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Substructural Fuzzy-Relevance Logic

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Abstract This paper proposes a new topic in substructural logic for use in research joining the fields of relevance and fuzzy logics. For this, we consider old and new relevance principles. We first introduce fuzzy systems satisfying an old relevance principle, that is, Dunn's weak relevance principle. We present ways to obtain relevant companions of the weakening-free uninorm (based) systems introduced by Metcalfe and Montagna and fuzzy companions of the system **R** of relevant implication (without distributivity) and its neighbors. The algebraic structures corresponding to the systems are then defined, and completeness results are provided. We next consider fuzzy systems satisfying new relevance principles introduced by Yang. We show that the weakening-free uninorm (based) systems and some extensions and neighbors of **R** satisfy the new relevance principles.

1 Introduction

The purpose of this paper is to extend the world of fuzzy logic to the realm of relevance logic, and vice versa. For this purpose, recall first some historical facts associated with fuzzy and relevance logics. Fuzzy logic based on t-norms has a distinguished history, the most famous examples being **Ł** (Łukasiewicz logic), **G** (Gödel–Dummett logic), $\prod$ (product logic), **BL** (basic fuzzy logic), and **MTL** (monoidal t-norm logic). These logic types are generally called *t-norm (based) logic*. T-norm logic is not a type of relevance logic because, while such logic proves the weakening (W) $\varphi \to (\psi \to \varphi)$, an arbitrary logic with (W) and modus ponens admits of a theorem $\varphi \to \psi$ such that φ and ψ are *irrelevant* to each other. The system **RM** (the **R** of relevant implication with mingle) has been considered as a type of relevance logic. In particular, Dunn [7] investigated **RM** capturing the tautologies on *denumerable* infinite sets of truth values and showed that **RM** is complete with respect to (w.r.t.) linearly ordered Sugihara matrices. According to Cintula [5], a (weakly implicative)

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logic **L** is said to be *fuzzy* if it is complete w.r.t. linearly ordered matrices (or algebras). Then, though **RM** is not a t-norm logic, it is still a fuzzy logic in Cintula's sense and is thus a type of fuzzy and relevance logic.

One interesting point is that Sugihara algebra with identity on the real unit interval [0, 1] is a uninorm, a generalization of t-norms where the identity can lie anywhere in [0, 1] (see Yager and Rybalov [21]). Metcalfe [15] introduced the uninorm logic **UL**, which captures the tautologies of *left-continuous conjunctive* uninorms and their residua, as a weakening of **MTL** and a strengthening of **MAILL** (multiplicative additive intuitionistic linear logic). Recently, Metcalfe and Montagna [16] have investigated **UL** and several axiomatic extensions of it as *substructural* fuzzy logical systems lacking structural rules like weakening or contraction. They introduced the weakening-free uninorm systems **UL**, **IUL** (involutive uninorm logic), **UML** (uninorm mingle logic), and **IUML** (involutive uninorm mingle logic). Among them, **IUML** is the $\mathbf{RM^T}$ (**RM** plus constants **T**, **F** and the corresponding axioms) with (FP) $\mathbf{t} \leftrightarrow \mathbf{f}$ (see Definition 14 below).[1] Thus, the system **IUML** may be regarded as a uninorm (based) version of **RM** or $\mathbf{RM^T}$ in the sense that the logic characterized by models based on [0, 1] is **IUML** but not $\mathbf{RM^{(T)}}$.[2] Furthermore, since the system **UL** is a weakening of $\mathbf{RM^T}$, both $\mathbf{RM^T}$ and **UL** seem to be not merely fuzzy, but relevant. (Note that relevance systems such as **R**, **RM**, **E** of entailment and **T** of ticket entailment all reject (W).) Therefore, it makes sense to wonder if the weakening-free uninorm systems introduced in [16] are all relevant.

The answer depends on the circumstances. Under the following situation, the answer is "no." We most frequently call a system *relevant* if it satisfies the *strong* relevance principle (SRP) in Anderson and Belnap [1] that $\varphi \to \psi$ is a theorem only if φ and ψ share a propositional variable, and sometimes if it satisfies the *weak* relevance principle (WRP) in Dunn [7] that $\varphi \to \psi$ is a theorem only if either (i) φ and ψ share a propositional variable, or (ii) both $\neg\varphi$ and ψ are theorems. For instance, the system **R** is strongly relevant in that it satisfies the principle SRP, and the system **RM** is weakly relevant in that it satisfies the principle WRP. However, the system **UL** is neither strongly nor weakly relevant because it proves such formulas as (α) $(\varphi \wedge \neg\varphi) \to (\psi \vee \neg\psi)$. Instead, although proving (α), the system **IUML** seems to be weakly relevant because it proves (EM) $\varphi \vee \neg\varphi$, and so the statements $\neg(\varphi \wedge \neg\varphi)$ and $\psi \vee \neg\psi$ are both theorems of **IUML**. However, since **IUML** also proves (β) $((\varphi \to \mathbf{F}) \wedge \varphi) \to \psi$, it still does not satisfy WRP. Therefore, none of the weakening-free uninorm systems are relevant in the sense that they satisfy neither SRP nor WRP.

Next, consider the circumstances under which the answer is "yes." Very recently, the present author [22] introduced new strong and weak relevance principles because the principles SRP and WRP do not work on relevance systems with propositional constants (see Galatos, Jipsen, Kowalski, and Ono [11] and Restall [20]). According to him, a system is said to be *strongly relevant* if it satisfies the new *strong* relevance principle (NSRP) in [22] that $\varphi \to \psi$ is a theorem only if φ and ψ either explicitly or strong implicitly share a propositional variable, and *weakly relevant* if it satisfies the new *weak* relevance principle (NWRP) that $\varphi \to \psi$ is a theorem only if either (i) φ and ψ share either explicitly or strong implicitly share a propositional variable, or (ii) both $\neg\varphi$ and ψ are theorems (see Section 3 below).[3] The weakening-free systems in [16] are all relevant in the sense that they satisfy the principle NSRP or the principle NWRP since theorems such as (α) and (β) strong implicitly share at least one propositional variable. Therefore, they are all relevant in that they satisfy NSRP or NWRP.

Let us call the relevance principles in [1] and [7] *old* relevance principles, and the relevance principles in [22] *new* relevance principles. Here, we introduce logics being both *fuzzy* in Cintula's sense and *relevant* in the old and new senses.

Related to the old relevance principles, one interesting point is that, while **IUML** proves the sentence (β), the **IUML** without constants **T**, **F** and the corresponding axioms (**RIUML**) does not. This shows that, among the systems obtained from the weakening-free uninorm systems introduced in [16] by omitting constants **T**, **F** and the corresponding axioms, systems proving (EM) satisfy WRP and so are weakly relevant in the old sense (see Theorem 2 in Section 2). This provides a way to obtain relevant companions of the weakening-free uninorm systems in [16], that is, the method that *drop constants **T**, **F** (and the corresponding axioms) from a weakening-free uninorm logic **L**, but instead add (EM) to **L** if it does not prove it*. Using this method, we can obtain weakening-free fuzzy systems satisfying WRP and so weakly (but not strongly) relevant in the old sense. We verify this by introducing several systems. More exactly, in Section 2.1, we first introduce the relevant uninorm logic **RUL**, the logic for a relevant companion of the uninorm logic **UL**, and its axiomatic extensions (i.e., relevant companions of **IUL**, **UML**, and **IUML**) as *substructural relevant fuzzy logics*. We can analogously consider fuzzy companions of relevance systems by adding an axiom ensuring prelinearity (together with (EM)) to relevance systems. Next, we introduce a method to obtain fuzzy companions of the relevance systems **RW** (the **R** without contraction), **R**, **RM**, or of their distributivity-free systems **LRW**, **LR**, and **LRM**, respectively. More precisely, we next introduce the contractionless fuzzy relevant logic **FRW**, the logic for a fuzzy companion of both **LRW** and **RW** (briefly **(L)RW**), and its axiomatic extensions **FR**, **FRM** (i.e., fuzzy companions of **(L)R**, **(L)RM**) as *substructural fuzzy relevant logics*. We will call these two types of logic *substructural fuzzy-relevance logics*. The results will show that the fuzzy relevant logics are also relevant fuzzy logics because the systems are between the weakest relevant fuzzy logic **RUL** and the strongest logic **RIUML**.

Many logics with the "prelinearity" axiom ($\mathrm{PL}_{\mathbf{t}}$) (A11 in Definition 2 below) are complete w.r.t. linearly ordered algebras (or matrices).[4] For example, the system **UL** is obtained by adding ($\mathrm{PL}_{\mathbf{t}}$) to **MAILL** and complete w.r.t. linearly ordered **UL**-algebras. In Section 2.2, we define algebraic structures corresponding to the systems introduced in Section 2.1 and then prove their completeness. In fact, since the method of the algebraic completeness proof is standard, we will instead show that they are *weakly implicative fuzzy logics*, the class of which is presented in [5]. This implies that they are all fuzzy logics in Cintula's sense. Furthermore, we prove that they also satisfy WRP. This will ensure that they are all weakly relevant in the old sense and so both fuzzy and relevant. Therefore, the study will introduce, in the view of substructural logic, one new research area—*fuzzy-relevance logic*—bridging fuzzy logic and relevance logic, each of which has been independently investigated. (Namely, it introduces the logics belonging to the intersection of the families of fuzzy and relevance logics.)

In addition, in Section 2.3, we briefly consider the fuzzy-relevance systems eliminating ($\mathrm{PL}_{\mathbf{t}}$) as both *weakly implicative logics*, the class of which is also presented in [5], and *substructural relevance logics*.

The fuzzy-relevance systems introduced in Section 2 would not be interesting to some (or many) fuzzy logicians because they are not t-norm systems nor even uninorm (based) systems. Instead, the principles NSRP and NWRP may be interesting to such logicians because these principles ensure that some weakening-free uninorm systems are relevant. More precisely, in Section 3, we preliminarily introduce NSRP and NWRP because they are not familiar to the readers. In Section 4.1, we prove that the weakening-free uninorm systems in [16] and the fuzzy companions of the relevance systems with constants **T** and **F** $\mathbf{FRW^T}$, $\mathbf{FR^T}$, $\mathbf{FRM^T}$ all satisfy NSRP or NWRP. In Section 4.2, we introduce a method to obtain strong relevant companions from **UL** and its weakening-free extensions. We also call the systems introduced in Sections 4.1 and 4.2 *substructural fuzzy-relevance logics*. In addition, as in Section 2.3, in Section 4.3, we consider substructural relevance logics obtained from the systems by omitting ($\mathrm{PL}_{\mathbf{t}}$).

All the systems (i.e., the systems with and without ($\mathrm{PL}_{\mathbf{t}}$)) introduced in Sections 2 and 4 are *substructural logics* placed somewhere over $\mathbf{FL}_e$ (full Lambek logic with exchange) (see Remark 3 below). Thus, all the (fuzzy-)relevance logics investigated here are substructural logics. For simplicity, we henceforth call them (fuzzy-)relevance logics and not substructural (fuzzy-)relevance logics.

For convenience, we adopt the notation and terminology similar to that in Cintula [5], Esteva and Godo [9], [10], Hájek [12], and Metcalf and Montagna [16] and assume familiarity with them (together with the results found therein).

2 Fuzzy-Relevance Logics (I)

In this section, we introduce several fuzzy-relevance systems satisfying the principle WRP and their corresponding non-fuzzy-relevance systems.

2.1 Syntax We base (fuzzy-)relevance logics on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\to, \&, \wedge, \vee$, and constants $\mathbf{f}, \mathbf{t}$, with defined connectives:

df1. $\neg\varphi := \varphi \to \mathbf{f}$, and

df2. $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

We moreover define $\varphi^n_{\mathbf{t}}$ as $\varphi_{\mathbf{t}} \,\&\, \cdots \,\&\, \varphi_{\mathbf{t}}$, n factors, where $\varphi_{\mathbf{t}} := \varphi \wedge \mathbf{t}$, and similarly for φ^n.

For the remainder of the paper, we will utilize the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of **RMAILL** (relevant multiplicative additive intuitionistic linear logic) as the basic relevance logic defined here.[5]

Definition 1 **RMAILL** consists of the following axiom schemes and rules:

A1. $\varphi \to \varphi$ (self-implication, SI)

A2. $(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$ ($\wedge$-elimination, $\wedge$-E)

A3. $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$ ($\vee$-elimination, $\vee$-E)

A6. $(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi)$ (&-commutativity, &-C)

A7. $(\varphi \,\&\, \mathbf{t}) \leftrightarrow \varphi$ (push and pop, PP)

A8. $(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \,\&\, \psi) \to \chi)$ (residuation, RE)

A9. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ (suffixing, SF)

A10. $\varphi \vee \neg\varphi$ (excluded middle, EM)
$\varphi \to \psi, \varphi \vdash \psi$ (modus ponens, mp)
$\varphi, \psi \vdash \varphi \wedge \psi$ (adjunction, adj)

Relevant uninorm logic **RUL**, the basic relevant fuzzy logic defined here, is **RMAILL** extended with the "prelinearity" axiom scheme below.

Definition 2 **RUL** is **RMAILL** plus
A11. $(\varphi \to \psi)_{\mathbf{t}} \vee (\psi \to \varphi)_{\mathbf{t}}$ ($\mathrm{PL}_{\mathbf{t}}$)

Relevant fuzzy logics are defined by extending **RUL** with suitable axiom schemes as follows.

Definition 3 A logic is an *axiomatic extension* (*extension* for short) of **L** if and only if (iff) it results from the addition of axiom scheme(s) to **L**. In particular, the following are relevant fuzzy logics extending **RUL**:

- involutive RUL **RIUL** is **RUL** plus (DNE) $\neg\neg\varphi \to \varphi$;
- idempotent RUL **RUML** is **RUL** plus (ID) $(\varphi \,\&\, \varphi) \leftrightarrow \varphi$;
- involutive RUML **RIUML** is **RIUL** plus (ID) and (FP) $\mathbf{t} \leftrightarrow \mathbf{f}$.[6]

Remark 1 By eliminating the axiom (EM) from the systems **RUL**, **RIUL**, **RUML**, and **RIUML** and adding propositional constants **T**, **F** (and the corresponding axioms, i.e., A12 and A13 in Section 4), we obtain the weakening-free uninorm systems **UL**, **IUL**, **UML**, and **IUML**, respectively, as introduced in [16]. However, note that, since **IUML** proves (EM), we obtain **IUML** simply by adding constants **T**, **F** (and the corresponding axioms) to **RIUML**.

The system **LRW** is the $\mathbf{FL}_e$ with (DNE). Fuzzy relevant logics are defined by extending **LRW** or **RMAILL** with suitable axiom schemes as follows.

Definition 4 The following are fuzzy relevant logics extending **LRW**:

- fuzzy LRW **FRW** is **LRW** plus A11 and (EM);
- fuzzy LR **FR** is **FRW** plus (SIN) $\varphi \to (\varphi \,\&\, \varphi)$;
- fuzzy LRM **FRM** (= **RM**) is **FR** plus (SDE) $(\varphi \,\&\, \varphi) \to \varphi$.

We may instead consider the systems **FRW**, **FR**, and **FRM** as fuzzy relevant logics extending **RW**. (Note that the system **RW** is **LRW** plus (distributivity, D) $(\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$, and so we can obtain **FRW**, **FR**, and **FRM** by adding the axiom scheme(s) in Definition 4 to the systems **RW**, **R**, and **RM**, respectively.) However, here we introduce them as extensions of **LRW** in place of **RW** because the basic relevance logic **RMAILL** is a distributivity-free system.

Remark 2 By eliminating both A11 and (EM) from the system **FRW**, we obtain the distributivity-free relevance logic **LRW**; by eliminating A11 from the systems **FR** and **FRM**, we get **LR** and **LRM**, respectively. Note that **LR** and **LRM** each prove (EM). Note also that the systems **FRW** and **FRM** are the same as **RIUL** and **RM**, respectively, and that the system **RIUML** is **FRM** (= **RM**) plus (FP). Therefore, the fuzzy relevant logics **FRW**, **FR**, and **RM** are all between **RUL** and **RIUML** and so are relevant fuzzy logics extending **RUL**. Tables 1 and 2 summarize some axiom schemes and the extensions of **RMAILL** introduced above.

For easy reference, we let Ls be the set of fuzzy-relevance logics defined previously.

Table 1 Some axiom schemes in fuzzy-relevance logics.

Axiom schema	Name
$\neg\neg A \to A$	Double negation elimination (DNE)
$A \to A \,\&\, A$	Square increasing (SIN)
$A \,\&\, A \leftrightarrow A$	Idempotence (ID)
$\mathbf{t} \leftrightarrow \mathbf{f}$	Fixed-point (FP)
$(\varphi \to \psi)_{\mathbf{t}} \vee (\psi \to \varphi)_{\mathbf{t}}$	Prelinearity (PL$_{\mathbf{t}}$)

Table 2 Some extensions of **RMAILL** obtained by adding the corresponding additional axiom schemes.

Logic	Additional axiom schemes
RUL	(PL$_{\mathbf{t}}$)
RIUL (= **FRW**)	(PL$_{\mathbf{t}}$) and (DNE)
FR	(PL$_{\mathbf{t}}$), (DNE) and (SIN)
RUML	(PL$_{\mathbf{t}}$) and (ID)
RM (= **FRM**)	(PL$_{\mathbf{t}}$), (DNE) and (ID)
RIUML	(PL$_{\mathbf{t}}$), (DNE), (ID) and (FP)

Definition 5 Ls = {**RUL**, **RIUL** (= **FRW**), **RUML**, **RIUML**, **FR**, **RM** (= **FRM**)}.

A *theory* is a set of formulas. A *proof* in a theory T over L ($\in$ Ls) is a sequence s of formulas such that each element of s is either an axiom of L, a member of T, or is derivable from previous elements of s by means of a rule of L. $T \vdash \varphi$, more exactly $T \vdash_L \varphi$, means that φ is *provable* in T w.r.t. L, that is, there is an L-proof of φ in T. If $\vdash_L \varphi$, that is, $T = \emptyset$, φ is said to be a *theorem* of L. A theory T is said to be *inconsistent* if $T \vdash \mathbf{F}$; otherwise it is *consistent*.

The relevant (local) deduction theorem (R(L)DT) for L is as follows.

Proposition 1 *Let T be a theory over L ($\in$ Ls), and let φ, ψ be formulas.*

(i) *(RLDT) $T \cup \{\varphi\} \vdash_L \psi$ iff there is n such that $T \vdash_L \varphi_t^n \to \psi$.*
(ii) *(RDT) For L with (SIN), $T \cup \{\varphi\} \vdash_L \psi$ iff $T \vdash_L \varphi_t \to \psi$.*

Proof For (i), see Novak [19, Theorem 9, Corollary 1]; (ii) is the enthymematic deduction theorem (see Meyer, Dunn, and Leblanc [18, Lemma 4]). □

An easy computation shows the following.

Proposition 2

(i) ***RMAILL*** *proves*

(1) $(\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ *(prefixing, PF)*
(2) $(\varphi \to \psi) \to ((\varphi \,\&\, \chi) \to (\psi \,\&\, \chi))$ *(monotonicity, MT)*
(3) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$ *(permutation, PM)*
(4) $(\varphi \,\&\, (\psi \,\&\, \chi)) \to ((\varphi \,\&\, \psi) \,\&\, \chi)$ *(associativity, AS)*
(5) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ *(contraposition, CP)*
(6) $\varphi \to \neg\neg\varphi$ *(double negation introduction, DNI)*
(7) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$ *(de MorganI, DM1)*
(8) $(\neg\varphi \vee \neg\psi) \to \neg(\varphi \wedge \psi)$ *($\vee\neg$)*
(9) $(\varphi \,\&\, (\psi \vee \chi)) \leftrightarrow ((\varphi \,\&\, \psi) \vee (\varphi \,\&\, \chi))$ *($\&\vee$-distributivity, $\&\vee$-D)*

(10) $(\varphi \,\&\, (\psi \wedge \chi)) \to ((\varphi \,\&\, \psi) \wedge (\varphi \,\&\, \chi))$ *($\&\wedge$)*
(11) $(\varphi_t \,\&\, \varphi_t) \to \varphi_t$ *(**t**-square decreasing, SDE$_t$)*
(12) $\varphi \leftrightarrow (\boldsymbol{t} \to \varphi)$
(13) $\neg(\varphi \wedge \neg\varphi)$, $\neg\varphi \vee \neg\neg\varphi$

(ii) ***RUL*** *proves*
(1) $(\neg\varphi \vee \neg\psi) \leftrightarrow \neg(\varphi \wedge \psi)$ *(de MorganII, DM2)*
(2) $(\varphi \,\&\, (\psi \wedge \chi)) \leftrightarrow ((\varphi \,\&\, \psi) \wedge (\varphi \,\&\, \chi))$ *($\&\wedge$-distributivity, $\&\wedge$-D)*
(3) $(\varphi \to \psi) \vee (\psi \to \varphi)$ *(PL)*
(4) *For each n,* $(\varphi \to \psi)^n_t \vee (\psi \to \varphi)^n_t$ *(^n_t-prelinearity, PL^n_t)*
(5) $(\varphi \wedge (\varphi \to \boldsymbol{f})) \to (\psi \vee (\psi \to \boldsymbol{f}))$, *that is,* $(\varphi \wedge \neg\varphi) \to (\psi \vee \neg\psi)$
(6) $(\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$ *(distributivity, D)*

(iii) ***RUL*** *having (SIN) proves*
(1) $(\varphi \wedge \psi) \to (\varphi \,\&\, \psi)$

(iv) ***RUML*** *proves*
(1) $(\varphi \,\&\, \psi) \to (\varphi \vee \psi)$
(2) $((\varphi \vee \psi) \to (\varphi \,\&\, \psi)) \vee ((\varphi \,\&\, \psi) \to (\varphi \wedge \psi))$
(3) $((\varphi \,\&\, \psi) \leftrightarrow (\varphi \wedge \psi)) \vee ((\varphi \,\&\, \psi) \leftrightarrow (\varphi \vee \psi))$

(v) ***RIUL*** *(=* ***FRW****) proves*
(1) $\varphi \leftrightarrow \neg\neg\varphi$ *(double negation, DN)*

(vi) ***FR*** *without (EM) proves*
(1) $\varphi \vee \neg\varphi$ *(excluded middle, EM)*
(2) $\neg(\varphi \wedge \neg\varphi)$ *(noncontradiction, NC)*

(vii) ***RM*** *(=* ***FRM****) proves*
(1) $(\varphi \to \psi) \to ((\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi))$
(2) $\neg(\varphi \to \psi) \to ((\varphi \to \psi) \leftrightarrow (\neg\varphi \wedge \psi))$
(3) $(\varphi \to \neg\psi) \to ((\varphi \,\&\, \psi) \leftrightarrow (\varphi \wedge \psi))$
(4) $\neg(\varphi \to \neg\psi) \to ((\varphi \,\&\, \psi) \leftrightarrow (\varphi \vee \psi))$
(5) $\neg(\varphi \to \varphi) \to (\psi \to \psi)$
(6) $\boldsymbol{f} \to \boldsymbol{t}$

In L ($\in$ Ls), the negation $\neg$ may be taken as a primitive connective, and the constant **f** can be instead defined as $\neg$**t** (see Proposition 2(i)(12)).

For convenience, $\neg$, $\wedge$, $\vee$, and $\to$ are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meanings.

2.2 Semantics Suitable algebraic structures for the (fuzzy-)relevance logics are obtained as varieties of residuated lattices in the sense of [11].

Definition 6 A *pointed commutative residuated lattice* is a structure $(A, t, f, \wedge, \vee, *, \to)$ such that:[7]

(I) $(A, \wedge, \vee)$ is a lattice,
(II) $(A, *, t)$ is a commutative monoid,
(III) $y \leq x \to z$ iff $x * y \leq z$, for all $x, y, z \in A$ (residuation),
(IV) f is an arbitrary element of A.

As φ^n in Section 2.1, by x^n, we denote $x * \cdots * x$, n factors.

Note that the class of pointed commutative residuated lattices characterizes the system $\mathbf{FL_e}$. Thus, we henceforth call such residuated lattices $\boldsymbol{FL_e}$*-algebras*.

Definition 7 Let $\neg x := x \to f$, and let $x_t := x \wedge t$.

(i) (RMAILL-algebra) An *RMAILL-algebra* is a pointed commutative residuated lattice satisfying the condition
(EM) $t \leq x \vee \neg x$.

(ii) (RUL-algebra) An *RUL-algebra* is an RMAILL-algebra satisfying the condition
(PL$_t$) $t \leq (x \to y)_t \vee (y \to x)_t$.

In an analogy to Definition 7, we can define algebras corresponding to the systems introduced in Definitions 3 and 4. When we define the FL$_e$-algebra, we can use in place of (III) a family of equations as in [11] and [12].

An RMAILL-algebra is said to be *linearly ordered* if the ordering of its algebra is linear; that is, $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y. Note that, in RMAILL-algebras, the operator $\neg$ can be defined using $\to$ and f, as above. Thus, an RMAILL-algebra $(A, t, f, \neg, \wedge, \vee, *, \to)$ may be abbreviated to $(A, t, f, \wedge, \vee, *, \to)$.

As in Section 2.1, for brevity, by *L-algebra(s)*, we henceforth ambiguously express algebras corresponding to all L systems, if we need not distinguish them, but context should determine which algebras are intended.

Definition 8 (Evaluation) Let $\mathcal{A}$ be an L-algebra. An $\mathcal{A}$-*evaluation* is a function $v : FOR \to \mathcal{A}$ satisfying: $v(\varphi \to \psi) = v(\varphi) \to v(\psi)$, $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$, $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$, $v(\varphi \,\&\, \psi) = v(\varphi) * v(\psi)$, $v(\mathbf{t}) = t$, $v(\mathbf{f}) = f$ (and hence $v(\neg\varphi) = \neg v(\varphi)$).

Definition 9 ([5, Definitions 16–17]) Let $\mathcal{A}$ be an L-algebra, let T be a theory, let φ be a formula, and let $\mathcal{K}$ be a class of L-algebras.

(i) (Tautology) φ is a t-*tautology* in $\mathcal{A}$, briefly an $\mathcal{A}$-*tautology* (or $\mathcal{A}$-*valid*), if $v(\varphi) \geq t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an $\mathcal{A}$-*model* of T if $v(\varphi) \geq t$ for each $\varphi \in T$. By $Mod(T, \mathcal{A})$, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) φ is a *semantic consequence* of T w.r.t. $\mathcal{K}$, denoted by $T \models_{\mathcal{K}} \varphi$, if $Mod(T, \mathcal{A}) = Mod(T \cup \{\varphi\}, \mathcal{A})$ for each $\mathcal{A} \in \mathcal{K}$.

Definition 10 (L-algebra, [5]) Let $\mathcal{A}$, T, and φ be as in Definition 9. $\mathcal{A}$ is an ***L****-algebra* if, whenever φ is L-provable in any T (i.e., $T \vdash_L \varphi$, L an L logic), it is a semantic consequence of T w.r.t. $\{\mathcal{A}\}$ (i.e., $T \models_{\{\mathcal{A}\}} \varphi$, $\mathcal{A}$ a corresponding L-algebra). By $MOD(L)$, we denote the class of **L**-algebras; by $MOD^l(L)$, the class of linearly ordered **L**-algebras. Finally, we write $T \models_L \varphi$ and $T \models^l_L \varphi$ in place of $T \models_{MOD(L)} \varphi$ and $T \models_{MOD^l(L)} \varphi$, respectively.

Note that, since each condition for an RMAILL-algebra (more generally, L-algebra) has the form of an equation or can be defined in an equation, it can be ensured that the class of all RMAILL-algebras (L-algebras) is a variety. Then, as in [16], we can show that L is complete w.r.t. an algebraic semantic given by a variety of L-algebras. We instead show that L is a weakly implicative fuzzy logic. This also implies the completeness of L.

Let a theory T be *linear* if, for each pair φ, ψ of formulas, $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$. We denote the class of all RUL-algebras as RUL and the class of all L-algebras as L. The system **RUL** is an algebraizable logic in the sense of Blok and

Pigozzi (see Czelakowski [6]). Then, since it implies that all axiomatic extensions of **RUL** are also algebraizable and that their equivalent algebraic semantics are the subvarieties of RUL defined by the translations of the axioms into equations, L is also algebraizable, and its equivalent algebraic semantic L is the subvariety of RUL.

Cintula [5] defined *weakly implicative logic* (WIL) as a logic satisfying A1, (mp), transitivity ($\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$), and congruence w.r.t connectives and called a WIL **L** a *fuzzy* logic (i.e., a weakly implicative fuzzy logic or WIFL) if it is complete w.r.t. linearly ordered (corresponding) matrices. He also showed that, for a finitary WIL **L**, the following are equivalent.

(1) **L** is a fuzzy logic.
(2) **L** has the linear extension property; that is, for each theory T, if $T \nvdash \varphi$, then there is a consistent linear theory $T' \supseteq T$ such that $T' \nvdash \varphi$.
(3) **L** has the prelinearity property, that is, for each theory T, if $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$, then $T \vdash \chi$.
(4) **L** has the subdirect decomposition property; that is, each ordered **L**-matrix is a subdirect product of linearly ordered **L**-matrices.

We can easily show that the system **RMAILL** is a WIL. Let a logic extending **RMAILL** be a relevant multiplicative additive intuitionistic linear logic (briefly, an RMAILL). For an RMAILL **L**, Proposition 2(ii)(4) is the condition for **L** to be a fuzzy logic.

Theorem 1 *Let* **L** *be an RMAILL. Then* **L** *is a fuzzy logic iff, for each n,* $\vdash_L (\varphi \to \psi)^n_t \vee (\psi \to \varphi)^n_t$.

Proof The left-to-right direction is obvious. For the right-to-left direction, we show that **L** has the prelinearity property. Let $T, \varphi \to \psi \vdash_{\mathbf{L}} \chi$ and $T, \psi \to \varphi \vdash_{\mathbf{L}} \chi$. By the theorem (RLDT), for some n, m, $T \vdash_{\mathbf{L}} (\varphi \to \psi)^n_{\mathbf{t}} \to \chi$ and $T \vdash_{\mathbf{L}} (\psi \to \varphi)^m_{\mathbf{t}} \to \chi$. Let $m \leq n$. Proposition 2(i)(11) ensures that we can obtain $T \vdash_{\mathbf{L}} (\varphi \to \psi)^n_{\mathbf{t}} \to \chi$ and $T \vdash_{\mathbf{L}} (\psi \to \varphi)^n_{\mathbf{t}} \to \chi$. Then, by A5 (together with (adj) and (mp)), we get $T \vdash_{\mathbf{L}} ((\varphi \to \psi)^n_{\mathbf{t}} \vee (\psi \to \varphi)^n_{\mathbf{t}}) \to \chi$. Thus, by Proposition 2(ii)(4) and (mp), $T \vdash_{\mathbf{L}} \chi$, as desired. □

Then, from Theorem 1, we establish the following corollaries.

Corollary 1 (Strong completeness) *Let T be a theory over L ($\in$ Ls), and let φ be a formula. Then $T \vdash_L \varphi$ iff $T \models^l_L \varphi$.*

Corollary 2 *L is a fuzzy logic (in Cintula's sense).*

Now we verify the relevance of L.

Theorem 2

(i) *L does not satisfy SRP (in [1]).*
(ii) *L satisfies WRP (in [7]).*

Proof (i) This directly follows from Proposition 2(ii)(5).

(ii) We prove this contrapositively. Namely, we assume that φ, ψ share no propositional variables and either $\nvdash_L \neg\varphi$ or $\nvdash_L \psi$ and show that $\nvdash_L \varphi \to \psi$. For this, consider an algebra $\mathcal{A} = (\{1, \frac{1}{2}, 0\}, \frac{1}{2}, \frac{1}{2}, \min, \max, *, \to)$, where, letting $\neg x$ (i.e., $x \to \frac{1}{2}$) $= 1 - x$,

$$x * y = \begin{cases} \min(x, y) & \text{if } x \leq \neg y, \\ \max(x, y) & \text{otherwise,} \end{cases}$$

and

$$x \to y = \begin{cases} \max(\neg x, y) & \text{if } x \leq y, \\ \min(\neg x, y) & \text{otherwise.} \end{cases}$$

The axioms of L are $\mathcal{A}$-tautologies for all assignments of values to the variables, and the rules preserve this property. This ensures soundness; that is, if $\vdash_L \varphi$, then $\models_{\mathcal{A}} \varphi$. Let φ and ψ share no propositional variables, and let either $\nvdash_L \neg\varphi$ or $\nvdash_L \psi$. If $\vdash_L \neg\varphi$ and $\nvdash_L \psi$, then assign the values yielding $v(\varphi) = 1/2$ to all the variables of φ and the values yielding $v(\psi) = 0$ to all the variables of ψ; if $\nvdash_L \neg\varphi$ and $\vdash_L \psi$, then assign the values yielding $v(\varphi) = 1$ to all the variables of φ and the values yielding $v(\psi) = 1/2$ to all the variables of ψ; otherwise, that is, if $\nvdash_L \neg\varphi$ and $\nvdash_L \psi$, assign the values yielding $v(\varphi) = 1$ to all the variables of φ and the values yielding $v(\psi) = 0$ to all the variables of ψ. Then, in each case, $v(\varphi \to \psi)$ is 0. Thus, $\not\models_{\mathcal{A}} \varphi \to \psi$. Therefore, by soundness, $\nvdash_L \varphi \to \psi$, as required. □

Corollary 3 *L is a relevance logic (in the weak sense of [7]).*

Corollary 4 *L is both a fuzzy logic and a relevance logic.*

2.3 Substructural relevance logics (I) Let **RMAILL**, **RMALL** (= **LRW** plus (EM)), **LR**, **RMAILML**, **LRM**, and **RMALML** be the systems excluding ($PL_{\mathbf{t}}$) from the systems **RUL**, **RIUL** (= **FRW**), **FR**, **RUML**, **RM**, and **RIUML**, respectively. We let Ls^- be the set of these systems; that is, we have the following.

Definition 11 Ls^- = {**RMAILL**, **RMALL**, **LR**, **RMAILML**, **LRM**, **RMALML**}.

Since L^- ($\in \mathrm{Ls}^-$) is an RMAILL and so a WIL, Theorem 1 shows that L ($\in$ Ls) is a WIFL (see Corollary 2); moreover, L is the *weakest* fuzzy logic extending L^-. (Note that the systems **LRM** and **RMALML** are not the same as **RM** and **RIUML**, respectively, because the former systems exclude (D), i.e., Proposition 2(ii)(6).) More exactly, [5, Theorem 1] says that, for a WIL **L**, $T \vdash \varphi$ iff $T \models \varphi$, and so we obtain the following corollary.

Corollary 5 (Strong completeness) *For each theory T over L^- ($\in Ls^-$) and formula φ, $T \vdash_{L^-} \varphi$ iff $T \models_{L^-} \varphi$.*

Let us verify the relevance of L^- ($\in \mathrm{Ls}^-$). Before verifying this, we first note that an easy computation shows the following.

Proposition 3 ***LRM** and **RMALML** each prove $\neg(\varphi \to \varphi) \to (\psi \to \psi)$, that is, Proposition 2(vii)(5).*

Corollary 6 *For $L^- \in \{$**LRM**, **RMALML**$\}$, L^- does not satisfy SRP (in [1]).*

We can further show the following.

Theorem 3

(i) *For $L^- \in \{$**RMAILL, RMALL, RMAILML, LR**$\}$, L^- satisfies SRP (in [1]).*
(ii) *For $L^- \in \{$**LRM, RMALML**$\}$, L^- satisfies WRP (in [7]).*

Proof (i) First note that L^- ($\in$ {**RMAILL**, **RMALL**, **LR**}) satisfies the matrices considered in [2] in order to show that $\mathbf{R^0}$, the **t**-free fragment of **R**, satisfies SRP. Thus, we prove that the system **RMAILML** satisfies SRP. We assume that φ, ψ share no propositional variables and show that $\nvdash_{\mathbf{RMAILML}} \varphi \to \psi$. For this, consider the algebra $\mathcal{A}$ in the proof of Theorem 2(ii). The axioms of **RMAILML** are $\mathcal{A}$-tautologies for all assignments of values to the variables, and the rules preserve this property. This ensures soundness. Let φ and ψ share no propositional variables. Assign the values yielding $v(\varphi) = 1/2$ to all the variables of φ and the values yielding $v(\psi) = 0$ to all the variables of ψ. Then, $v(\varphi \to \psi)$ is 0. Thus, $\not\models_{\mathcal{A}} \varphi \to \psi$. Therefore, by soundness, $\nvdash_{\mathbf{RMAILML}} \varphi \to \psi$, as required.

(ii) This proof is analogous to that of Theorem 2. □

Corollary 7 *L^- is a relevance logic (in the strong or weak sense of [1], [7]).*

Remark 3 As is known, the relevance logics **RW**, **R**, **RM** and fuzzy logics such as **UL** and **BL** are all *substructural logics* extending $\mathbf{FL}_e$. Since the system **RMAILL** is the $\mathbf{FL}_e$ with (EM), the (fuzzy-)relevance systems introduced in Definitions 5 and 11 are all substructural logics extending $\mathbf{FL}_e$. Thus, the system **RMAILL** may instead be expressed as $\mathbf{FL}_e^{em}$ (in the substructural logic tradition) and analogously for the other systems. Note that, in the literature of substructural logic, $\mathbf{FL}_{ec}$ is already introduced as $\mathbf{FL}_e$ plus contraction and **RW** and **R** as $\mathbf{InDFL}_e$ ($= \mathbf{FL}_e$ plus distributivity and involution) and $\mathbf{InDFL}_{ec}$, respectively (see [11]). Thus, the introduction of $\mathbf{FL}_e^{em}$ (including $\mathbf{FL}_e$) and its extensions in Definition 11 is a step in the natural evolution of relevance logic (in particular associated with fuzzy logic) in the search for weaker systems.

3 Propositional Constants and New Relevance Principles

This section summarizes our work in [22]. More exactly, we briefly recall new strong and weak relevance principles introduced in [22], that is, NSRP and NWRP, because they are unfamiliar to the readers. (As we mentioned in footnote 2, we assumed that formulas such as $(\varphi \wedge \mathbf{t}) \to (\mathbf{t} \vee \psi)$ satisfy SRP in L ($\in$ Ls) because its antecedent and consequent implicitly share at least one propositional variable. The introduction of NSRP and NWRP will show the reason for this assumption. However, the principles NSRP and NWRP were introduced without such an assumption (see [22]).) Before introducing these principles, we consider intensional features of propositional constants **t**, **f**, **T**, **F** and some (defective) weak versions of the new principles.

Many logical systems with propositional constants are equivalent to those without propositional constants because, in the latter systems, propositional constants are definable. For instance, the constants **F** and **T** can be defined as $\varphi \wedge \neg\varphi$ and $\varphi \vee \neg\varphi$, respectively, in classical logic (CL). However, such equivalences are not valid in the systems **R** and $\mathbf{R}^{\mathbf{T}}$ (**R** plus constants **T**, **F**, and axioms A12, A13 below) because we cannot define the propositional constants **t**, **f** or **T**, **F**, respectively, using propositional language from which they are excluded.

One interesting fact to mention is that, in the literature of relevance logic (see, e.g., [2], [8]), the constant **t** is interpreted as the conjunction of all true sentences, and similarly for the others. More exactly, the propositional constants in **R** and $\mathbf{R}^{\mathbf{T}}$ can be interpreted as follows (see Anderson and Belnap [2, Section 27.1.2] and Dunn [8, Section 1.3]):[8]

df3. **t** = the conjunction of all true sentences;
df4. **f** = the disjunction of all false sentences;
df5. **T** = the disjunction of all sentences;
df6. **F** = the conjunction of all sentences.

Here, df3 and df4 are semantic interpretations. The corresponding syntactic interpretations can be provided as follows:[9]

df3′. $\mathbf{t}$ = the conjunction of all theorems;
df4′. $\mathbf{f}$ = the negation of $\mathbf{t}$.

These are not *object*-definitions, but *meta*definitions, which cannot be provided by means of the object-language for the $\mathbf{t}$-free fragment of $\mathbf{R}$ (calling it here $\boldsymbol{R}^{\mathbf{0}}$). That is, the propositional constants $\mathbf{t}$, $\mathbf{f}$, $\mathbf{T}$, $\mathbf{F}$ are not object-definable in $\mathbf{R}^{\mathbf{0}}$; $\mathbf{T}$, $\mathbf{F}$ are not in $\mathbf{R}$. Thus, we cannot eliminate those constants in $\mathbf{R}$ or $\mathbf{R}^{\mathbf{T}}$. (We use the word *object-definition* for a clear distinction from *metadefinition*.) Although df3 to df6, df3′, and df4′ are not object-definitions of propositional constants, they cannot be defined as we please because the definitions are based on (syntactic and semantical) roles of the constants in the proof of soundness and completeness for a relevance logic. In fact, the above interpretations are given according to such roles of the constants in $\mathbf{R}$ and $\mathbf{R}^{\mathbf{T}}$. (Note that, as mentioned in [22], the propositional constants $\mathbf{t}$, $\mathbf{T}$, $\mathbf{f}$, $\mathbf{F}$ correspond to the least true element t, the greatest (true) element $\top$, and their negations, i.e., the greatest false element f, and the least (false) element $\bot$, respectively, in algebraic semantics; see [2], [7], [8].)

The relevance principle is a metacriterion or metaprinciple, by virtue of which we can examine whether a system is a relevance logic. Similarly, metadefinitions or interpretations can be used to assess variable sharing between the antecedent and consequent of an implication (as a theorem), as far as they are given as above (i.e., according to syntactic and semantical roles of the constants). Let metadefinitions of propositional constants be given like this, that is, based on the above roles of constants; let the antecedent and consequent φ, ψ of a statement $\varphi \to \psi$ *implicitly* share a propositional variable by virtue of the metadefinitions. For example, in $\mathbf{R}$, the antecedent $p \wedge \mathbf{t}$ and the consequent $\mathbf{t} \vee q$ of the theorem $(p \wedge \mathbf{t}) \to (\mathbf{t} \vee q)$ implicitly share (the conjunction of) "all true sentences (or theorems)" by virtue of df3 (or df3′).

Before introducing new relevance principles, we introduce their weak versions and related facts in order to help the readers better understand them.

Definition 12 ([22, Definition 6])

(i) (The implicit strong relevance principle, ISRP) $\varphi \to \psi$ is a theorem only if φ and ψ implicitly share a propositional variable where the word *implicitly* means that we can identify a sharing variable by means of metadefinitions or interpretations such as df3 to df6, df3′, and df4′.

(ii) (The implicit weak relevance principle, IWRP) $\varphi \to \psi$ is a theorem only if (a) φ and ψ implicitly share a propositional variable or (b) both $\neg\varphi$ and ψ are theorems.

Let $\varphi \to \psi$ *satisfy* the relevance principle ISRP (resp., IWRP) in a logic $\mathbf{L}$ if it is a theorem of $\mathbf{L}$ and its antecedent φ and consequent ψ implicitly share a propositional variable (or both the negation of its antecedent and its consequent are theorems). Then we can prove the following.

Proposition 4 (see [22, Proposition 4])

(i) $(\varphi \wedge \boldsymbol{t}) \to (\boldsymbol{t} \vee \psi)$ *satisfies ISRP in* $\boldsymbol{R}$.

(ii) $((\varphi \to \boldsymbol{F})\,\&\,\varphi) \to \psi$ *satisfies ISRP in* $\boldsymbol{R}^{\boldsymbol{T}}$ *and* $\boldsymbol{UL}$.

(iii) $((\varphi \to \boldsymbol{F}) \wedge \varphi) \to \psi$ *satisfies ISRP in* $\boldsymbol{R}^{\boldsymbol{T}}$.

(iv) $((\varphi \to \varphi) \to \boldsymbol{f}) \to (\psi \to \psi)$ *satisfies ISRP in* ***IUML***.
(v) $(\varphi \wedge \neg\varphi) \to (\psi \vee \neg\psi)$ *satisfies IWRP in* ***RM***.
(vi) $\neg(\varphi \to \varphi) \to (\psi \to \psi)$ *satisfies IWRP in* ***RM*** *and* $\boldsymbol{RM^T}$.

Proof We can easily prove (i) to (v) using df3 (or df3′), df4 (or df4′), and df6. Then, (vi) follows from Proposition 2(vii)(5), Proposition 5(iv)(1) and the second condition of IWRP. □

The principles ISRP and IWRP, however, do not prevent us from giving metadefinitions of propositional constants to the systems having their object-definitions. (Note that the constants **T** and **F** can still be interpreted as df5 and df6, respectively, in CL.) For instance, if CL has not only the object-definitions (df7) $\mathbf{T} := \varphi \vee \neg\varphi$, (df8) $\mathbf{F} := \varphi \wedge \neg\varphi$, but also the metadefinitions df5, df6, then the statement $(p \wedge \neg p) \to q$ satisfies ISRP and IWRP in CL since the sentences **T**, $\mathbf{F} \to \mathbf{F}$, and $\mathbf{T} \leftrightarrow (\mathbf{F} \to \mathbf{F}) \leftrightarrow ((p \wedge \neg p) \to q)$ are all theorems of CL. (We generally regard the statement $(p \wedge \neg p) \to q$ as an example of paradoxes of material implication, and many irrelevant logics such as CL, intuitionistic logic, etc., can allow (2), from which an irrelevance between the antecedent and consequent of an implication arises.)

Note that a propositional constant in a logic **L** is said to be *object-definable* in that it can be defined by means of the object-language for **L** and *metadefinable* in that it can be defined by virtue of metadefinitions such as df3 to df6, df3′, and df4′. Let a propositional constant be *strongly metadefinable* in a logic **L** if it is metadefinable but not object-definable in **L**, for example, the constants **t** and **f** in **R** and **T** and **F** in $\mathbf{R^T}$; let the antecedent and consequent of an implication *strong implicitly* share a propositional variable if we can establish variable sharing between them by virtue of strong metadefinitions.

The new strong and weak relevance principles introduced in [22] are defined as follows.[10]

Definition 13 ([22, Definition 8])

(i) (The new strong relevance principle, NSRP) $\varphi \to \psi$ is a theorem only if φ and ψ either explicitly or strong implicitly share a propositional variable.
(ii) (The new weak relevance principle, NWRP) $\varphi \to \psi$ is a theorem only if (i) φ and ψ either explicitly or strong implicitly share a propositional variable, or (ii) both $\neg\varphi$ and ψ are theorems.

The principles NSRP and NWRP prevent logics with both object-definable and metadefinable propositional constants from satisfying the relevance principles. For instance, CL satisfies neither NSRP nor NWRP since, in CL, the constants **T** and **F** can be defined not merely by df7 and df8, but by df5 and df6.

4 Fuzzy-Relevance Logics (II)

4.1 Fuzzy-relevance logics with constants T, F In this section, we introduce several substructural fuzzy-relevance systems satisfying new relevance principles, that is, NSRP and NWRP. First, we provide axiomatizations of the L with constants **T**, **F**.

Definition 14

(i) **UL** is **RUL** minus (EM) plus constants **F**, **T**, and
A12. $\mathbf{F} \to \varphi$ (*ex falso quodlibet*, EF)
A13. $\varphi \to \mathbf{T}$ (*verum ex quodibet*, VE)

(ii) **IUL** (= $\mathbf{FRW^T}$) is **UL** plus (DNE).
(iii) $\mathbf{FR^T}$ is **IUL** plus (SIN).
(iv) **UML** is **UL** plus (ID).
(v) $\mathbf{RM^T}$ is **UML** plus (DNE).
(vi) **IUML** is $\mathbf{RM^T}$ plus (FP).

Definition 15 $\mathrm{Ls^T} = \{\mathbf{UL}, \mathbf{IUL}\ (= \mathbf{FRW^T}), \mathbf{FR^T}, \mathbf{UML}, \mathbf{RM^T}, \mathbf{IUML}\}$.

An easy computation shows the following.

Proposition 5

(i) *L* ($\in$ *Ls*) *proves*
  (1) $(\varphi \wedge \boldsymbol{t}) \to (\boldsymbol{t} \vee \psi)$.
(ii) L^T ($\in Ls^T$) *proves*
  (1) $(\varphi \wedge \neg\varphi) \to (\psi \vee \neg\psi)$;
  (2) $((\varphi \to \boldsymbol{F})\ \&\ \varphi) \to \psi$.
(iii) $\boldsymbol{FR^T}$, ***UML***, $\boldsymbol{RM^T}$, *and* ***IUML*** *each prove*
  (1) $((\varphi \to \boldsymbol{F}) \wedge \varphi) \to \psi$.
(iv) $\boldsymbol{RM^T}$ *and* ***IUML*** *each prove*
  (1) $\neg(\varphi \to \varphi) \to (\psi \to \psi)$.

First, note that, using the standard technique, we can provide algebraic completeness results for $\mathrm{L^T}$ ($\in \mathrm{Ls^T}$). For this, it suffices to note that $\mathrm{L^T}$-algebras are obtained as varieties of pointed *bounded* commutative residuated lattices; that is, for $\mathrm{L^T}$-algebras, it suffices to replace the condition (I) in Definition 6 with

(I′) $(A, \top, \bot, \wedge, \vee)$ is a *bounded* lattice with top element $\top$ and bottom element $\bot$.

Since the condition (I′) can be defined in equations, it is clear that the class of $\mathrm{L^T}$-algebras forms a variety. Then, as in [16], we can show that $\mathrm{L^T}$ is complete w.r.t. an algebraic semantic given by a variety of $\mathrm{L^T}$-algebras. Moreover, as in Section 2.2, we can prove the following.

Theorem 4 *Let* $\mathbf{L}$ *be a multiplicative additive intuitionistic linear logic. Then* $\mathbf{L}$ *is a fuzzy logic iff, for each n,* $\vdash_L (\varphi \to \psi)^n_t \vee (\psi \to \varphi)^n_t$.

Proof See Theorem 1. □

Corollary 8 (Strong completeness) *Let T be a theory over* L^T ($\in Ls^T$)*, and let* φ *be a formula. Then* $T \vdash_{L^T} \varphi$ *iff* $T \models^l_{L^T} \varphi$.

Corollary 9 L^T *is a fuzzy logic (in Cintula's sense).*

For the relevance of $\mathrm{L^T}$, note that the constants **T**, **F**, **t**, **f** are not object-definable in $\mathrm{L^T}$ ($\in \mathrm{Ls^T}$) without the constants. Note also that, in $\mathrm{L^T}$, the constant **t** can instead be interpreted as df3′ (or df3) and the constant **f** as its negation (see df2 and Proposition 2(i)(12)); and the constants **T** and **F** as df5 and df6, respectively (see A12 and A13 in Definition 14). Then, we can show the following.

Proposition 6

(i) $(\varphi \wedge \boldsymbol{t}) \to (\boldsymbol{t} \vee \psi)$ *satisfies NSRP in L* ($\in Ls$) *and* L^T ($\in Ls^T$).
(ii) $(\varphi \wedge (\varphi \to \boldsymbol{f})) \to (\psi \vee (\psi \to \boldsymbol{f}))$ *satisfies NSRP in* L^T ($\in Ls^T$).
(iii) $((\varphi \to \boldsymbol{F})\ \&\ \varphi) \to \psi$ *satisfy NSRP in* L^T ($\in Ls^T$).

(iv) $((\varphi \to \boldsymbol{F}) \wedge \varphi) \to \psi$ *satisfy NSRP in* $L^{\boldsymbol{T}} \in \{\boldsymbol{FR^T}, \boldsymbol{UML}, \boldsymbol{RM^T}, \boldsymbol{IUML}\}$.
(v) $\neg(\varphi \to \varphi) \to (\psi \to \psi)$ *satisfies NSRP in* $\boldsymbol{IUML}$.
(vi) $\neg(\varphi \to \varphi) \to (\psi \to \psi)$ *satisfies NWRP in* $\boldsymbol{RM^T}$.

Proof See [22, Proposition 5]. □

Proposition 6 shows that theorems strong implicitly sharing variable(s) can satisfy NSRP. Thus, we verify the relevance of $\mathrm{L}^{\mathbf{T}}$ ($\in \mathrm{Ls}^{\mathbf{T}}$) as follows.

Theorem 5

(i) *For* $L^{\boldsymbol{T}} \in Ls^{\boldsymbol{T}} \setminus \{\boldsymbol{RM^T}\}$, $L^{\boldsymbol{T}}$ *satisfies NSRP.*
(ii) $\boldsymbol{RM^T}$ *satisfies NWRP.*

Proof (i) We can prove this by using the matrices considered in Theorem 2 w.r.t **UML** and **IUML**. Let $\mathrm{L} \in \{\mathbf{UL}, \mathbf{IUL}, \mathbf{FR^T}\}$. Contrapositively, assume that φ and ψ share neither propositional variables nor propositional constants and show that $\nvdash_L \varphi \to \psi$. For this, consider an algebra $\mathcal{A} = (\{1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0\}, 1, 0, \frac{1}{3}, \frac{2}{3}, \min, \max, *, \to)$, where: Letting $\neg x$ (*i.e.*, $x \to \frac{2}{3}$) $= 1 - x$,

$\to$	1^+	$\frac{5}{6}^+$	$\frac{2}{3}^+$	$\frac{1}{2}^+$	$\frac{1}{3}^+$	$\frac{1}{6}$	0
1^+	1	0	0	0	0	0	0
$\frac{5}{6}^+$	1	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0
$\frac{2}{3}^+$	1	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0
$\frac{1}{2}^+$	1	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	0
$\frac{1}{3}^+$	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0
$\frac{1}{6}$	1	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	0
0	1	1	1	1	1	1	1

$*$	1^+	$\frac{5}{6}^+$	$\frac{2}{3}^+$	$\frac{1}{2}^+$	$\frac{1}{3}^+$	$\frac{1}{6}$	0
1^+	1	1	1	1	1	1	0
$\frac{5}{6}^+$	1	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{1}{6}$	0
$\frac{2}{3}^+$	1	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0
$\frac{1}{2}^+$	1	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	0
$\frac{1}{3}^+$	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0
$\frac{1}{6}$	1	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0
0	0	0	0	0	0	0	0

In each matrix, for $\to$ and $*$, the superscript $+$ indicates designated values. The axioms of L are $\mathcal{A}$-tautologies for all assignments of values to the variables, and the rules preserve this property. This ensures soundness; that is, if $\vdash_L \varphi$, then $\models_{\mathcal{A}} \varphi$. Let φ and ψ share neither propositional variables nor propositional constants. If $\vdash_L \psi$, assign the values yielding $v(\varphi) > 1/3$ to all the variables of φ and the values yielding $v(\psi) = 1/3$ to all the variables of ψ. Otherwise, that is, if $\nvdash_L \psi$, assign the values yielding $v(\varphi) = 1/3$ to all the variables of φ and the values yielding $v(\psi) < 1/3$ to all the variables of ψ. Then, in each case, $v(\varphi \to \psi)$ is $1/6$ or 0. Thus, $\not\models_{\mathcal{A}} \varphi \to \psi$. Therefore, by soundness, $\nvdash_L \varphi \to \psi$, as required.

(ii) This proof is analogous to that of Theorem 2. □

Corollary 10

(i) *For $L^T \in Ls^T \setminus \{\boldsymbol{RM^T}\}$, L^T is a relevance logic (in the new strong sense of [22]).*
(ii) *$\boldsymbol{RM^T}$ is a relevance logic (in the new weak sense of [22]).*

Then, from Corollaries 9 and 10, the next corollary directly follows.

Corollary 11 *L^T is both a fuzzy logic and a relevance logic.*

4.2 Strong relevant companions The system **IUML** is $\mathbf{RM^T}$ plus (FP). Here, we note that, while **IUML** satisfies NSRP, $\mathbf{RM^T}$ does not. This fact gives an insight into a method to obtain relevant companions of weakening-free uninorm (based) logics. We introduce here a way to obtain *strong* relevant companions of weakening-free logics extending **UL**. Let L be the system **UL**.

Definition 16 We introduce several extensions of L as follows.

- Involutive L IL is L plus (DNE) $\neg\neg\varphi \to \varphi$.
- Square increasing L L_c is L plus (SIN) $\varphi \to (\varphi \,\&\, \varphi)$.
- Involutive L_c IL_c is L_c plus (DNE).
- Square decreasing L L_p is L plus (SDE) $(\varphi \,\&\, \varphi) \to \varphi$.
- Involutive L_p IL_p is L_p plus (DNE).
- Idempotent L L_{cp} is L_c plus (SDE).
- Involutive L_{cp} IL_{cp} is L_{cp} plus (DNE).

Tables 3 and 4 summarize some axiom schemes and the extensions of **UL** introduced in Definition 16.

Definition 17

(i) $\mathrm{ELs} = \{\mathrm{L}, \mathrm{IL}, \mathrm{L}_c, \mathrm{IL}_c, \mathrm{L}_p, \mathrm{IL}_p, \mathrm{L}_{cp}, \mathrm{IL}_{cp}\}$.
(ii) For EL ($\in$ ELs), EL^f is EL plus (FP).
(iii) $\mathrm{ELs}^f = \{\mathrm{L}^f, \mathrm{IL}^f, \mathrm{L}_c^f, \mathrm{IL}_c^f, \mathrm{L}_p^f, \mathrm{IL}_p^f, \mathrm{L}_{cp}^f, \mathrm{IL}_{cp}^f\}$.

Table 3 Some axiom schemes in fuzzy-relevance logics.

Axiom schema	Name
$\neg\neg A \to A$	Double negation elimination (DNE)
$A \to A \,\&\, A$	Square increasing (SIN)
$A \,\&\, A \to A$	Square decreasing (SDE)
$A \,\&\, A \leftrightarrow A$	Idempotence (ID)

Table 4 Some extensions of L (= **UL**) obtained by adding the corresponding additional axiom schemes.

Logic	Additional axiom schemes
IL	(DNE)
L_c	(SIN)
IL_c	(DNE) and (SIN)
L_p	(SDE)
IL_p	(DNE) and (SDE)
L_{cp}	(ID) (= (SIN) + (SDE))
IL_{cp}	(DNE) and (ID)

Proposition 7 *$((\varphi \to \varphi) \to \boldsymbol{f}) \to (\psi \to \psi)$ satisfies NSRP in each of L^f_p, IL^f_p, L^f_{cp}, and IL^f_{cp}.*

Proof The proof is immediate since (FP) $\mathbf{t} \leftrightarrow \mathbf{f}$ is an axiom in each system, and so the antecedent and consequent of $((\varphi \to \varphi) \to \mathbf{f}) \to (\psi \to \psi)$ implicitly share the statement $\psi \to \psi$. □

Theorem 6 *For $EL^f \in ELs^f$, EL^f satisfies NSRP.*

Furthermore, as in Section 2, we can prove the completeness of EL^f.

Theorem 7 (Strong completeness) *Let T be a theory over EL^f ($\in ELs^f$), and let φ be a formula. Then $T \vdash_{EL^f} \varphi$ iff $T \models^l_{EL^f} \varphi$.*

From Theorems 6 and 7, we obtain the following corollary.

Corollary 12 *EL^f is both a fuzzy logic (in Cintula's sense) and a relevance logic (in the new strong sense of [22]).*

4.3 Substructural relevance logics (II) Let **MAILL**, **MALL** ($=$ $\mathbf{LRW^T}$), $\mathbf{LR^T}$, **MAILML**, $\mathbf{LRM^T}$, and **MALML** be the systems eliminating ($\mathrm{PL}_\mathbf{t}$) from the systems **UL**, **IUL** ($=$ $\mathbf{FRW^T}$), $\mathbf{FR^T}$, **UML**, $\mathbf{RM^T}$, and **IUML**, respectively. We let $\mathrm{Ls}^{\mathbf{T}-}$ be the set of these systems; that is, we have the following.

Definition 18 $\mathrm{Ls}^{\mathbf{T}-} = \{\mathbf{MAILL}, \mathbf{MALL}, \mathbf{LR^T}, \mathbf{MAILML}, \mathbf{LRM^T}, \mathbf{MALML}\}$.

Since $\mathrm{L}^{\mathbf{T}-}$ ($\in \mathrm{Ls}^{\mathbf{T}-}$) is a WIL, Theorem 4 shows that $\mathrm{L}^{\mathbf{T}}$ ($\in \mathrm{Ls}^{\mathbf{T}}$) is a WIFL (see Corollary 9); moreover, $\mathrm{L}^{\mathbf{T}}$ is the *weakest* fuzzy logic extending $\mathrm{L}^{\mathbf{T}-}$. More exactly, [5, Theorem 1] says that, for a WIL **L**, $\mathrm{T} \vdash \varphi$ iff $\mathrm{T} \models \varphi$, and so we obtain the following corollary.

Corollary 13 (Strong completeness) *For each theory T over L^T ($\in Ls^{T-}$) and formula φ, $T \vdash_{L^{T-}} \varphi$ iff $T \models_{L^{T-}} \varphi$.*

Let us verify the relevance of $\mathrm{L}^{\mathbf{T}-}$ ($\in \mathrm{Ls}^{\mathbf{T}-}$). However, we first note that an easy computation shows the following.

Proposition 8 *$\boldsymbol{LRM^T}$ and **MALML** each prove $\neg(\varphi \to \varphi) \to (\psi \to \psi)$, that is, Proposition 2(vii)(5).*

Corollary 14 *$\boldsymbol{LRM^T}$ does not satisfy NSRP (in [22]).*

Furthermore, as in Section 4.1, we can show the following.

Theorem 8

(i) *For $L^{T-} \in \{\boldsymbol{MAILL}, \boldsymbol{MALL}, \boldsymbol{LR^T}, \boldsymbol{MAILML}, \boldsymbol{MALML}\}$, L^- satisfies NSRP (in [22]).*

(ii) *$\boldsymbol{LRM^T}$ satisfies NWRP (in [22]).*

Proof

(i) The proof is immediate since $\mathrm{L}^{\mathbf{T}-}$ also satisfies the matrices considered in [2] in order to show that the system $\mathbf{R^0}$ satisfies SRP (w.r.t. **MAILL**, **MALL**, and $\mathbf{LR^T}$) or the matrices considered in Theorem 2 (w.r.t. **MAILML** and **MALML**).

(ii) This proof is analogous to that of Theorem 2. □

Corollary 15 *L^{T-} is a relevance logic (in the new strong or weak sense of [22]).*

Remark 4 Since the system **MAILL** is the $\mathbf{FL}_{e\perp}$, the (fuzzy-)relevance systems introduced in Definitions 15 and 18 are all substructural logics extending $\mathbf{FL}_{e\perp}$. Thus, the introduction of $\mathbf{FL}_{e\perp}$ and its extensions in Definition 18 is also a step in the natural evolution of relevance logic (in particular associated with fuzzy logic) in the search for weaker systems.

5 Concluding Remarks

We introduced several fuzzy-relevance logics with and without constants **T**, **F** and provided completeness results for them by showing that such logics are WIFLs. We furthermore proved that they satisfy old and new relevance principles. In addition, we considered relevance logics obtained from the fuzzy-relevance logics by omitting prelinearity. All of the systems investigated here are extensions of the substructural logic $\mathbf{FL}_e$, and so they are all substructural logics. They also have the associative intensional conjunction (so-called *fusion*) &. Therefore, such systems all can be called *associative (fuzzy-)relevance logics*.

The fuzzy-relevance logics without constants **T** and **F** are not characterized by models based on uninorms. Note that the uninorm-based systems introduced in [16] have constants **T** and **F**, and the systems with **T** and **F** investigated here are not relevant in the old senses. This implies that, as far as *uninorm* (based) systems have **T** and **F**, they cannot be relevant in the old senses and so are not fuzzy-relevance logics in the old senses.

Notes

1. For the introduction of propositional constants **t**, **f**, **T**, and **F**, see Sections 2 and 4. In particular, in order to interpret these constants in relevance logic, see Definitions 3–6 in Section 3.

2. While Sugihara matrices (as a semantic for **RM**) need not have a fixed point, such matrices on $[0, 1]$ (as a semantic for **IUML**) have such a point, for example, $1/2$ in the standard involutive negation $1 - x$ so that the logic **IUML** requires the corresponding axiom (FP) (see [7], [16]). Note that Sugihara matrices with an odd number of elements introduced in [7] have a fixed point corresponding to (FP).

3. Here, we regard $\mathbf{R}^{\mathbf{t}}$ (the **R** with the constant **t**) as **R**. Often in the literature of relevance logic, **R** is used for the **t**-free fragment of $\mathbf{R}^{\mathbf{t}}$. One reason for that is that $\mathbf{R}^{\mathbf{t}}$ proves formulas such as (γ) $(\varphi \wedge \mathbf{t}) \rightarrow (\mathbf{t} \vee \psi)$ and so seems not to satisfy the old relevance principles (see [11]). However, we have to mention that, in the literature of relevance logic (e.g., Anderson and Belnap [2]), the constant **t** is interpreted as the conjunction of all true sentences. Thus, (γ) does implicitly satisfy SRP, and so the relevance principles in a sense do not fail in $\mathbf{R}^{\mathbf{t}}$. Hence, here we assume that such formulas satisfy SRP. We will, in Section 3, introduce NSRP and NWRP as principles allowing implicit variable sharing.

4. This is not a necessity, however, as some logics, for example, **psBL** (pseudo-**BL**) and **psMTL** (pseudo-**MTL**), are not (see Hájek [13], [14]).

5. The systems **RMAILL** and **MAILL** are the $\mathbf{FL}_e$ with (EM) and $\mathbf{FL}_{e\perp}$, respectively (see [11]). While **RMAILL** does not prove formulas such as (δ) $((\varphi \to \mathbf{F})\,\&\,\varphi) \to \psi$, **MAILL** does. Thus, the system **MAILL** is neither strongly nor weakly relevant in the old senses. Here, we introduce **RMAILL** as a relevant companion of **MAILL**. Note that the system $\mathbf{FL}_e$ does not prove (δ) and so is strongly relevant in the old sense. Then, we may establish $\mathbf{FL}_e$ as the basic relevance logic. However, if this is so, **RUL** must require not merely ($\mathrm{PL}_\mathbf{t}$), but the additional axiom (EM), which is irrelevant to *fuzziness*. Here, we want to use **RUL** as the *weakest* fuzzy logic obtained from some weakly implicative logic by adding ($\mathrm{PL}_\mathbf{t}$) (but no more).

6. Roughly speaking, in **IUML**, **t** and **f** both correspond to a fixed point, for example, the element $1/2$ in the real unit interval $[0, 1]$, because the involutive negation $\neg x$ may be defined as $1 - x$ (see [16, Proposition 19]). Since $\mathbf{t} = \neg\mathbf{f}$ in involutive uninorm logics, the sentence $\mathbf{t} \leftrightarrow \mathbf{f} \leftrightarrow \neg\mathbf{f}$ is a theorem in **RIUML**.

7. A lattice does not have to have top and bottom elements $\top$ and $\bot$, and so t and f need not be the same as $\top$ and $\bot$, respectively, in pointed commutative residuated lattices. Note that lattices having $\top$ and $\bot$ are called *bounded* lattices (see (I′) in Section 4).

8. Here, "true sentences" correspond to t-tautologies in $\mathcal{A}$, that is, $\mathcal{A}$-tautologies, as in Section 2.2. Note that, if propositional quantification is possible, each constant can be defined as follows: $\mathbf{t} := (\forall p)(p \to p)$, $\mathbf{f} := (\exists p)\neg(p \to p)$, $\mathbf{T} := (\exists p)p$, and $\mathbf{F} := (\forall p)p$ (see Anderson, Belnap, and Dunn [3] and Beal and Restall [4]).

9. The interpretation df4′ is given, for example, in [8, p. 131]. The constant **t** is generally interpreted as in endnote 8 (see Meyer [17, p. 173]). We use df3′ here because it corresponds more exactly to df3 when we consider the soundness and completeness of **R**.

10. Here, the phrase "φ and ψ *explicitly* share a propositional variable" is the same as "φ and ψ share a propositional variable" in the SRP and WRP in [1] and [7] (see [22]).

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