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# Weak One-Basedness

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**Abstract** We study the notion of weak one-basedness introduced in recent work of Berenstein and Vassiliev. Our main results are that this notion characterizes linearity in the setting of geometric þ-rank 1 structures and that lovely pairs of weakly one-based geometric þ-rank 1 structures are weakly one-based with respect to þ-independence. We also study geometries arising from infinite-dimensional vector spaces over division rings.

## 1 Introduction

An independence relation $\mathop{\smile\!\!\!\!\!\!|}$ is a ternary relation on the set of small subsets of a sufficiently saturated structure $M$. Roughly speaking, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ is intended to mean that "$A$ is independent from $B$ over $C$." More precisely, $\mathop{\smile\!\!\!\!\!\!|}$ satisfies certain axioms. For convenience, we list the axioms which we shall be using in Definition 1.1. Six of these are taken from Kim and Pillay [10, Definition 4.1], and for notational convenience, we have added the normality axiom (see, e.g., Adler [1]). Note that the axioms in [10] are stated to suit the situation where $A$ is a finite tuple. We follow Adler in [1] in expressing them here without that restriction. We have also phrased the invariance axiom without the aid of automorphisms to avoid having to make a strong homogeneity assumption. The penultimate paragraph of this section gives notational conventions which apply throughout this section.

**Definition 1.1** Let $M$ be a sufficiently saturated structure. Let $\mathop{\smile\!\!\!\!\!\!|}$ be a ternary relation on the small subsets of $M$. Then $\mathop{\smile\!\!\!\!\!\!|}$ is an independence relation on $M$ if, for all small $A, B, C \subseteq M$, the following conditions are satisfied.

**Invariance:** For all $A', B', C' \subseteq M$ such that $\mathrm{tp}(A, B, C) = \mathrm{tp}(A', B', C')$, if $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $A' \mathop{\smile\!\!\!\!\!\!|}_{C'} B'$.

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**Symmetry:** If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $B \mathop{\smile\!\!\!\!\!\!|}_C A$.
**Transitivity:** For all $D \subseteq C$, if $C \subseteq B$, then $A \mathop{\smile\!\!\!\!\!\!|}_D B$ if and only if both $A \mathop{\smile\!\!\!\!\!\!|}_D C$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$.
**Extension:** There exists $A' \models \mathrm{tp}(A/C)$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.
**Normality:** We have $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}_C B \cup C$.
**Finite character:** We have $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if, for all finite $B' \subseteq B$, $A \mathop{\smile\!\!\!\!\!\!|}_C B'$.
**Local character:** There exists $D \subseteq B$ such that $|D| \leq |Th(M)| \cdot |A|$ and $A \mathop{\smile\!\!\!\!\!\!|}_D B$.

Well-known examples of independence relations include the case where $M$ is a vector space and $\mathop{\smile\!\!\!\!\!\!|}$ is linear independence or where $M$ is an algebraically closed field and $\mathop{\smile\!\!\!\!\!\!|}$ is algebraic independence. An important difference between these two examples is that linear independence is linear while algebraic independence (in an algebraically closed field) is not. Various conditions, which characterize linearity of an independence relation $\mathop{\smile\!\!\!\!\!\!|}$ in certain situations, have been defined. For the rest of this section we fix a sufficiently saturated infinite structure $M$ and an independence relation $\mathop{\smile\!\!\!\!\!\!|}$ on $M$. So "$A \subseteq M$" means "$A \subseteq M$ and $A$ is small." An important role is played by the notion of modularity which we now recall.

**Definition 1.2** $(M, \mathop{\smile\!\!\!\!\!\!|})$ is modular if, for all $A, B \subseteq M$, $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(A) \cap \mathrm{acl}(B)} B$.

Here acl is the model-theoretic algebraic closure operator in $M$. There is a much-studied weaker notion called *local modularity*. A further weakening is given by Berenstein and Vassiliev in [3] and is as follows.

**Definition 1.3** $(M, \mathop{\smile\!\!\!\!\!\!|})$ is weakly locally modular if, for all $A, B \subseteq M$, there exists $C \subseteq M$ such that $C \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} AB$ and $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(AC) \cap \mathrm{acl}(BC)} B$.

There is a related notion called *one-basedness* which Berenstein and Vassiliev weaken in [4] to obtain the following (in which we treat the finite tuple $\bar{a}$ as a small set by forgetting the order of its entries).

**Definition 1.4** $(M, \mathop{\smile\!\!\!\!\!\!|})$ is weakly one-based if, for all $\bar{a} \in M^n$ and $B \subseteq M$, there exists $C \subseteq M$ such that $B \subseteq C$, $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_B C$, and, for all $\bar{a}' \models \mathrm{tp}(\bar{a}/C)$, if $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}'$, then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}'} C$.

There are several other relevant properties to consider, including one which is actually called *linearity*. We recall its definition in Section 3. Some background is given at the beginning of [4], and we recall some of that now. For the remainder of this paragraph, the independence relation in question is always $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ (see Definition 2.3) which is known to coincide with $\mathop{\smile\!\!\!\!\!\!|}^{þ}$, the independence relation which comes from þ-forking (see Onshuus [11]), for the structures under consideration. When $M$ is strongly minimal, it is known that local modularity, one-basedness, and linearity all coincide. In the more general setting where $M$ is simple with SU-rank 1, one-basedness and linearity are known to coincide and to be strictly weaker than local modularity. In this setting it is proved in [3] (using a result from Vassiliev [13]) that weak local modularity is equivalent to one-basedness and linearity. In the even more general setting where $M$ is geometric and has þ-rank 1, it is shown in [4] that weak local modularity is equivalent to weak one-basedness. Also in this setting, it is shown in [4] that weak one-basedness is equivalent to a notion called generic linearity and implies linearity. We prove in Section 3, in this geometric þ-rank 1 setting, that linearity implies generic linearity and therefore that weak one-basedness is equivalent to linearity. This is proved in [4] under the assumption that $M$ is dense o-minimal.

The notion of þ-rank for a formula is given in Ealy and Onshuus [8, Definition 4.3]. We follow the convention that $M$ has þ-rank 1 if and only if the formula $x = x$ has þ-rank 1 in $M$. We recall what it means for $M$ to be (pre)geometric in Definition 2.3.

In Section 2 we observe that the equivalence between weak one-basedness and weak local modularity, which is proved in [4] in the case where $M$ is pregeometric and $\mathop{\smile\!\!\!\!\!\!|}\, = \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}$, extends to the general setting of an arbitrary sufficiently saturated infinite structure $M$ and an arbitrary independence relation $\mathop{\smile\!\!\!\!\!\!|}$ on $M$, provided that one uses an appropriately modified definition of weak local modularity.

The notion of a lovely pair $(N, P(N))$ of geometric structures has been extensively studied (see [3] and Boxall [5]). It consists of a geometric structure $N$ expanded by a unary predicate $P$ which names a well-behaved elementary substructure $P(N)$. A nice example is given by the real field together with a predicate for the subfield of all real algebraic numbers (see van den Dries [12]). Lovely pairs of geometric structures play an interesting role in the history of the topics we are considering here. In [3] weak local modularity of $\mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}$ in a sufficiently saturated geometric structure is characterized in terms of the modularity of an independence relation in a corresponding lovely pair. Lovely pairs, being an especially well-behaved kind of expansion, also provide a test of the robustness of the notion of weak one-basedness. It is proved in [5] that if $M$ is geometric with þ-rank 1, then the corresponding theory of lovely pairs is rosy, and so sufficiently saturated models of it are equipped with the independence relation $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$. We prove in Section 4 that if $M$ is geometric with þ-rank 1 and $(M, \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}})$ is weakly one-based, then $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ in a sufficiently saturated model of the corresponding theory of lovely pairs will also be weakly one-based. Berenstein and Vassiliev prove this in [4] under an additional assumption.

An earlier version of [4] had contained several questions about the theory of the projective geometry of an infinite-dimensional vector space over a division ring. See Section 4 of [4] to understand the relevance of these geometries. In the appendix we address the issue of stability, showing that the theory of the projective geometry of an infinite-dimensional vector space over an infinite division ring is stable if and only if the theory of the division ring is stable. We do this by proving a quantifier elimination result for the vector space in an appropriate language. We do not claim that these results are new. Indeed they seem to be essentially well known. However, we thought it would be useful to give a presentation of them here.

Our terminology and notation are fairly standard. The following applies to the first four sections. Parameter sets (as opposed to definable sets) are denoted by the letters $A, B, C$, or $D$ or by variants of them such as $A'$. We always work in a sufficiently saturated infinite structure, and so all such sets are automatically assumed to be small. When we say that two tuples of parameter sets *have the same type* (possibly over some other parameter set), for example, $\text{tp}(A, B, C/D) = \text{tp}(A', B', C'/D)$, we mean that this is true for some well ordering of each of these parameter sets. Elements of $M^n$, for some finite $n$, are denoted by $\bar{a}, \bar{b}, \bar{c}$, or $\bar{d}$ (or $\bar{a}'$, etc.). We use $e$ to denote an imaginary element (an element of $M^{\text{eq}}$). We use $x, y, z$ as real variables, $\bar{x}, \bar{y}, \bar{z}$ as finite tuples of real variables, and $w$ as an imaginary variable. We usually just write sets or tuples next to each other to indicate their union or the tuple obtained by writing one before the other. The conventions in use in the appendix are made clear in the appendix.

## 2 Some Equivalent Notions

Throughout this section, $M$ is a sufficiently saturated infinite structure, and $\mathop{\smile\!\!\!\!\!\!\mid}$ is an independence relation on $M$. We recalled one formulation of weak one-basedness in Definition 1.4. The following alternative version is also given in [4] (but here we give it a slightly different name for ease of reference).

**Definition 2.1** $(M, \downarrow)$ is *very weakly one-based* if, for all $\bar{a} \in M^n$ and $B \subseteq M$, there exists $\bar{a}' \models \mathrm{tp}(\bar{a}/B)$ such that $\bar{a} \downarrow_B \bar{a}'$ and $\bar{a} \downarrow_{\bar{a}'} B$.

**Remark 2.2** If $(M, \downarrow)$ is very weakly one-based then, for all $\bar{a} \in M^n$ and $B \subseteq M$, there exists $\bar{a}'' \models \mathrm{tp}(\bar{a}/B)$ such that $\bar{a}'' \downarrow_B \bar{a}$ and $\bar{a}'' \downarrow_{\bar{a}} B$.

**Proof** Suppose that for $\bar{a} \in M^n$ and $B \subseteq M$ we have $\bar{a}' \models \mathrm{tp}(\bar{a}/B)$ such that $\bar{a} \downarrow_B \bar{a}'$ and $\bar{a} \downarrow_{\bar{a}'} B$. Then as $\bar{a} \models \mathrm{tp}(\bar{a}'/B)$ we have an $\bar{a}''$ such that $\mathrm{tp}(\bar{a}\bar{a}'/B) = \mathrm{tp}(\bar{a}''\bar{a}/B)$. The result follows by invariance. □

We use acl to denote model-theoretic algebraic closure in the structure $M$. Recall the following standard notions.

**Definition 2.3** $M$ is *pregeometric* if acl has the Steinitz exchange property (in which case we say that $(M, \mathrm{acl})$ is a *pregeometry*). In this case, for $\bar{a} \in M^n$ and $B, C \subseteq M$, we say that $\bar{a} \downarrow^{\mathrm{acl}}_B C$ if $\dim(\bar{a}/B) = \dim(\bar{a}/BC)$, where this notion of dimension is obtained from acl analogously to the way that transcendence degree is obtained from the algebraic closure operator in an algebraically closed field. For $A, B, C \subseteq M$, we say that $A \downarrow^{\mathrm{acl}}_C B$ if $\bar{a} \downarrow^{\mathrm{acl}}_C B$ for all finite tuples $\bar{a}$ from $A$. If in addition $\mathrm{Th}(M)$ eliminates the quantifier $\exists^\infty$, we say that $M$ is *geometric*.

It is well known that $\downarrow^{\mathrm{acl}}$ is an independence relation on $M$ when $M$ is pregeometric. It is proved in [4] that weak local modularity, weak one-basedness, and very weak one-basedness are all equivalent when $M$ is pregeometric and $\downarrow = \downarrow^{\mathrm{acl}}$. We would like to extend this equivalence to the general setting which we are considering here, that of an arbitrary sufficiently saturated infinite structure $M$ and an arbitrary independence relation $\downarrow$ on $M$. However, we do not see how to make this work with weak local modularity as in Definition 1.3, and so we consider the following version instead.

**Definition 2.4** $(M, \downarrow)$ is very weakly locally modular if, for all $\bar{a} \in M^n$ and $B \subseteq M$, there exists $C \subseteq M$ such that $C \downarrow_{\bar{a}} B$, $C \downarrow_B \bar{a}$, and $\bar{a} \downarrow_{\mathrm{acl}(\bar{a}C)\cap\mathrm{acl}(BC)} B$.

It is easy to check that very weak local modularity coincides with weak local modularity when $M$ is pregeometric and $\downarrow = \downarrow^{\mathrm{acl}}$. After replacing weak local modularity with very weak local modularity, Berenstein and Vassiliev's equivalence extends to our general setting. However, some preliminary work is required. The argument in [4] makes use of the following property, which we do not claim to be true generally.

**Property 2.5** Let $\bar{a} \in M^n$ and $B \subseteq M$. Let $\bar{a}' \models \mathrm{tp}(\bar{a}/B)$. If $\bar{a} \downarrow_B \bar{a}'$ and $\bar{a} \downarrow_{\bar{a}'} B$, then $\bar{a}' \downarrow_{\bar{a}} B$.

It is proved in [4] that $(M, \mathop{\smile\!\!\!\!\!\mid}\,)$ has Property 2.5 when $M$ is pregeometric and $\mathop{\smile\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{acl}}$. The argument may be phrased in terms of additivity of $U$-rank and so extended to apply to any independence relation which has a $U$-rank which is finite on every type. However, we do not see how to stretch it to the general setting of an arbitrary sufficiently saturated infinite structure $M$ and an arbitrary independence relation $\mathop{\smile\!\!\!\!\!\mid}$ on $M$. This does not matter since we require only the following strengthening of very weak one-basedness which can be proved, in this general setting, by other means.

**Lemma 2.6** *Suppose that $(M, \mathop{\smile\!\!\!\!\!\mid}\,)$ is very weakly one-based. Let $\bar{a} \in M^n$ and $B \subseteq M$. Then there exists $\bar{a}' \models \mathrm{tp}(\bar{a}/B)$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_B \bar{a}'$, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}'} B$, and $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} B$.*

**Proof** By very weak one-basedness and Remark 2.2, there exists $\bar{a}_1 \models \mathrm{tp}(\bar{a}/B)$ such that $\bar{a}_1 \mathop{\smile\!\!\!\!\!\mid}_B \bar{a}$ and $\bar{a}_1 \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} B$. Using very weak one-basedness and Remark 2.2 again, there exists $\bar{a}_2 \models \mathrm{tp}(\bar{a}_1/B\bar{a})$ such that $\bar{a}_2 \mathop{\smile\!\!\!\!\!\mid}_{B\bar{a}} \bar{a}_1$ and $\bar{a}_2 \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}_1} B\bar{a}$. Continuing in this way, for $i = 2, 3, 4, \ldots$, we obtain $\bar{a}_{i+1} \models \mathrm{tp}(\bar{a}_i/B\bar{a}\bar{a}_1 \cdots \bar{a}_{i-1})$ such that $\bar{a}_{i+1} \mathop{\smile\!\!\!\!\!\mid}_{B\bar{a}\bar{a}_1 \cdots \bar{a}_{i-1}} \bar{a}_i$ and $\bar{a}_{i+1} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}_i} B\bar{a}\bar{a}_1 \cdots \bar{a}_{i-1}$. Let $\bar{a}_0 = \bar{a}$.

Having constructed the sequence $(\bar{a}_i)_{i<\omega}$, we could continue the process and extend it to a sequence $(\bar{a}_i)_{i<\alpha}$, for any small infinite ordinal $\alpha$. Then, for each $n < \omega$ and $i_0 < i_1 < \cdots < i_n < i_{n+1} < \alpha$, we would have $\bar{a}_{i_{n+1}} \mathop{\smile\!\!\!\!\!\mid}_{B\bar{a}_{i_0}\bar{a}_{i_1}\cdots\bar{a}_{i_{n-1}}} \bar{a}_{i_n}$ and $\bar{a}_{i_{n+1}} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}_{i_n}} B\bar{a}_{i_0}\bar{a}_{i_1} \cdots \bar{a}_{i_{n-1}}$. (Both of these independences follow, by transitivity, normality, and invariance, from the construction of $(\bar{a}_i)_{i<\alpha}$.)

By choosing $\alpha$ large enough and then applying a well-known consequence of the Erdős–Rado theorem (which is stated as Adler [2, Theorem 1] where references are also given), we obtain a sequence indexed by $\omega$ with all the properties stated for our original sequence $(\bar{a}_i)_{i<\omega}$ and with the additional property of being indiscernible over $B$. Therefore we may assume that our original sequence $(\bar{a}_i)_{i<\omega}$ is indiscernible over $B$, and from now on we do so.

For all $n < \omega$, by transitivity and normality, we have

$$\bar{a}_1 \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}_0} B, \ldots, \bar{a}_n \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}_0 \cdots \bar{a}_{n-1}} B,$$

and so we get $\bar{a}_1 \cdots \bar{a}_n \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}_0} B$ using transitivity, normality, and symmetry.

Let $\lambda$ be $(|\mathrm{Th}(M)| \cdot |B|)^+$ considered as an ordinal. Then there is an indiscernible sequence $(\bar{c}_i)_{i<\lambda}$ of tuples from $M$ such that $\mathrm{tp}(\bar{c}_{i_1} \cdots \bar{c}_{i_n}/B) = \mathrm{tp}(\bar{a}_{j_1} \cdots \bar{a}_{j_n}/B)$ for all natural numbers $n$ and all $i_1 < i_2 < \cdots < i_n < \lambda$ and $j_n < j_{n-1} < \cdots < j_1 < \omega$. Given that $\lambda$ is a limit ordinal with cofinality greater than $|\mathrm{Th}(M)| \cdot |B|$, it is a well-known consequence of the local character axiom (in conjunction with transitivity, symmetry, and normality) that there cannot exist a sequence $(\bar{d}_i)_{i<\lambda}$ of finite tuples from $M$ such that $\bar{d}_{i+1} \not\mathop{\smile\!\!\!\!\!\mid}_{\bar{d}_0 \cdots \bar{d}_i} B$ for all $i < \lambda$. Therefore there will be some $i < \lambda$ such that $\bar{c}_{i+2} \mathop{\smile\!\!\!\!\!\mid}_{\bar{c}_0 \cdots \bar{c}_{i+1}} B$. We also have $\bar{c}_0 \cdots \bar{c}_i \mathop{\smile\!\!\!\!\!\mid}_{\bar{c}_{i+1}} B$. We then get $\bar{c}_{i+2} \mathop{\smile\!\!\!\!\!\mid}_{\bar{c}_{i+1}} B$. We also have $\bar{c}_{i+1} \mathop{\smile\!\!\!\!\!\mid}_{\bar{c}_{i+2}} B$ and $\bar{c}_{i+1} \mathop{\smile\!\!\!\!\!\mid}_B \bar{c}_{i+2}$. Since $\bar{a} \models \mathrm{tp}(\bar{c}_{i+1}/B)$, there exists $\bar{a}' \models \mathrm{tp}(\bar{a}/B)$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_B \bar{a}'$, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}'} B$, and $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} B$. □

We are now in a position to prove the main result of this section, which extends Berenstein and Vassiliev's equivalence to the setting where $M$ is an arbitrary sufficiently saturated infinite structure and $\mathop{\smile\!\!\!\!\!\vert}$ is an arbitrary independence relation on $M$.

**Theorem 2.7** *The following are equivalent:*

*(1)* $(M, \mathop{\smile\!\!\!\!\!\vert})$ *is very weakly locally modular,*
*(2)* $(M, \mathop{\smile\!\!\!\!\!\vert})$ *is weakly one-based,*
*(3)* $(M, \mathop{\smile\!\!\!\!\!\vert})$ *is very weakly one-based.*

**Proof** After Lemma 2.6, the rest of the proof of Theorem 2.7 is essentially given in [4]. For convenience we write it out here. We use the axioms of an independence relation, as stated in Definition 1.1, and well-known consequences of them freely and without specific reference.

Assume (1). Let $\bar{a} \in M^n$ and $B \subseteq M$. Let $C \subseteq M$ be such that $C \mathop{\smile\!\!\!\!\!\vert}_B \bar{a}$, $C \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}} B$, and $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\mathrm{acl}(\bar{a}C)\cap\mathrm{acl}(BC)} B$. Let $\bar{a}' \models \mathrm{tp}(\bar{a}/\mathrm{acl}(BC))$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{BC} \bar{a}'$. Then $C \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} B$, and so $\mathrm{acl}(\bar{a}'C) \cap \mathrm{acl}(BC) \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} B$. But $\mathrm{acl}(\bar{a}'C) \cap \mathrm{acl}(BC) = \mathrm{acl}(\bar{a}C) \cap \mathrm{acl}(BC)$. So $\mathrm{acl}(\bar{a}C) \cap \mathrm{acl}(BC) \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} B$. We also have $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_C B\bar{a}'$, and so $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\mathrm{acl}(\bar{a}C)\cap\mathrm{acl}(BC)} B\bar{a}'$ and $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{(\mathrm{acl}(\bar{a}C)\cap\mathrm{acl}(BC))\bar{a}'} B$. Therefore $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} B$. We also have $\bar{a}' \models \mathrm{tp}(\bar{a}/B)$ and $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_B \bar{a}'$. Therefore we have (3).

Assume (3). Let $\bar{a} \in M^n$ and $B \subseteq M$. By assumption and Lemma 2.6, there exists $\bar{a}' \models \mathrm{tp}(\bar{a}/B)$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_B \bar{a}'$, $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} B$, and $\bar{a}' \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}} B$. Let $C = \bar{a}'$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\mathrm{acl}(\bar{a}C)\cap\mathrm{acl}(BC)} B$. Therefore we have (1).

Assume (3). Let $\bar{a} \in M^n$ and $B \subseteq M$. By assumption and Lemma 2.6, there exists $\bar{a}'' \models \mathrm{tp}(\bar{a}/B)$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_B \bar{a}''$, $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}''} B$, and $\bar{a}'' \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}} B$. Let $C = B\bar{a}''$. Let $\bar{a}' \models \mathrm{tp}(\bar{a}/C)$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_C \bar{a}'$. Then $\bar{a}'' \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} B$. Also $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}''\bar{a}'} B$, since $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}''} B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{B\bar{a}''} \bar{a}'$. So $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} B$. From $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_B C$ and $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_C \bar{a}'$ we get $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{B\bar{a}'} C\bar{a}'$. Therefore $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} C$. Therefore we have (2).

Assume (2). Let $\bar{a} \in M^n$ and $B \subseteq M$. Let $C \subseteq M$ be such that $B \subseteq C$, $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_B C$, and, for all $\bar{a}' \models \mathrm{tp}(\bar{a}/C)$, if $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_C \bar{a}'$, then $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} C$. There is some $\bar{a}' \models \mathrm{tp}(\bar{a}/C)$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_C \bar{a}'$. We then have $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_B \bar{a}'$ and $\bar{a} \mathop{\smile\!\!\!\!\!\vert}_{\bar{a}'} B$. Clearly $\bar{a}' \models \mathrm{tp}(\bar{a}/B)$. Therefore we have (3). □

## 3 Notions of Linearity

For this section we continue to assume that $M$ is a sufficiently saturated, infinite structure and $\mathop{\smile\!\!\!\!\!\vert}$ is an independence relation on $M$. In addition, throughout this section, we assume $M$ is geometric and $\mathop{\smile\!\!\!\!\!\vert} = \mathop{\smile\!\!\!\!\!\vert}^{\mathrm{acl}}$. Recall that a *family of plane curves* is given by a pair of formulas $\varphi(x, y, w)$ and $\psi(w)$, possibly with parameters from $M$, such that $x$ and $y$ are variables of the home sort (real variables) but $w$ possibly belongs to an imaginary sort and, for each $e \in M^{\mathrm{eq}}$ such that $M^{\mathrm{eq}} \models \psi(e)$, the subset of $M^2$ defined by $\varphi(x, y, e)$ has acl-dimension 1 (i.e., it is infinite and no element $ab$ of it is acl-independent over $e$ together with the parameters in $\varphi$). Recall too that a family of plane curves is said to be *normal* if, for any two distinct $e, e' \in M^{\mathrm{eq}}$ such that $M^{\mathrm{eq}} \models \psi(e)$ and $M^{\mathrm{eq}} \models \psi(e')$, the set defined by $\varphi(x, y, e) \wedge \varphi(x, y, e')$ is finite. In [4] (which refers back to Hasson, Onshuus, and Peterzil [9]), a family of plane curves given by $\varphi(x, y, \bar{z})$ and $\psi(\bar{z})$, where $\bar{z}$ is a tuple of real variables, is said to be *almost normal* if, for all $\bar{c} \models \psi(\bar{z})$, there exist only finitely many $\bar{c}' \models \psi(\bar{z})$ such that the set defined by $\varphi(x, y, \bar{c}) \wedge \varphi(x, y, \bar{c}')$ is infinite.

In the following definition, linearity is a standard notion and generic linearity is defined in [4] (which refers back to [9]). The dimension referred to in the definition of linearity is some extension of acl-dimension from $M$ to $M^{\mathrm{eq}}$. We only consider linearity in situations where a well-behaved such extension is known to exist such as when $M$ has þ-rank 1.

**Definition 3.1**

(1) $(M, \mathop{\smile\!\!\!\!\mid}^{\mathrm{acl}})$ is *linear* if, for every normal family $\varphi(x, y, w)$ and $\psi(w)$ of plane curves, the set defined by $\psi(w)$ has dimension less than 2.

(2) $(M, \mathop{\smile\!\!\!\!\mid}^{\mathrm{acl}})$ is *generically linear* if, for every almost normal family $\varphi(x, y, \bar{z})$ and $\psi(\bar{z})$ of plane curves (where $\bar{z}$ is a tuple of real variables), the set defined by $\psi(\bar{z})$ has acl-dimension less than 2 (i.e., $\dim(\bar{c}/\bar{d}) < 2$ for all $\bar{c} \models \psi(\bar{z})$, where $\bar{d}$ are the parameters in $\psi$ and dim is as in Definition 2.3).

Berenstein and Vassiliev prove in [4] that $(M, \mathop{\smile\!\!\!\!\mid}^{\mathrm{acl}})$ is weakly one-based if and only if it is generically linear. Assuming that $M$ has þ-rank 1, they also prove that $(M, \mathop{\smile\!\!\!\!\mid}^{\mathrm{acl}})$ is linear if it is weakly one-based. We reverse this implication using the following variation on the theme of linearity.

**Definition 3.2**

(1) Let $\bar{a} \in M^n$. Let $E$ be a $\emptyset$-definable equivalence relation on $M^n$. Let $e = \bar{a}/E$ be the imaginary which corresponds to the equivalence class of $\bar{a}$ with respect to $E$. We call $e$ a *finite set imaginary* (an FSI) if every equivalence class of $E$ is finite. In this case we call the sort to which $e$ belongs an *FSI-sort*.

(2) We say that a normal family $\varphi(x, y, w)$ and $\psi(w)$ of plane curves is *FSI-normal* if the variable $w$ ranges over an FSI-sort.

(3) We say that $(M, \mathop{\smile\!\!\!\!\mid}^{\mathrm{acl}})$ is *FSI-linear* if, for every FSI-normal family $\varphi(x, y, w)$ and $\psi(w)$ of plane curves, the set defined by $\psi(w)$ has acl-dimension less than 2.

Here we speak of acl-dimension, even though we are talking about a set of imaginary elements. On this occasion it is perfectly safe to do so because there will be a definable set $Z \subseteq M^n$ and a $\emptyset$-definable (in the sense of $M^{\mathrm{eq}}$) function $f$ (the function which quotients out by $E$) such that $f(Z)$ is the set defined by $\psi(w)$ and each fiber of $f$ is finite. We take the acl-dimension of the set defined by $\psi(w)$ to be equal to the acl-dimension of $Z$. As is well known, this agrees with the extension of acl-dimension to $M^{\mathrm{eq}}$ that we get using $\mathop{\smile\!\!\!\!\mid}^{þ}$ when $M$ has þ-rank 1.

The "only if" part of the following result is essentially proved in [4].

**Theorem 3.3** *$(M, \mathop{\smile\!\!\!\!\mid}^{\mathrm{acl}})$ is generically linear if and only if $(M, \mathop{\smile\!\!\!\!\mid}^{\mathrm{acl}})$ is FSI-linear.*

**Proof** ($\Rightarrow$) Suppose that $(M, \mathop{\smile\!\!\!\!\mid}^{\mathrm{acl}})$ is not FSI-linear. Let $\varphi(x, y, w)$ and $\psi(w)$ be an FSI-normal family of plane curves which witnesses this. Let $Z$ and $f$ be as in the paragraph after Definition 3.2. Let $\psi'(\bar{z})$ define $Z$. Let $\varphi'(x, y, \bar{z})$ be such that $M^{\mathrm{eq}} \models (\forall x y \bar{z})[\psi'(\bar{z}) \rightarrow (\varphi'(x, y, \bar{z}) \leftrightarrow \varphi(x, y, f(\bar{z})))]$. Then $\varphi'(x, y, \bar{z})$ and $\psi'(\bar{z})$ form an almost normal family of plane curves (since, for $\bar{c}, \bar{c}' \models \psi'(\bar{z})$, either the set defined by $\varphi'(x, y, \bar{c}) \wedge \varphi'(x, y, \bar{c}')$ is finite or $f(\bar{c}) = f(\bar{c}')$; recall that the fibers of $f$ are finite). The set defined by $\psi(w)$ has acl-dimension at least 2, by assumption, and so the set defined by $\psi'(\bar{z})$ has acl-dimension at least 2. Therefore $(M, \mathop{\smile\!\!\!\!\!|}^{\text{acl}})$ is not generically linear.

($\Leftarrow$) Suppose that $(M, \mathop{\smile\!\!\!\!\!|}^{\text{acl}})$ is not generically linear. Let $\varphi(x, y, \bar{z})$ and $\psi(\bar{z})$ be an almost normal family of plane curves which witnesses this. Fix some $\bar{c} \models \psi(\bar{z})$ such that $\dim(\bar{c}/\bar{d}) \geq 2$, where dim is as in Definition 2.3 and $\bar{d}$ are the parameters in $\psi$ and (we may assume also) in $\varphi$. Let $m < \omega$ be maximal subject to there being distinct $\bar{c}_1, \ldots, \bar{c}_m \models \psi(\bar{z})$ such that $\bar{c} = \bar{c}_1$ and the formula $\varphi(x, y, \bar{c}_1) \wedge \cdots \wedge \varphi(x, y, \bar{c}_m)$ defines an infinite set. For each $k < \omega$, let $\psi_k(\bar{z}_1, \ldots, \bar{z}_k)$ be the formula $(\exists^\infty xy)[\varphi(x, y, \bar{z}_1) \wedge \cdots \wedge \varphi(x, y, \bar{z}_k) \wedge \psi(\bar{z}_1) \wedge \cdots \wedge \psi(\bar{z}_k) \wedge \bigwedge_{i \neq j} \bar{z}_i \neq \bar{z}_j]$. Let $\psi'(\bar{z}_1, \ldots, \bar{z}_m)$ be $\psi_m(\bar{z}_1, \ldots, \bar{z}_m) \wedge (\forall \bar{z}_{m+1})[\neg\psi_{m+1}(\bar{z}_1, \ldots, \bar{z}_{m+1})]$. Let $E$ be the equivalence relation which says that the order of the $\bar{z}_i$'s in $\bar{z}_1 \cdots \bar{z}_m$ does not matter. Then $E$ is $\emptyset$-definable in $M$. Let $f$ be the function which quotients out by $E$. Then $f$ is $\emptyset$-definable in $M^{\text{eq}}$. Let $\psi''(w)$ be such that $M^{\text{eq}} \models (\forall \bar{z}_1 \cdots \bar{z}_m)[\psi''(f(\bar{z}_1, \ldots, \bar{z}_m)) \leftrightarrow \psi'(\bar{z}_1, \ldots, \bar{z}_m)]$. Let $\varphi'(x, y, w)$ be such that

$$M^{\text{eq}} \models (\forall xy\bar{z}_1 \cdots \bar{z}_m)\big[\varphi'\big(x, y, f(\bar{z}_1, \ldots, \bar{z}_m)\big) \leftrightarrow \varphi(x, y, \bar{z}_1) \wedge \cdots \wedge \varphi(x, y, \bar{z}_m)\big].$$

Then $\varphi'(x, y, w)$ and $\psi''(w)$ form a normal family of plane curves. Since the fibers of $f$ are finite, $\varphi'(x, y, w)$ and $\psi''(w)$ form an FSI-normal family of plane curves. Let $Z$ be the set defined by $\psi'(\bar{z}_1, \ldots, \bar{z}_m)$. We have $\bar{c}_1 \cdots \bar{c}_m \in Z$ such that $\bar{c}_1 = \bar{c}$. Clearly $Z$ is definable over the parameters in $\varphi$ and $\psi$. Therefore $Z$ has acl-dimension at least 2. The set defined by $\psi''(w)$ is $f(Z)$ and so it too has acl-dimension at least 2. Therefore $(M, \mathop{\smile\!\!\!\!\!|}^{\text{acl}})$ is not FSI-linear. □

It is clear that linearity implies FSI-linearity when $M$ has þ-rank 1. So, combining Theorem 3.3 with the results from [4] referred to above, we get the following.

**Corollary 3.4** *Suppose that the geometric structure $M$ has þ-rank 1. Then $(M, \mathop{\smile\!\!\!\!\!|}^{\text{acl}})$ is weakly one-based if and only if $(M, \mathop{\smile\!\!\!\!\!|}^{\text{acl}})$ is linear.*

This was proved in [4] under the assumption that $M$ is dense o-minimal, via an argument which overlaps to some extent with our proof of the "if" part of Theorem 3.3.

## 4 Lovely Pairs

Throughout this section we assume that $M$ is a sufficiently saturated infinite structure which is also geometric and has þ-rank 1. Furthermore, $P$ is a new unary predicate which is interpreted in $M$ so that the expansion $N = (M, P(M))$ is a sufficiently saturated model of the theory of lovely pairs of models of $\text{Th}(M)$ (see [3] for the relevant definitions). We shall use $\mathop{\smile\!\!\!\!\!|}^{\text{þ}}$ to denote þ-independence in the structure $N$. It was proved in [5] that $\text{Th}(N)$ is superrosy. A well-known consequence of this is that $\mathop{\smile\!\!\!\!\!|}^{\text{þ}}$ is an independence relation on $N$. We continue to use $\mathop{\smile\!\!\!\!\!|}^{\text{acl}}$ to denote acl-independence in the structure $M$. One might wonder whether weak one-basedness of $(M, \mathop{\smile\!\!\!\!\!|}^{\text{acl}})$ would imply weak one-basedness of $(N, \mathop{\smile\!\!\!\!\!|}^{\text{þ}})$. This is proved in [4] under an additional assumption (namely, their Assumption 5.8). In this section we show that this additional assumption is not needed.

For the rest of this section we assume that $(M, \mathop{\smile\!\!\!\!\!|}^{\text{acl}})$ is weakly one-based. Recall from [4] that it is then also weakly locally modular. It is proved in [3] that then the algebraic closure operator in $N$ coincides with acl in $M$, and so $\mathop{\smile\!\!\!\!\!|}^{\text{acl}}$ is also an independence relation on $N$.

In addition to the independence relations $\perp^{\text{þ}}$ on $N$ and $\perp^{\mathrm{acl}}$ on either $M$ or $N$, we shall also want to use the independence relation $\perp^{\mathrm{scl}}$ on $N$. The closure operator scl is defined in [3] as follows. Given $a \in M$ and $B \subseteq M$, we have $a \in \mathrm{scl}(B)$ if and only if $a \in \mathrm{acl}(B \cup P(M))$. Then $\perp^{\mathrm{scl}}$ is obtained from scl analogously to the way that $\perp^{\mathrm{acl}}$ is obtained from acl.

Let $\bar{a} \in M^n$, and let $B, C \subseteq M$. It is proved in [3] that $\bar{a} \perp^{\text{þ}}_C B$ if both $\bar{a} \perp^{\mathrm{acl}}_C B$ and $\bar{a} \perp^{\mathrm{scl}}_C B$. It is well known, and clear from the definition of $\perp^{\text{þ}}$, that $\bar{a} \perp^{\mathrm{acl}}_C B$ if $\bar{a} \perp^{\text{þ}}_C B$. So the relation $\perp^{\text{þ}}$ lies, in strength, somewhere between $\perp^{\mathrm{acl}} \wedge \perp^{\mathrm{scl}}$ and $\perp^{\mathrm{acl}}$.

The following fact is an immediate consequence of the definition of the theory of lovely pairs of models of $\mathrm{Th}(M)$, together with the saturation assumption (see [3]).

**Fact 4.1** Let $a \in M$ and $B \subseteq M$. Suppose that $a \notin \mathrm{acl}(B)$. Then $\mathrm{tp}_M(a/B)$ has a realization in $P(M)$.

We now prove the main theorem of this section. There is some overlap between the argument presented here and that used to obtain the corresponding result, [4, Proposition 5.9] (where their Assumption 5.8 was used). We use the axioms of an independence relation, as stated in Definition 1.1, and well-known consequences of them freely and without specific reference.

**Theorem 4.2** *Suppose that* $(M, \perp^{\mathrm{acl}})$ *is weakly one-based. Then* $(N, \perp^{\text{þ}})$ *is weakly one-based.*

**Proof** Let $\bar{a} \in M^n$ and $B \subseteq M$. Using the weak one-basedness of $(M, \perp^{\mathrm{acl}})$, let $D' \supseteq B$ be such that $\bar{a} \perp^{\mathrm{acl}}_B D'$ and, for all $\bar{a}' \models \mathrm{tp}_M(\bar{a}/D')$, if $\bar{a} \perp^{\mathrm{acl}}_{D'} \bar{a}'$, then $\bar{a} \perp^{\mathrm{acl}}_{\bar{a}'} D'$. Using Fact 4.1, let $D \models \mathrm{tp}_M(D'/B\bar{a})$ be such that $D \subseteq \mathrm{scl}(B)$. We may assume $\bar{a} = \bar{a}_0\bar{a}_1$, where $\bar{a}_0$ is scl-independent over $D$ and $\bar{a}_1 \subseteq \mathrm{scl}(D\bar{a}_0)$. Let $\bar{p} \in P(M)^m$ be such that $\bar{a}_1 \subseteq \mathrm{acl}(D\bar{a}_0\bar{p})$. Using the weak one-basedness of $(M, \perp^{\mathrm{acl}})$ again, let $C' \supseteq D$ be such that $\bar{a}\bar{p} \perp^{\mathrm{acl}}_D C'$ and, for all $\bar{a}'\bar{p}' \models \mathrm{tp}_M(\bar{a}\bar{p}/C')$, if $\bar{a}\bar{p} \perp^{\mathrm{acl}}_{C'} \bar{a}'\bar{p}'$, then $\bar{a}\bar{p} \perp^{\mathrm{acl}}_{\bar{a}'\bar{p}'} C'$. Using Fact 4.1 again, let $C \models \mathrm{tp}_M(C'/D\bar{a}\bar{p})$ be such that $C \subseteq \mathrm{scl}(D)$. Then $\bar{a} \perp^{\mathrm{acl}}_B C$ and, since $C \subseteq \mathrm{scl}(B)$, $\bar{a} \perp^{\mathrm{scl}}_B C$. Therefore $\bar{a} \perp^{\text{þ}}_B C$.

Let $\bar{a}' \models \mathrm{tp}_N(\bar{a}/C)$ be such that $\bar{a} \perp^{\text{þ}}_C \bar{a}'$. We may assume that $\bar{p}$ was chosen so that $\bar{a}\bar{p} \perp^{\text{þ}}_C \bar{a}'$. Let $\bar{p}'$ be such that $\bar{a}'\bar{p}' \models \mathrm{tp}_N(\bar{a}\bar{p}/C)$. We may assume $\bar{a}\bar{p} \perp^{\text{þ}}_C \bar{a}'\bar{p}'$. Then $\bar{a}\bar{p} \perp^{\mathrm{acl}}_C \bar{a}'\bar{p}'$, and so $\bar{a}\bar{p} \perp^{\mathrm{acl}}_{\bar{a}'\bar{p}'} C$. We also have $\bar{a} \perp^{\mathrm{acl}}_C \bar{a}'$. We then get $\bar{a} \perp^{\mathrm{acl}}_D \bar{a}'$, and so also $\bar{a} \perp^{\mathrm{acl}}_{\bar{a}'} D$ and then $\bar{a} \perp^{\mathrm{acl}}_{\bar{a}'} C$.

Let $\bar{a}_2$ be a maximal subtuple of $\bar{a}_0$ such that $\bar{a}_2$ is scl-independent over $\bar{a}'$. Using $\bar{a}\bar{p} \perp^{\mathrm{acl}}_{\bar{a}'\bar{p}'} C$ we then get $\bar{a} \subseteq \mathrm{scl}(\bar{a}_2\bar{a}')$. So we have $\bar{a} \perp^{\mathrm{scl}}_{\bar{a}_2\bar{a}'} C$. We also have $\bar{a} \perp^{\mathrm{acl}}_{\bar{a}_2\bar{a}'} C$. Therefore $\bar{a} \perp^{\text{þ}}_{\bar{a}_2\bar{a}'} C$. Since $\bar{a}_2$ is a subtuple of $\bar{a}_0$, we have $\bar{a}_2 \perp^{\mathrm{scl}}_{\emptyset} C$ and $\bar{a}_2 \perp^{\mathrm{acl}}_{\emptyset} C$. Therefore $\bar{a}_2 \perp^{\text{þ}}_{\emptyset} C$. Using $\bar{a}_2 \perp^{\text{þ}}_C \bar{a}'$ we then get $\bar{a}_2 \perp^{\text{þ}}_{\emptyset} C\bar{a}'$ and so $\bar{a}_2 \perp^{\text{þ}}_{\bar{a}'} C$.

From $\bar{a} \perp^{\text{þ}}_{\bar{a}_2\bar{a}'} C$ and $\bar{a}_2 \perp^{\text{þ}}_{\bar{a}'} C$ we get $\bar{a} \perp^{\text{þ}}_{\bar{a}'} C$. Therefore $(N, \perp^{\text{þ}})$ is weakly one-based. □

## Appendix: Infinite-Dimensional Projective Geometries

Let $F$ be an infinite division ring, and let $V$ be an infinite-dimensional vector space over $F$. We use Geom$(V)$ to refer to the structure $(G, (\text{`}x \in cl(y_1, \ldots, y_n)\text{'})_{n \geq 1})$, which is the geometry associated with the pregeometry $(V, \text{span})$. It is a classical result in projective geometry that $F$ is definable in Geom$(V)$ (see, e.g., Part 5 of the notes Csikós [6]).

We aim to show that Th(Geom$(V)$) is stable if and only if Th$(F)$ is stable. To that end, we consider the two-sorted structure $(V, F)$. We refer to the sort of $V$ as the *vector sort* and to the sort of $F$ as the *field sort*. A natural choice of language $L$ for the structure $(V, F)$ consists of the ring language on the field sort for the division ring structure on $F$, the abelian group language on the vector sort for vector addition on $V$, and a function from the Cartesian product of the field sort and the vector sort to the vector sort for scalar multiplication on $V$. We shall use variables $x, y, z$ for the field sort and variables $u, v, w$ for the vector sort. Clearly, Geom$(V)$ is interpretable in $(V, F)$.

We will expand the structure $(V, F)$ in a natural way to prove a quantifier elimination result for Th$(V, F)$ which we can then apply to count the number of types. As we have already said, we do not claim that the results of this section are original. However, we feel that a presentation of them here might be useful. In addition it is worth mentioning that the functions $\lambda_i$, which we are about to define, have been used by Francoise Delon and Luis Pinto, in a similar way to how we use them, to obtain quantifier elimination results in certain settings. Delon works with pairs of fields (see Delon [7]), as does Pinto, who also works with vector spaces over certain fields.

We extend $L$ as follows.

- For every formula $\varphi(\bar{x})$ in the ring language, we add a predicate $P_\varphi(\bar{x})$ on the field sort.
- For every $n \geq 1$, $1 \leq i \leq n$, we add a new $(n+1)$ary function symbol $\lambda_i^n(u_1, \ldots, u_n, v)$ from the appropriate Cartesian product of the vector sort to the field sort.

Call this language $L_{F,\lambda}$.

We make $(V, F)$ into an $L_{F,\lambda}$-structure as follows.

- For every formula $\varphi(\bar{x})$ in the ring language, we interpret $P_\varphi$ as the solution set of $\varphi(\bar{x})$ in $F$.
- For every $n \geq 1$, $1 \leq i \leq n$, and $s_1, \ldots, s_n, r \in V$,

$$(\lambda_i^n)^{(V,F)}(s_1, \ldots, s_n, r) = \begin{cases} 0 & \text{if } s_1, \ldots, s_n \text{ are not linearly independent,} \\ 0 & \text{if } s_1, \ldots, s_n, r \text{ are linearly independent,} \\ a_i & \text{if } s_1, \ldots, s_n \text{ are linearly independent and} \\ & r = \sum_{i=1}^n a_i s_i. \end{cases}$$

Let $T_{F,\lambda}$ be the theory of $(V, F)$ as an $L_{F,\lambda}$-structure. Note that $T_{F,\lambda}$ is a definitional expansion of the $L$-theory of $(V, F)$. In particular, every $L$-structure $(U, K)$ with $(U, K) \equiv (V, F)$ can be uniquely expanded to an $L_{F,\lambda}$-structure satisfying $T_{F,\lambda}$.

**Theorem A.1** *$T_{F,\lambda}$ has quantifier elimination.*

**Proof** We shall use the well-known fact that a complete (first-order) theory $T$ has quantifier elimination if, whenever $M$ and $N$ are $\omega$-saturated models of $T$, the collection of finite partial isomorphisms between $M$ and $N$ has the back-and-forth property.

Note first that because of the functions $\lambda_i^n$, finitely generated substructures of a model $(U, K)$ of $T_{F,\lambda}$ consist exactly of pairs $(S, A)$ where either $A$ is a finitely generated subring of $K$ and $S = \{0\}$ or $A$ is a finitely generated division subring of $K$ and $S$ is a finite-dimensional $A$-subspace of $U$ such that any $A$-linearly independent subset of $S$ is also $K$-linearly independent. (Note that if $0 \neq s \in U$, then $a^{-1} = \lambda_1^1(as, s)$ for any nonzero $a \in K$, and if $s_1, \dots, s_n \in S \setminus \{0\}$ are not linearly independent over $K$, then, after reordering if necessary, we may assume that for some $1 \leq m < n$, $s_1, \dots, s_m$ are linearly independent over $K$ and $s_{m+1} = \sum_{i=1}^m a_i s_i$ for some $a_1, \dots, a_m \in K$ not all zero. But then $a_i = (\lambda_i^n)^{(U,K)}(s_1, \dots, s_m, s_{m+1}) \in A$, and so $s_1, \dots, s_n$ are not linearly independent over $A$ either.)

Now let $(U, K)$ and $(W, J)$ be $\omega$-saturated models of $T_{F,\lambda}$, and let $f : (S, A) \to (R, B)$ be a finite partial isomorphism from $(U, K)$ to $(W, J)$. We may assume that $A$ is a division subring of $K$. Then we can find $\bar{a} \in A^n$ such that $A$ is the division subring of $K$ generated by $\bar{a}$ and $\bar{s} \in S^m$, such that $S$ is the $A$-subspace of $U$ generated by $\bar{s}$ and $\bar{s}$ is linearly independent over $K$. We need to show that whenever $a \in K \setminus A$ or $s \in U \setminus S$, there is a finite partial isomorphism $g$ extending $f$ with $a \in \mathrm{dom}(g)$ or $s \in \mathrm{dom}(g)$, respectively. (Similarly, we can then also extend $f$ so that its image contains any given element from $(W, J)$.)

First, let $a \in K$ be given. By Morleyization of the field sort and $\omega$-saturation, we can find $b \in J$ such that $\bar{a}a$ and $f(\bar{a})b$ have the same division ring type. Let $A'$ be the division subring of $K$ generated by $\bar{a}a$, let $S'$ be the $A'$-submodule of $U$ generated by $\bar{s}$, let $B'$ be the division subring of $J$ generated by $f(\bar{a})b$, and let $R'$ be the $B'$-submodule of $W$ generated by $f(\bar{s})$. Then $(S', A')$ and $(R', B')$ are obviously again closed under all the functions $\lambda_i^n$ and are thus again (finitely generated) substructures of $(U, K)$ and $(W, J)$, respectively, and $f \cup \{(a, b)\}$ extends to a partial isomorphism $g : (S', A') \to (R', B')$.

Now let $s \in U$ be given. There are two cases.

**Case 1:** $s$ is $K$-linearly independent from $S$. Since $W$ is infinite-dimensional over $J$ we can choose $r \in W$ $J$-linearly independent from $R$. Let $S'$ be the $A$-submodule of $U$ generated by $S \cup \{s\}$, and let $R'$ be the $B$-submodule of $W$ generated by $R \cup \{r\}$. Then obviously, $(S', A)$ and $(R', B)$ are closed under all the functions $\lambda_i^n$ and thus are again (finitely generated) substructures of $(U, K)$ and $(W, J)$, respectively, and $f \cup \{(s, r)\}$ extends to a partial isomorphism $g$ from $(S', A)$ to $(R', B)$.

**Case 2:** $s$ is not $K$-linearly independent from $S$. Let $c_i = (\lambda_i^m)^{(U,K)}(\bar{s}, s)$, $i = 1, \dots, m$. By Morleyization of the field sort and $\omega$-saturation of $(W, J)$, there are $d_1, \dots, d_m \in J$ such that $\bar{a}, c_1, \dots, c_m$ and $f(\bar{a}), d_1, \dots, d_m$ have the same division ring type. Let $r = \sum_{i=1}^m d_i f(s_i)$. Then obviously, $f \cup \{(s, r)\}$ extends to a partial isomorphism between the substructure of $(U, K)$ generated by $(\bar{s}s, \bar{a})$ and the substructure of $(W, J)$ generated by $(f(\bar{s})r, f(\bar{a}))$. (Note that the substructure generated by $(\bar{s}s, \bar{a})$ is exactly $(S', A')$ where $A'$ is the division subring generated by $\bar{a}, c_1, \dots, c_m$ and $S'$

is the $A'$-subspace of $U$ generated by $\bar{s}s$, and similarly for the substructure generated by $(f(\bar{s})r, f(\bar{a}))$.) □

**Proposition A.2** *If* Th$(F)$ *is stable (superstable, totally transcendental), then* Th$(V, F)$ *is stable (superstable, totally transcendental).*

**Proof** Clearly it is enough to prove, for every infinite cardinal $\kappa$, that $T_{F,\lambda}$ is $\kappa$-stable if Th$(F)$ is $\kappa$-stable.

Let $\kappa$ be an infinite cardinal such that Th$(F)$ is $\kappa$-stable. Let $(U, K)$ be a model of $T_{F,\lambda}$ of cardinality $\kappa$ (i.e., both $U$ and $K$ have cardinality $\kappa$). We count the number of 1-types in each sort over $(U, K)$.

Let $a$ be an element of the field sort of an elementary extension $(U', K')$ of $(U, K)$. We may assume that $a \in K' \setminus K$. By Theorem A.1, the type of $a$ over $(U, K)$ is then already determined by the division ring type of $a$ over $K$ since all the solutions of $\lambda_i^n(s_1, \ldots, s_n, r) = x$ with $s_1, \ldots, s_n, r \in U$ lie already within $K$ so that the vector sort cannot contribute any new information about $a$. Thus, the number of 1-types in the field sort is $\kappa$ by $\kappa$-stability of Th$(F)$.

Now let $s$ be an element of the vector sort of an elementary extension $(U', K')$ of $(U, K)$. Again we may assume $s \notin U$. Then there are two cases.

**Case 1:** $s$ is $K'$-linearly independent from $U$. Then by Theorem A.1, the type of $s$ over $(U, K)$ is uniquely determined by this. So in this case, we have a unique type.

**Case 2:** $s$ is not $K'$-linearly independent from $U$. Then there exist linearly independent $s_1, \ldots, s_n \in U$ such that $s$ is not linearly independent from $s_1, \ldots, s_n$. Then again by Theorem A.1, it is easily seen that tp$(s/(U, K))$ is fully determined by this information (including the choice of $s_1, \ldots, s_n$) together with the division ring type of $\lambda_1^n(s_1, \ldots, s_n, s), \ldots, \lambda_n^n(s_1, \ldots, s_n, s)$ over $K$, for which there are, by the assumption of $\kappa$-stability of Th$(F)$, only $\kappa$ many possibilities. Therefore, we count a total of $\aleph_0\kappa\kappa$, where the first two factors $\aleph_0\kappa$ correspond to the number of choices of finite linearly independent tuples from $U$ and the last $\kappa$ to the choice of the division ring type over $K$ of the coefficients in the linear combination.

Thus, we count a total of

$$\kappa + \aleph_0\kappa\kappa + 1 = \kappa$$

many 1-types in the vector space sort over $(U, K)$. (The first summand $\kappa$ counts the realized types.)

This also shows that $T_{F,\lambda}$ is superstable (totally transcendental) if Th$(F)$ is superstable (totally transcendental). □

**Proposition A.3** Th(Geom$(V)$) *is stable (superstable, totally transcendental) if and only if* Th$(F)$ *is stable (superstable, totally transcendental).*

**Proof** The left-to-right direction follows immediately from definability of $F$ in Geom$(V)$. The right-to-left direction follows immediately from interpretability of Geom$(V)$ in $(V, F)$ and Proposition A.2. □

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