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Internal Categoricity in Arithmetic and Set Theory

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Abstract We show that the categoricity of second-order Peano axioms can be proved from the comprehension axioms. We also show that the categoricity of second-order Zermelo–Fraenkel axioms, given the order type of the ordinals, can be proved from the comprehension axioms. Thus these well-known categoricity results do not need the so-called "full" second-order logic, the Henkin second-order logic is enough. We also address the question of "consistency" of these axiom systems in the second-order sense, that is, the question of existence of models for these systems. In both cases we give a consistency proof, but naturally we have to assume more than the mere comprehension axioms.

1 The Set-Theory View and the Second-Order View

Second-order logic was originally considered an innocuous variant of first-order logic. Gödel's completeness and incompleteness theorems revealed that the analogy with first-order logic does not do full justice to the character of second-order logic. Second-order logic truly transcends first-order logic in terms of strength and can be more appropriately compared with (first-order) set theory.

There is a debate between the "set-theory view" and the "second-order view" in the foundations of mathematics (see, e.g., Väänänen [5]). The set-theory view holds that mathematics is best formalized using first-order set theory. The second-order view holds that mathematics is best formalized in second-order logic.

Two important issues in this debate are completeness and categoricity. A clear merit of the set-theory view is that first-order logic has a complete proof calculus, while second-order logic does not. An equally clear merit of the second-order view is that second-order theories of classical structures (e.g., $(\mathbb{N}, +, \cdot, 0, 1)$, $(\mathbb{R}, +, \cdot, 0, 1)$) are categorical, while the corresponding first-order theories allow nonstandard models. More precisely, for classical structures $\mathfrak{A}$ there is a second-order sentence $\theta_{\mathfrak{A}}$

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that characterizes $\mathfrak{A}$ uniquely in the following way:

$$\mathfrak{A} \models \theta_{\mathfrak{A}}, \tag{1}$$

$$\forall \mathfrak{B} \forall \mathfrak{C} \big((\mathfrak{B} \models \theta_{\mathfrak{A}} \wedge \mathfrak{C} \models \theta_{\mathfrak{A}}) \to \mathfrak{B} \cong \mathfrak{C} \big). \tag{2}$$

However, a closer inspection shows that the matter is more subtle than a simple trade-off between completeness and categoricity. First of all, the claim that the second-order view is inferior to the set-theory view because it lacks completeness is unwarranted. It is true that *full* second-order logic does not have a complete proof calculus, but for many reasons it is more appropriate to use *Henkin second-order logic* instead of full second-order logic in a foundational quest (see [5]). Henkin second-order logic, introduced in Henkin [3], is the extension of the usual logical axioms with the *comprehension axioms*:

CA: $\exists X \forall \vec{x} (X(\vec{x}) \leftrightarrow \varphi(\vec{x}))$ for any second-order formula φ not containing X free.

A model of Henkin second-order logic, a *Henkin model*, consists of a pair $(\mathfrak{A}, \mathcal{G})$, where $\mathfrak{A}$ is a model in the usual first-order sense and $\mathcal{G}$ is a collection of relations, functions, and subsets of A. The set $\mathcal{G}$ serves as the range of the second-order variables in $(\mathfrak{A}, \mathcal{G})$. The schema CA asserts that all definable subsets of the model in question are in $\mathcal{G}$. Note that every full model $(\mathfrak{M}, \mathcal{G})$ (i.e., model of full second-order logic) is a Henkin model. We denote full Henkin models by $(\mathfrak{M}, \mathcal{G}_{\mathrm{full}})$. Therefore Henkin second-order logic adds to full second-order logic the comprehension axiom, and at the same time its semantics permits a broader class of models. Importantly, Henkin second-order logic is complete with respect to this extended class of models. Both the set-theory view and the second-order logic view have a complete underlying logic, and they characterize classical structures to the same level of categoricity (see [5]).

The aim of this paper is to synthesize completeness and categoricity in the second-order view. We work within the framework of Henkin second-order logic. We want to restore the idea that second-order logic should provide unique characterizations of classical structures. We want something like (1) and (2) to be still true.

Our first innovation is the notion of *internal categoricity*. Internal categoricity is a generalization of the notion of categoricity, and was proposed in [5]. We say that a theory T is *internally categorical* if all models of T within a common Henkin model are witnessed to be isomorphic by the model. We will make this definition more intelligible through examples in what follows. For a detailed account of internal categoricity, and the motivation behind it, see [5].

In this paper we prove that second-order theories of arithmetic and set theory are internally categorical, although they are not categorical in the classical sense. This fact suggests that nonstandard models and categoricity can exist in harmony. This restores (2).

On the other hand there is the question of consistency, that is, existence of models. There is a marked difference between the second-order consistency (i.e., existence of a model) and the first-order consistency (i.e., nonexistence of a proof of contradiction). By Gödel's completeness theorem, the nonexistence of a proof of contradiction is equivalent to the existence of a model. In second-order set theory this means that the *formalized* second-order theory (e.g., arithmetic) does not permit a proof of contradiction if and only if there is a set—in the sense of (second-order) set theory—that is a model of the theory. Having relations that satisfy the axioms of arithmetic is not enough for a construction of a set which is a model of arithmetic. But this is not our concern here. We ask whether there are relations which satisfy the axioms of a given theory, for example, arithmetic.

If we take second-order logic as a foundation, then the status of (1) is not clear at first glance. What exactly does it mean that $\mathfrak{A} \models \theta_{\mathfrak{A}}$? It is tempting to say that the meaning of $\mathfrak{A} \models \theta_{\mathfrak{A}}$ is given by Tarski's truth definition. However, Tarski's truth definition presupposes that we can read off the truth value of $\mathfrak{A} \models \theta_{\mathfrak{A}}$ in the metatheory—in this case set theory. It would undermine the second-order view if the meaning of such basic notions relies on set theory. The second task of this paper is thus to prove the existence of classical structures based on more logical grounds.

Suppose that $\theta(R_1, \ldots, R_n)$ is a potential second-order characterization of some structure $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \ldots, R_n^{\mathfrak{A}})$. If we can prove under certain assumptions Γ that there exists a model of θ, and that θ is internally categorical, then we have, at least to some extent, restored (1) and (2):

$$\mathrm{CA} + \Gamma \vdash \exists R_1, \ldots, R_n \theta(R_1, \ldots, R_n), \tag{3}$$

$$\mathrm{CA} \vdash \forall R_1, \ldots, R_n \forall R_1, \ldots, R_n \big(\big(\theta(R_1, \ldots, R_n) \wedge \theta(R_1', \ldots, R_n')\big) \to \mathfrak{A} \cong \mathfrak{A}' \big). \tag{4}$$

In this paper we work out two prime examples of this scheme: arithmetic and set theory. We prove (3) and (4) for second-order arithmetic P^2 and second-order set theory ZFC^2, respectively. In Section 2 we prove internal categoricity of P^2. In Section 3 we prove, under the assumption that the underlying domain is infinite, that there is a model of P^2. In Section 4 we prove internal categoricity of ZFC^2. In Section 5, the most extensive part of this paper, we prove, under certain large domain assumptions, that there exists a model of ZFC^2.

2 Internal Categoricity of Arithmetic

The axiom system P^2 is the second-order version of Peano arithmetic (see Dedekind [2]). For the purpose of this paper, we consider the relativized version of P^2. Let $L = (N, S, 0)$ be the language of arithmetic. Intuitively, N denotes the underlying domain. The axioms of P^2 consist of

P0: $\forall x(x \in N \to S(x) \in N)$,
P1: $\forall x \in N \neg S(x) = 0$,
P2: $\forall x \in N \forall y \in N(S(x) = S(y) \to x = y)$,
P3: $\forall X((X(0) \wedge \forall x \in N(X(x)) \to X(S(x)) \to \forall x(N(x) \to X(x))$.

It is well known that P^2 characterizes $\mathbb{N}$ up to isomorphism in full second-order logic. For Henkin second-order logic, this is no longer the case. A counterexample can be provided by an application of the completeness theorem. Expand the language with a new constant symbol c, and let Σ be the theory $P^2 \cup \{c > S^n 0 : n \in \mathbb{N}\}$. Clearly Σ is finitely satisfiable. Since Henkin second-order logic has a complete proof system, it satisfies the compactness property. Hence Σ has a model $(\mathfrak{M}, \mathcal{G})$, which is a nonstandard model of P^2. Note that $(\mathfrak{M}, \mathcal{G})$ is a Henkin model but necessarily not a full one: in particular the standard part is not in $\mathcal{G}$, for otherwise it would contradict the induction clause P3.

Now we investigate the notion of internal categoricity. Let

$$L = \{N, S, 0, N', S', 0'\}$$

consist of two copies of the language of arithmetic. An essential feature of P^2 is that it replaces the induction schema of first-order Peano axioms by a second-order quantification over subsets. This renders P^2 a finitely axiomatizable theory. Consequently the interpretation of $(N, S, 0)$ being a model of P^2 can be written as a sentence $P^2(N, S, 0)$ in second-order logic. Similarly $P^2(N', S', 0')$ has the intuitive meaning that the interpretation of $(N', S', 0')$ satisfies P^2. That R is an isomorphism from the interpretation of $(N, S, 0)$ to the interpretation of $(N', S', 0')$ can also be written as a second-order sentence $\mathrm{ISO}(R, (N, S, 0), (N', S', 0'))$. We say that P^2 is *internally categorical* if whenever a Henkin model contains two copies of P^2, that is,

$$(\mathfrak{M}, \mathcal{G}) \models P^2(N, S, 0) \wedge P^2(N', S', 0'),$$

this model "sees" that these two copies are isomorphic, that is,

$$(\mathfrak{M}, \mathcal{G}) \models \exists R\ \mathrm{ISO}\big(R, (N, S, 0), (N', S', 0')\big).$$

We prove that P^2 is internally categorical under this definition. We emphasize that we do not assume any a priori connection between $(N, S, 0)$ and $(N', S', 0')$ below, apart from the comprehension axioms, just that they both satisfy P^2.

Theorem 1 *Let L consist of two copies of the language of arithmetic, that is, $L = \{N, S, 0, N', S', 0'\}$. Let* CA *denote the comprehension axioms in this language. Then*

$$\mathrm{CA} \vdash \big(P^2(N, S, 0) \wedge P^2(N', S', 0')\big) \to \exists R\ \mathrm{ISO}\big(R, (N, S, 0), (N', S', 0')\big).$$

Proof Suppose that $(\mathfrak{M}, \mathcal{G}) \models \mathrm{CA}$ and that

(1) $(\mathfrak{M}, \mathcal{G}) \models P^2(N, S, 0)$,
(2) $(\mathfrak{M}, \mathcal{G}) \models P^2(N', S', 0')$,
(3) $N, S, N', S' \in \mathcal{G}$.

We want to show that there is an $R \in \mathcal{G}$ such that $R : (N, S, 0) \cong (N', S', 0')$. Let

$$R = \bigcap\big\{P \in \mathcal{G} : P(0, 0') \wedge \forall x \in N \forall y \in N'\big(P(x, y) \to P\big(S(x), S'(y)\big)\big)\big\}.$$

Note that R is a definable subset of M, so $R \in \mathcal{G}$. For any $c, d \in M$ we have

$$R(c, d) \leftrightarrow \forall P\Big(\big(P(0, 0') \wedge \forall x \in N \forall y \in N'\big(P(x, y) \to P\big(S(x), S'(y)\big)\big)\big) \\ \to P(c, d)\Big).$$

By the comprehension axiom, $R \in \mathcal{G}$. It is easy to verify that $R(0, 0')$ and that $\forall x \in N \forall y \in N'(R(x, y) \to R(S(x), S'(y)))$. From these we prove that R is an isomorphism from $(N, S, 0)$ to $(N', S', 0')$.

We claim that R is total. By the definition of R, $0 \in \mathrm{dom}(R)$ and $\forall x \in N(x \in \mathrm{dom}(R) \to S(x) \in \mathrm{dom}(R))$. Hence by P3, $\mathrm{dom}(R) = N$. Surjectivity is proved similarly. Next we show that R is functional. Let $X = \{x \in N : \exists! y\ R(x, y)\}$. We prove $X = N$ by induction. For the base case, suppose $R(0, a)$ for $a \neq 0'$. Now we define $R' = R - \{(0, a)\}$. Note that R' also satisfies $R'(0, 0') \wedge \forall x \in N \forall y \in N'(R'(x, y) \to R'(S(x), S'(y)))$, contradicting the minimality of R. The induction case is similar. Injectivity is proved similarly. Finally, the homomorphism property of R follows directly from the definition of R. □

For a recent investigation of the reverse mathematical status of second-order categoricity of arithmetic, see Simpson and Yokoyama [4].

Note that if $(\mathfrak{M}, \mathcal{G})$ and $(\mathfrak{M}', \mathcal{G}')$ are two nonisomorphic models of the comprehension axioms CA, there is no reason to believe that $(N, S, 0)$ which satisfies P^2 in $(\mathfrak{M}, \mathcal{G})$ would be isomorphic to any $(N', S', 0')$ which satisfies P^2 in $(\mathfrak{M}', \mathcal{G}')$. The two models $(N, S, 0)$ and $(N', S', 0')$ are not models of P^2 in the same *sense*. The first is a model of P^2 in the sense of $(\mathfrak{M}, \mathcal{G})$, the second in the sense of $(\mathfrak{M}', \mathcal{G}')$. Categoricity holds only with respect to structures that are models of P^2 in the same sense. The *true* sense of being a model of P^2 is captured by the full models $(\mathfrak{M}, \mathcal{G}_{\text{full}})$. If $(N, S, 0)$ satisfies P^2 in a full model, then $(N, S, 0)$ is isomorphic to the standard model of arithmetic. In particular, all such are isomorphic to each other by the transitivity of isomorphism. Internal categoricity says that it is not only the models of P^2 in the full models that are isomorphic but even all models of P^2 in any fixed—full or not—model of CA.

3 Model Existence for Arithmetic

Our next concern is the second-order consistency of P^2, that is, the existence of relations that satisfy the axioms of P^2. CA alone cannot prove that there is a model of arithmetic: in particular, all finite full models are models of CA, but they cannot contain models of P^2. Therefore we make the additional assumption that the underlying domain of our model of CA is infinite. More precisely, we assume that the model contains a nonsurjective injective mapping F.

Theorem 2 *Let* $(\mathfrak{M}, \mathcal{G})$ *be a Henkin model. Suppose*

$$(\mathfrak{M}, \mathcal{G}) \models \exists F \exists z \big(\forall x \forall y \big(F(x) = F(y) \rightarrow x = y\big) \wedge z \notin \operatorname{ran}(F)\big).$$

Then

$$(\mathfrak{M}, \mathcal{G}) \models \exists N \exists S \exists w \; P^2(N, S, w).$$

Proof Choose $F \in \mathcal{G}$, $a \in M$ such that F is injective and a is not in the range of F. We take the closure of a under F. Let

$$N = \bigcap \big\{X \subset M : X(a) \wedge \forall x \big(X(x) \rightarrow X\big(F(x)\big)\big)\big\}.$$

By the same reasoning as in the proof of Theorem 1, N is definable, so the comprehension axiom implies $N \in \mathcal{G}$. It is easy to see that $(\mathfrak{M}, \mathcal{G}) \models P^2(N, F, a)$. □

The assumption of a one-one function which is not onto is very natural for constructing a model of arithmetic, because it directly gives the successor function. One can start with any other second-order assumption and try to use the comprehension axioms and, when needed, axioms of choice (see AC below), to derive the existence of a model of arithmetic. Possible candidates for such second-order assumptions are "There is a linear order without a last element," "There is a dense linear order," "There is an onto map from a proper subset of the universe onto the universe," "There is a linear order which is not isomorphic to its inverse order," and so on. Probably the investigation of such assumptions would closely resemble elementary set theory even if it is carried out entirely in second-order logic.

4 Internal Categoricity of Set Theory

In the same fashion as P^2, we have the second-order counterpart ZFC^2 of the ZFC axioms of set theory. Consider the second-order language of set theory consisting of a single nonlogical symbol $\in$. ZFC^2 has the same axioms for extensionality, union, pair, power set, infinity, regularity, and choice as ZFC. As for separation and replacement, ZFC^2 replaces the axiom schemata with second-order versions. Separation and replacement then read as follows:[1]

Sep: $\forall X \forall x \exists y (\forall z (z \in y \leftrightarrow (z \in x \wedge X(z))))$,
Rep: $\forall x (\forall u \forall z \forall z' ((u \in x \wedge X(u,z) \wedge X(u,z')) \to z = z') \to \exists y \forall z (z \in y \leftrightarrow \exists u (u \in x \wedge X(u,z))))$.

It is not reasonable to suggest internal categoricity for ZFC^2 as it is. Zermelo proved that the (full) models of ZFC^2 are—up to isomorphism—the structures[2] V_κ for κ an inaccessible cardinal (see Zermelo [6]). For two different inaccessibles κ and λ, V_κ and V_λ are not isomorphic. However, if we assume that two models of ZFC^2 are "of the same height," that is, there is an isomorphism between the ordinals in the first model and the ordinals in second model, we can prove that they are isomorphic.

Below we use E to denote the $\in$-relation of set theory and V to denote the universe of set theory. By $\mathrm{ZFC}^2(V,E)$ we mean the relativization of ZFC^2 to the predicate V with E as the $\in$-relation.

Theorem 3 *Let $L = \{V, E, V', E'\}$ consist of two copies of the language of set theory. Let $(\mathfrak{M}, \mathcal{G})$ be a Henkin model in the vocabulary L. Suppose*

(1) $(\mathfrak{M}, \mathcal{G}) \models \mathrm{ZFC}^2(V,E)$,
(2) $(\mathfrak{M}, \mathcal{G}) \models \mathrm{ZFC}^2(V',E')$,
(3) $V, V', E, E' \in \mathcal{G}$,
(4) $(\mathfrak{M}, \mathcal{G}) \models \exists \pi \ \mathrm{ISO}(\pi, \mathrm{Ord}, \mathrm{Ord}')$, *where* Ord *and* Ord$'$ *denote order types of the ordinals in (V,E) and (V',E'), respectively.*

Then $(\mathfrak{M}, \mathcal{G}) \models \exists R \ \mathrm{ISO}(R, (V,E), (V',E'))$.

Proof Choose an isomorphism $\pi \in \mathcal{G} : \mathrm{Ord} \cong \mathrm{Ord}'$. We use a back-and-forth criterion to define an isomorphism between V and V'. Let

$$R = \bigcap \big\{ P \in \mathcal{G} : \pi \subset P, \forall x \in V \forall y \in V' \\ \big(\big(\forall z E x \exists u E' y P(z,u) \wedge \forall u E' y \exists z E x P(z,u)\big) \to P(x,y)\big)\big\}.$$

We may think of R as the minimal extension of π respecting E and E'. By construction R is in $\mathcal{G}$. Moreover, R satisfies

(a) $\pi \subset R$,
(b) $\forall x \in V \forall y \in V' ((\forall z E x \exists u E' y R(z,u) \wedge \forall u E' y \exists z E x R(z,u)) \leftrightarrow R(x,y))$.

The forward direction in (b) follows from the definition of R. The converse follows from the minimality of R. The property (b) gives us a general criterion to decide whether $R(x,y)$ holds. We proceed to prove that R is an isomorphism.

Claim 1 *The relation R, when defined, is an isomorphism onto its image.*

First we prove that R is functional. Let $x \in V$ be an E-minimal element such that there are $y, y' \in V'$ with $y \neq y'$, $R(x,y)$, and $R(x,y')$. By extensionality, without loss of generality there is $u \in V'$ such that $uE'y$, $\neg uE'y'$. Since we have $R(x,y)$, by property (b) above there is zEx such that $R(z,u)$. Similarly, there is $u'E'y'$ such that $R(z,u')$. Since $\neg uE'y$, $u \neq u'$. Now we have $R(z,u)$ and $R(z,u')$ for zEx, contradicting the minimality of x.

By exchanging the role of V and V' in the above argument we can prove that R is injective. That R respects the relations E and E' is clear. Hence R is an isomorphism onto its image when it is defined.

Below V_α refers to the cumulative hierarchy in the sense of (V, E); similarly V'_α refers to the cumulative hierarchy in the sense of (V', E'). The power-set operation in the sense of (V, E) is denoted by $\mathcal{P}$, and in the sense of (V', E') it is denoted by $\mathcal{P}'$.

Claim 2 *For each $\alpha \in \mathrm{Ord}$, $R : V_\alpha \to V'_{\pi(\alpha)}$ is an isomorphism.*

For the base case $R : \varnothing \to \varnothing'$ this is trivially true. For the successor case, suppose that $R : V_\alpha \to V'_{\pi(\alpha)}$ is an isomorphism; we aim to show that so is $R : V_{\alpha+1} \to V'_{\pi(\alpha+1)}$. Thanks to (i) it suffices to prove that R is defined on $V_{\alpha+1}$ and that $R \restriction V_{\alpha+1}$ is onto $V'_{\pi(\alpha+1)}$. Pick an arbitrary $y \in \mathcal{P}(V_\alpha)$, that is, $\mathrm{Ext}(y) \subset V_\alpha$, where $\mathrm{Ext}(x) = \{y \in M : yEx\}$, the "extension" of x. By induction hypothesis $R(\mathrm{Ext}(y)) \subset V_{\pi(\alpha)}$. Let $zE\mathcal{P}'(V'_{\pi(\alpha)})$ be such that $\mathrm{Ext}(z) = R(\mathrm{Ext}(y))$. It is straightforward to see by property (b) that $R(y,z)$. Since y is arbitrary, R is defined on $V_{\alpha+1}$. Symmetrically we can prove that $R \restriction V_{\alpha+1}$ is onto $V'_{\pi(\alpha+1)}$.

The limit case is straightforward.

Claim 3 *The relation $R : V \to V'$ is an isomorphism.*

Step (ii) implies that $R : V \to V'$ is an embedding. Since π is an isomorphism between the ordinals in V and in V', the dual argument using π^{-1} shows that R^{-1} is also an embedding. Therefore R is an isomorphism from V to V'. □

A special case of the internal categoricity, given the order type of the ordinals, of ZFC^2 is that ZFC^2 in the sense of any full Henkin model is categorical, given the (inaccessible) order type of the ordinals. In particular the continuum hypothesis (CH)

$$2^{\aleph_0} = \aleph_1$$

is famously decided by ZFC^2 as a full second-order theory. In other words, if a full Henkin model has any models of ZFC^2, they all agree about CH. However, it is an immediate consequence of the classical results of Paul Cohen on CH that if there is a Henkin model with a model for ZFC^2, then there are Henkin models $(\mathfrak{M}_1, \mathcal{G}_1)$ and $(\mathfrak{M}_2, \mathcal{G}_2)$ such that

(1) $(\mathfrak{M}_1, \mathcal{G}_1) \models$ "$\mathrm{ZFC}^2 \models \mathrm{CH}$",
(2) $(\mathfrak{M}_2, \mathcal{G}_2) \models$ "$\mathrm{ZFC}^2 \models \neg\mathrm{CH}$".

We can add to this list. Every full Henkin model $(\mathfrak{M}, \mathcal{G}_{\mathrm{full}})$ satisfies

(3) $(\mathfrak{M}, \mathcal{G}_{\mathrm{full}}) \models$ "$\mathrm{ZFC}^2 \models \mathrm{CH}$" $\vee$ "$\mathrm{ZFC}^2 \models \neg\mathrm{CH}$".

By the truth definition, full Henkin models divide into two classes:

$$K^1_{\mathrm{full}} = \{(\mathfrak{M}, \mathcal{G}_{\mathrm{full}}) : (\mathfrak{M}, \mathcal{G}_{\mathrm{full}}) \models \text{"}\mathrm{ZFC}^2 \models \mathrm{CH}\text{"}\},$$
$$K^2_{\mathrm{full}} = \{(\mathfrak{M}, \mathcal{G}_{\mathrm{full}}) : (\mathfrak{M}, \mathcal{G}_{\mathrm{full}}) \models \text{"}\mathrm{ZFC}^2 \models \neg\mathrm{CH}\text{"}\}.$$

We know that exactly one of these classes is nonempty, but we do not know which. By (1) and (2) above we know that the class of all Henkin models divides into two classes:

$$K^1 = \{(\mathfrak{M}, \mathcal{G}) : (\mathfrak{M}, \mathcal{G}) \models \text{“ZFC}^2 \models \text{CH”}\},$$
$$K^2 = \{(\mathfrak{M}, \mathcal{G}) : (\mathfrak{M}, \mathcal{G}) \models \text{“ZFC}^2 \models \neg\text{CH”}\},$$

and *both* classes are nonempty. Of course,

$$K^1_{\text{full}} \subset K^1, \qquad K^2_{\text{full}} \subset K^2.$$

Thus ZFC^2 decides CH, but which way it decides depends on our understanding of second-order logic. If we relegate decisions about second-order logic to set theory, we essentially enhance the comprehension axioms with new axioms which arise from contemplation in set theory. Unfortunately the discussion in set theory about CH is still open, so no clue comes from there. One day we may know which of K^1_{full} and K^2_{full} is nonempty. Most probably this is a result of enhancing the comprehension axioms, reflecting developments in set theory, with new axioms so that the concept of a Henkin model becomes sufficiently narrow to render one of K^1 or K^2 empty, and that is the reason why we can in that situation decide which of K^1_{full} and K^2_{full} is empty.

5 Model Existence for Set Theory

In this section we seek to establish the second-order consistency—that is, model existence—of ZFC^2 under certain large domain assumptions, just as we proved the consistency (model existence) for second-order arithmetic from the assumption of a one-one function which is not onto. The key ingredient here is a power-set operation which generates the set-theoretic structure by iteration.

Let $(\mathfrak{M}, \mathcal{G})$ be a Henkin model. To cope with the relevant set-theoretical terminology, in this section we refer to elements x of M as “sets” and subsets X of M in $\mathcal{G}$ as “classes.” Let $\mu(X)$ be the formula saying that X is of smaller cardinality than the universe or, in brief, “X is small”:

$$\mu(X) =_{\text{def}} \neg\exists F\big(\text{“}F \text{ is injective”} \wedge \forall x\ X\big(F(x)\big)\big).$$

Let $\eta(X, Y, E)$ be the formula saying that Y behaves like the power set of X, with E taken to be the intended membership relation:

$$\begin{aligned}\eta(X, Y, E) =_{\text{def}}\ & \forall x \forall y\big(xEy \to \big(X(x) \wedge Y(y)\big)\big)\\ & \wedge \forall x, y \in Y\big(\big(\forall z \in X(zEx \leftrightarrow zEy)\big) \to x = y\big)\\ & \wedge \forall Z \subset X \exists y \in Y \forall z\big(Z(z) \leftrightarrow zEy\big).\end{aligned}$$

Then the following sentence says that every small class has a power set:

$$\forall X\big(\mu(X) \to \exists Y, E\ \big(\mu(Y) \wedge \eta(X, Y, E)\big)\big). \tag{5}$$

We can also express that the cardinality of the universe is inaccessible, namely, that the union of a family of small sets indexed by a small set is always small:

$$\forall X \forall R\big(\mu(X) \wedge \forall x\big) \in X\mu\big(R(x, -)\big) \to \mu\big(R(X)\big), \tag{6}$$

where $R(x, -)$ denotes the image of x under R, and $R(X)$ denotes the image of the class X under R. Note that $R(X) = \bigcup_{x \in X} R(x, -)$.

The models of ZFC^2 are of the form V_κ for κ an inaccessible cardinal. For such κ we have $|V_\kappa| = \kappa$. Condition (5) implies that the cardinality of the universe is a strong limit cardinal; condition (6) implies that the cardinality of the universe is a regular cardinal. Quite naturally, our first guess is that (5) and (6) together would imply that there is a model of ZFC^2:

$$(5) + (6) \vdash \exists M, E\ \mathrm{ZFC}^2(M, E)?$$

Tempting as it is, a moment's reflection shows that this plan does not work. It is not enough to postulate that there is a power set for each small set, but we also need these power sets to be compatible with each other, in order to glue them together and generate the set-theoretic structure. For example, suppose $X \subset Y$. Ideally we would have $\mathcal{P}(X) \subset \mathcal{P}(Y)$, but this is not a priori a consequence of (5) and (6). In order to remedy this defect, we will impose a compatibility assumption on the power set operation. The price we pay is that the resulting postulate amounts to assuming the existence of a third-order object. However, this is in harmony with Gödel's second incompleteness theorem, which suggests that one has to go to a higher type to prove the consistency of a formal system.

Our postulates, stronger than (5) and (6) above, are as follows. We use a semantic formulation.

Definition 1 Suppose that $(\mathfrak{M}, \mathcal{G})$ is a Henkin model in a vocabulary with second-order function symbols P and $\mathcal{E}$. Let Γ consist of the following axioms, involving P and $\mathcal{E}$. For all small X, Y (letting E_X denote $\mathcal{E}(X)$) we have the following.

(a) The class $P(X)$ is small. Moreover,
  (i) $\eta(X, P(X), E_X)$, with η as defined in Section 4;
  (ii) $X \subset Y \to P(X) \subset P(Y)$;
  (iii) $X \subset Y \to \forall y \in P(X)(\{u \in X : uE_X y\} = \{u \in Y : uE_Y y\})$.

(b) The cardinality of the universe is regular;[3] that is,

$$\forall X \forall R\big(\mu(X) \wedge \forall x \in X \mu\big(R(x, -)\big) \to \mu\big(R(X)\big)\big).$$

(c) Each E_X is well founded;[4] that is,

$$\forall Y\left(\exists x Y(x) \to \exists x\big(Y(x) \wedge \forall z\big(Y(z) \to \neg z E_X y\big)\big)\right).$$

(d) There is a small transitive class X such that it is infinite; that is, there is an injective function into a proper subclass. A class X is *transitive*, if

$$\forall x\big(X(x) \to \forall y\big(\exists Y(y E_Y x) \to X(y)\big)\big).$$

If we allowed third-order quantification, we could existentially quantify P and $\mathcal{E}$ in Γ. Then Γ would become a higher-order axiom in the empty vocabulary.

We also assume the following formulation of the axiom of choice (AC) in second-order logic in addition to the comprehension axioms:

AC: $\forall R(\forall x \exists y R(x, y) \to \exists F \forall x R(x, F(x)))$.

The main result of this section is the following.

Theorem 4 *We have* $\mathrm{CA} + \mathrm{AC} + \Gamma \vdash \exists M, E\ \ \mathrm{ZFC}^2(M, E)$.

The proof extends over several lemmas and propositions. For any small X and $y \in P(X)$, define the *extension of y in X* to be $\mathbf{Ext}_X(y) = \{u \in X : uE_X y\}$. Requirement (a.iii) says that whenever Y is a superset of X and y is a member of $P(X)$ (and hence by (a.ii) a member of P(Y)), $\mathbf{Ext}_Y(y)$ is equal to $\mathbf{Ext}_X(x)$. Upon reflection this is indeed what is the case in the real set-theoretic universe. For any set

x there corresponds a power set $\mathscr{P}(x)$, and this $\mathscr{P}(x)$ will not change as we regard x as the subset of varying supersets.

Postulate (d) guarantees that the model we construct contains an infinite object. Without this assumption, the structure we end up with could very well be V_ω, satisfying every axiom of ZFC^2 except the axiom of infinity.

We now construct a model of ZFC^2 in much the same way as we did for P^2. We choose a witness C for postulate (d). We define V to be the closure of C under the power set operation P:

$$V = \bigcap\{N \in \mathscr{G} : C \subset N, \forall X \subset N\big(\mu(X) \to P(X) \subset N\big)\},$$

and we let the binary relation E be the union of the local relations:

$$E = \bigcup\{E_X : X \subset V, \mu(X)\}.$$

The comprehension axiom implies that $V, E \in \mathscr{G}$.

Now we set out to prove that $(V, E) \models \mathrm{ZFC}^2$, by which we mean $(\mathfrak{M}, \mathscr{G}) \models \mathrm{ZFC}^2(V, E)$. Below we usually mean by a formula its relativization to V unless otherwise specified.

First several lemmas. By the definition of E as the union of the classes E_X, we have $\mathrm{Ext}(y) = \bigcup_X \mathrm{Ext}_X(y)$. The following lemma shows that it suffices to consider one such X.

Lemma 1 *For any y and any X such that $y \in P(X)$,* $\mathrm{Ext}(y) = \mathbf{Ext}_X(y)$. *Equivalently, for any u, uEy if and only if uE_Xy.*

Proof By definition uE_Xy implies uEy. Now we prove the other direction. Suppose uEy; then uE'_Xy for some X'. Suppose also that u is not in X. Consider the set $X \cup X'$. By postulate (b) on regularity, the union of two small sets is small; hence P is defined on $X \cup X'$. By postulate (a.iii) we have $\mathbf{Ext}_X(y) = \mathbf{Ext}_{X\cup X'}(y) = \mathbf{Ext}_{X'}(y)$. But this is a contradiction, since $u \in \mathbf{Ext}_{X'}(y)$ yet $u \notin \mathbf{Ext}_X(y)$. □

Lemma 2 (Comprehension) *We have* $(V, E) \models \forall X(\mu(X) \to \exists y\ \mathrm{Ext}(y) = X)$.

Proof Choose $y \in P(X)$ such that $\mathbf{Ext}_X(y) = X$. This is guaranteed by (a.i). By Lemma 1, $X = \mathbf{Ext}_X(y) = \mathrm{Ext}(y)$. □

We refer to this lemma henceforth as the *comprehension lemma*. Together with CA, this lemma tells us that for any class X, if it is definable and small, then there is a set x with extension X. This will be our key apparatus for proving existential claims in the ZFC^2-axioms.

Lemma 3 (Every set is small) *We have* $(V, E) \models \forall x \mu(\mathrm{Ext}(x))$.

Proof We distinguish two cases. First suppose $x \notin C$. Recall that C is a fixed witness for postulate (d). Suppose that $\mathrm{Ext}(x)$ is not small. Then $x \notin P(X)$ for all small X, for otherwise $\mathrm{Ext}(x) = \mathbf{Ext}_X(x) \subset X$ and hence is small. Now we consider the model $V' = V - \{x\}$. V' is also closed under P, since x is not in any $P(X)$. Moreover, V' still contains C for $x \notin C$. This contradicts the construction of V as the minimal such class. On the other hand if $x \in C$, by the transitivity of C we have that $\mathrm{Ext}(x) \subset C$, and hence $\mathrm{Ext}(x)$ is small. □

We can now verify the axioms of set theory.

Proposition 1 (Extensionality) *We have* $(V, E) \models \forall x \forall y (\forall z (zEx \leftrightarrow zEy) \rightarrow x = y)$.

Proof Suppose $\mathrm{Ext}(x) = \mathrm{Ext}(y) = Z$. By Lemma 1, $\mathrm{Ext}(x) = \mathbf{Ext}_X(x)$ for some small X, and $\mathrm{Ext}(y) = \mathbf{Ext}_Y(y)$ for some small Y. Consider the set $X \cup Y$. We have $\mathbf{Ext}_{X \cup Y}(x) = \mathbf{Ext}_X(x) = \mathrm{Ext}(x) = \mathrm{Ext}(y) = \mathbf{Ext}_Y(y) = \mathbf{Ext}_{X \cup Y}(y)$. We know that extensionality holds locally by (a.i); hence $x = y$. □

Proposition 2 (Power set) *We have* $(V, E) \models \forall x \exists y \forall z (zEy \leftrightarrow \forall u (uEz \rightarrow uEx))$.

Proof The power-set axiom becomes $\exists y (\forall z (zEy \leftrightarrow X(z)))$, where X is defined by $\forall z (X(z) \leftrightarrow \forall w (wEz \rightarrow wEx))$. For any $z \in X$, $\mathrm{Ext}(z) \subset \mathrm{Ext}(x)$; therefore $z \in P(\mathrm{Ext}(x))$. Since z is arbitrary, we have $X \subset P(\mathrm{Ext}(x))$, and hence X is small. By the comprehension lemma there is a y such that $\mathrm{Ext}(y) = X$. □

The pairing, separation, and replacement axioms follow from the comprehension lemma. In each case, it is rather straightforward to verify that the set we desire is small.

Proposition 3 (Pair) *We have* $(V, E) \models \forall x \forall y \exists z \forall w (wEz \leftrightarrow (w = x \vee w = y))$.

Proposition 4 (Separation) *We have* $(V, E) \models \forall X \forall x \exists y (\forall z (zEy \leftrightarrow (zEx \wedge X(z))))$.

Proposition 5 (Replacement) *If a class F is a function, then for every set x, $F(x)$ is a set.*

Proof Suppose that x is a set and that F is a functional class. Let

$$Y = \{y \in V : \exists z (zEx \wedge F(z) = y)\}.$$

By the axiom of choice of second-order logic there is a functional class G that associates to each $y \in Y$ one of its preimages, that is, $F(G(y)) = y$. By the functionality of F, G is an injective mapping from Y into $\mathrm{Ext}(x)$. Since $\mathrm{Ext}(x)$ is small, Y is also small. Therefore by the comprehension lemma there is a set y with extension Y. This is the set we desired. □

The axiom of union requires the postulate on regularity.

Proposition 6 (Union) *We have* $(V, E) \models \forall x \exists y \forall z (zEx \leftrightarrow \exists w (wEx \wedge zEw))$.

Proof By the regularity postulate (b), the class $\bigcup_{wEx} \mathrm{Ext}(w)$ is small, and hence it is a set by the comprehension lemma. □

The proof of the axiom of regularity is based on postulate (c).

Proposition 7 (Regularity) $(V, E) \models$ *"Every set has an E-minimal element."*

Proof Choose an arbitrary element x, and suppose that $\mathrm{Ext}(x)$ does not have a minimal element. Then there is an infinite descending chain $\cdots x_n E x_{n-1} \cdots x_2 E x_1 E x_0$ in $\mathrm{Ext}(x)$. Let $Z = \{x_0, x_1, \ldots, x_n, \ldots\}$. Now for each x_n in Z $(n > 0)$ there is a small set Y_{x_n} such that $x_n E_{Y_{x_n}} x_{n-1}$. Take $Y = \bigcup_{z \in Z} Y_z$. Z is a subset of $\mathrm{Ext}(x)$ and hence is small; therefore by regularity Y is also small. Moreover, $E_{x_n} \subset E_Y$ for all $x_n \in Z$. Thus the sets x_n have become an infinite descending chain in the relation E_Y, contradicting postulate (c). Note that we have relied here on the axiom of choice. □

Now we can tackle the axiom of infinity.

Proposition 8 (Infinity) *We have $(V, E) \models$ "There exists an infinite set."*

Proof Consider the witness C for postulate (d). Postulate (d) asserts that $(\mathfrak{M}, \mathcal{G})$ thinks that C is transitive. Since "X is transitive" is equivalent to the universal formula

$$\forall x \forall y \forall Y \big((X(x) \wedge y E_Y x) \to X(y) \big),$$

it is preserved downward to the submodel V of $\mathfrak{M}$. Therefore V also thinks C is transitive. Since C is infinite, there is an injective class function F from C to its proper subset. Define the class X as follows:

$$\forall x \big(X(y, z) \leftrightarrow F(y) = z \big).$$

The class X has cardinality less than or equal to $C \times C$, and hence is small by regularity. Let f be the set the extension of which is X, and let c be the set the extension of which is C. Then f is an injective function in the sense of (V, E) from c to its proper subset; therefore c is an infinite set in V. □

Finally, we prove the axiom of choice. Note that we assume in second order logic the axiom of choice AC in addition to the comprehension axioms. However, what we will prove is the internalized version of AC, that is, in the sense of the model (V, E). This is similar in spirit to the proof of the axiom of infinity: we assume the existence of an infinite object in the sense of second-order logic, and the proof consists of internalizing the infinite object into (V, E).

Proposition 9 (Choice) *We have $(V, E) \models$ "For any set x, if the empty set is not a member of x, then there is a choice function on x."*

Proof Suppose that for each $y \in \mathrm{Ext}(x)$ there is $z \in \mathrm{Ext}(y)$. By AC there is a class function F such that for all $y \in \mathrm{Ext}(x)$, $F(y) \in \mathrm{Ext}(y)$, that is, $F(y) E y$. Similarly to the proof of the axiom of infinity, let f be the set the extension of which is F. This f is a choice function on x. □

We have checked all the axioms. This ends the proof of Theorem 4. □

The question arises to what extent the assumption Γ in Theorem 4 is really anything but another way of saying that some classes constitute a model of set theory. Let us then discuss the axiom Γ. The point of Γ is that it uses second-order logic to say that there are an inaccessible number of elements in the universe, an obvious prerequisite to building a model of ZFC^2. To say that the cardinality of the universe is an uncountable regular cardinal is easy enough. Likewise, to say that the number of elements in the universe is a strong limit cardinal is straightforward. This does not, however, seem to suffice. So we have to assume that the power sets, which demonstrate that the number of elements of the universe is a strong limit cardinal, cohere. But that is all we need.

In retrospect, what we have done is similar in spirit to what Burgess establishes in [1]. Burgess proves the consistency (model existence) of ZFC^2 under the following two assumptions.

(a) There are just as many individuals as small classes.
(b) There are indescribably many individuals.

Assumption (a) is essentially what we have achieved with Lemmas 2 and 3: to each small class there corresponds a set (i.e., an individual), and the extension of each set is a small class. The connection between (b) and our approach remains a topic for further investigation.

Notes

1. Note that ZFC^2 is a finite set of axioms.

2. More exactly $(V_\kappa, \in)$.

3. In the sense of the model, not necessarily in the "real" V.

4. In the sense of the model, not necessarily in the "real" V.

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