then by Theorem 2.7, \mathcal{C} is 1-maximum, and $|A \setminus \emptyset| = |A| = n + 1$. □

The above theorem shows that much of the nature of Shelah’s famous characterization of stable formulas by indiscernibles (see [15]) is already visible at the level of maximum traces. Unspooling the theorem reveals a structural characterization of stable maximum set systems, as we now show.

If X is a set, and $\mathcal{C} \subseteq \mathcal{P}(X)$, define the *ladder dimension* of \mathcal{C} to be the maximal $n \in \omega$ such that there are $x_1, \dots, x_n \in X$ and $B_1, \dots, B_n \in \mathcal{C}$ with $x_i \in B_j$ if and only if $i < j$. Say that \mathcal{C} is *stable* just in case it has finite ladder dimension.

If $A \subseteq X$, define $\mathcal{C} \Delta A = \{C \Delta A : C \in \mathcal{C}\}$.

Lemma 3.11 If $\mathcal{C} \subseteq \mathcal{P}(X)$ has $\text{LD}(\mathcal{C}) = n$, then for any $A \subseteq X$, we have $\text{LD}(\mathcal{C} \Delta A) \leq 2n$, and this bound is tight.

Proof Suppose that $\text{LD}(\mathcal{C} \Delta A) = 2n$ for an integer n . Then there exist x_1, \dots, x_{2n} in X and $C_1, \dots, C_{2n} \in \mathcal{C}$ such that for all $i, j \leq 2n$, $x_i \in C_j \Delta A$ if and only if $i < j$. Consider the case in which there are indices $i_1 < \dots < i_n$ such that for each $k = 1, \dots, n$, $x_{i_k} \notin A$. Then these elements, together with an appropriate choice of C_{j_1}, \dots, C_{j_n} , witness that $\text{LD}(\mathcal{C}) \geq n$. Now suppose that the opposite holds, namely, that there are indices $i_1 < \dots < i_{n+1}$ such that for each $k = 1, \dots, n + 1$, $x_{i_k} \in A$. Then $i_k < j$ and $x_{i_k} \notin C_j$ are equivalent, since both are equivalent to $x_{i_k} \in C_j \Delta A$. Reindexing by $C'_j = C_{2n-j}$ and taking an appropriate $j_1 < \dots < j_n$, we have that $x_{i_k} \in C'_{j_l}$ if and only if $k < l$. Thus $\text{LD}(\mathcal{C}) \geq n$. A similar argument shows that $\text{LD}(\mathcal{C}) \geq n$ in the case in which $\text{LD}(\mathcal{C} \Delta A) = 2n + 1$ is odd. This establishes the bound.

To see that the bound is tight, fix $n \in \omega$. Let X be the integers between $-n$ and n , inclusive, but not including zero. Let $\mathcal{C} = \{[0, i] \cap X : 0 < i \leq n\} \cup \{[-i, 0] \cap X : 0 < i \leq n\}$. Clearly $\text{LD}(\mathcal{C}) = n$. But $\text{LD}(\mathcal{C} \Delta [-n, -1]) = 2n$. □

Note that the example in the above proof is 1-maximum.

Corollary 3.12 *Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be d -maximum of ladder dimension n .*

1. *If $\emptyset \in \mathcal{C}$, then $\mathcal{C} \subseteq [X]^{\leq n}$.*
2. *$\mathcal{C} \Delta B \subseteq [X]^{\leq 2n}$ for any $B \in \mathcal{C}$.*
3. *$\mathcal{C} \subseteq [X]^{\leq 2n} \Delta B$ for any $B \in \mathcal{C}$.*

Proof The claim in (1) is clear from Theorem 3.10. For the claim in (2) note that $\emptyset \in \mathcal{C} \Delta B$ (because $B \in \mathcal{C}$), and the ladder dimension of $\mathcal{C} \Delta B$ is at most $2n$ by Lemma 3.11. Therefore $\mathcal{C} \Delta B \subseteq [X]^{\leq 2n}$ by (1). Claim (3) follows after applying ΔB to both sides of the containment in (2). \square

The $2n$ bound in Corollary 3.12(2) is tight, as the following example shows. Let $X = \{1, \dots, 2n\}$, and let $\mathcal{C} = [X]^{\leq n}$. Clearly $\text{LD}(\mathcal{C}) = n$. Now putting $B = \{1, \dots, n\}$ gives $\mathcal{C} \Delta B$ an element of cardinality $2n$.

On the other hand, it seems possible that for some $B \subseteq X$, it holds that $\mathcal{C} \Delta B \subseteq [X]^{\leq n}$, where the hypotheses are as in Corollary 3.12. However, since the hypotheses admit all finite maximum classes, this conjecture may be too optimistic. Such a result would clearly be the best possible.

It is also evident from the above that the stable maximum set systems are exactly those maximum set systems $\mathcal{C} \subseteq \mathcal{P}(X)$ which can be realized as $\mathcal{C} \subseteq [X]^m \Delta B$ for some $m \in \omega$ and $B \subseteq X$ (because the latter systems are clearly stable).

It would be useful to know whether every \mathcal{C} of ladder dimension n embeds into a $O(n)$ -maximum set system \mathcal{C}' (see [2] for relevant embedding notions). It is conjectured (see [4]) that the vast majority of set systems are *not* embeddable in maximum systems of the same VC-dimension, prompting the question of whether stable set systems, which are well behaved in so many respects, are also unusual in this way. Very little is known about model-theoretic criteria for when a definable family embeds in a maximum family, other than the easy observation that this is frequently possible in dimension 1.

Many nice properties of maximum set systems, in particular the existence of *compression schemes* (see [10] for definitions), are inherited under the relation of embedded substructure. Compression schemes emphasize the amount of information needed to represent a φ -type rather than the definability of the representation, and as a consequence they can be used to bound not only the VC-density of a set system, but also the size of the multiplicative constant in the definition of VC-density.

4 Conclusion

In model theory, much of the combinatorial content of theories comes from considering formulas restricted to indiscernible sequences. The existence of sequences of indiscernibles is guaranteed by Ramsey’s theorem (and compactness), though it is rarely required to exhibit a concrete sequence of indiscernibles.

When dealing with a certain formula $\varphi(\mathbf{x}, \mathbf{y})$ on a sequence $A = \langle \mathbf{a}_i \rangle_{i \in I}$, it is a weaker condition to assume that φ is maximum on A than to assume that A is indiscernible. However, as we have seen, if φ is maximum on A , that provides “enough” indiscernibility for some combinatorial notions to manifest. Namely, dp -rank, NIP, and stability can all be understood in terms of the maximum property. Avoiding the use of indiscernibles has the potential to make these concepts, particularly dp -rank, much more accessible to non-model theorists.

Unlike indiscernible sequences, maximum domains are frequently easily constructible. In the semilinear case, it follows from the work of Floyd and Dudley that a basic semilinear family will be maximum on a set of points which is in “general position,” for which it is sufficient to take a randomly selected set of points.

Moreover, the similarity of maximum families and formulas on indiscernible sequences provides a point of contact between work done in computational learning theory and model theory, where, especially recently, researchers are pursuing compatible combinatorial goals, but without a common framework.

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