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Real Closed Exponential Subfields of Pseudo-Exponential Fields

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Abstract In this paper, we prove that a pseudo-exponential field has continuum many nonisomorphic countable real closed exponential subfields, each with an order-preserving exponential map which is surjective onto the nonnegative elements. Indeed, this is true of any algebraically closed exponential field satisfying Schanuel's conjecture.

1 Introduction

For many decades, the first-order theory of complex exponentiation, that is, the theory of $\mathbb{C}_{\exp} := \langle \mathbb{C}, +, \cdot, 0, 1, e^z \rangle$ has been very difficult to study, and many questions stemming from model theory, geometry, and number theory remain open. One of the most famous of these problems is the following conjecture from the 1960s due to Schanuel.

Conjecture 1 (Schanuel's conjecture) *If* $\{z_1, \dots, z_n\} \subset \mathbb{C}$*, then* $\mathrm{td}_{\mathbb{Q}}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n})$*, where* $\mathrm{td}_{\mathbb{Q}}$ *is the transcendence degree over* $\mathbb{Q}$*, is at least the* $\mathbb{Q}$*-linear dimension of* $\{z_1, \dots, z_n\}$.

In 2001, Zilber combined this and many other open questions into one intriguing conjecture. In [7], Zilber constructs a class of exponential fields known as pseudo-exponential fields. A pseudo-exponential field, K, satisfies the following six properties.

1. K is an algebraically closed field of characteristic zero.
2. exp is a surjective homomorphism from the additive group of K onto the multiplicative group of K.
3. There is some transcendental ν so that $\ker(\exp_K) = \nu\mathbb{Z}$.

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4. Schanuel property: If $a_1, \ldots, a_n \in K$ are $\mathbb{Q}$-linearly independent, then $\mathrm{td}_{\mathbb{Q}}(a_1, \ldots, a_n, \exp(a_1), \ldots, \exp(a_n)) \geq n$. (Note that this is equivalent to Schanuel's conjecture for K.)
5. Exponential closure: We need the following definitions to state this property, but we will not refer to them for the remainder of the paper. Let $\alpha \in \mathbb{N}$ and $G_\alpha(K) := K^\alpha \times (K^*)\alpha$. For $[C] = (c_{i,j})$ an $r \times \alpha$ matrix of integers, let $[C] : G_\alpha(K) \to G_r(K)$ be the function which acts additively on the first α coordinates and multiplicatively on the last α coordinates, that is, $[C](\bar{z}, \bar{y}) = (u_1, \ldots, u_r, v_1, \ldots, v_r)$ where

$$u_i = \sum_{j=1}^{\alpha} c_{i,j} z_j \qquad \text{and} \qquad v_i = \prod_{j=1}^{\alpha} y_j^{c_{i,j}}.$$

An irreducible Zariski-closed $V \subseteq K^\alpha \times (K^*)^\alpha$ is *rotund* if $\dim([C](V)) \geq r$ for any $r \times \alpha$ matrix of integers C of rank r where $1 \leq r \leq \alpha$. We say that V is *free* if it is not contained in a closed set given by equations of the form

$$\left\{(\bar{u}, \bar{v}) : \prod_{i=1}^{\alpha} v_i^{m_i} = b\right\}$$

or

$$\left\{(\bar{u}, \bar{v}) : \sum_{i=1}^{\alpha} m_i u_i = b\right\}$$

for any $m_1, \ldots, m_\alpha \in \mathbb{Z}$ and $b \in K$.

Given these definitions, the exponential closure property can be stated as follows.

If $V \subseteq K^\alpha \times (K^*)^\alpha$ is irreducible, rotund, and free, then for any finite $A \in K$ there is $(a_1, \ldots, a_\alpha, \exp(a_1), \ldots, \exp(a_\alpha)) \in V$ a generic point in V over A.
6. Countable closure: We will state this property in terms of the Schanuel predimension δ. For finite $X \subset K$, let

$$\delta(X) := \mathrm{td}_{\mathbb{Q}}\big(X, \exp(X)\big) - \mathbb{Q}\text{-l.d.}(X)$$

where $\mathbb{Q}$-l.d.(X) is the $\mathbb{Q}$-linear dimension of the span of X; δ is a predimension. Notice that the Schanuel property implies that $\delta(X) \geq 0$. Therefore, the following is always defined:

$$d(X) = \min\big\{\delta(Y) : \text{ Y is finite and } X \subseteq Y \subseteq K\big\}.$$

We can now define the Schanuel closure of any set $S \subseteq K$:

$$\mathrm{scl}(S) = \big\{y \in K : \exists X \subseteq_{\text{fin}} S, \delta(Xy) = \delta(X)\big\}.$$

Then countable closure states that the Schanuel closure of a finite set is countable.

Note: Schanuel closure gives a pregeometry on K. (For the definition of pregeometry, see Marker [4].)

These axioms classify pseudo-exponential fields. In [7], Zilber proved the following.

Theorem 2 (Zilber) *For κ uncountable, there is a unique pseudo-exponential field of size κ, and it has 2^k isomorphisms. Furthermore, pseudo-exponential fields are quasi-minimal; that is, every definable subset of a pseudo-exponential field is countable or co-countable.*

This leads to the following question: Is $\mathbb{C}_{\text{exp}}$ the unique pseudo-exponential field of size continuum? Zilber conjectured that $\mathbb{C}_{\text{exp}}$ is indeed the pseudo-exponential field of size $2^{\aleph_0}$. It is clear that $\mathbb{C}_{\text{exp}}$ satisfies properties 1, 2, and 3. In [7] Zilber proved that $\mathbb{C}_{\text{exp}}$ satisfies countable closure. This paper explores a fundamental consequence of Zilber's conjecture.

From this point on, let $\mathcal{K}$ be a fixed pseudo-exponential field of size κ. If $\kappa = 2^{\aleph_0}$ and $\mathcal{K}$ is isomorphic to $\mathbb{C}_{\text{exp}}$, then $\mathcal{K}$ contains an exponential subfield isomorphic to $\mathbb{R}_{\text{exp}}$. Motivated by this observation, we will prove the following theorem.

Theorem 3 *Let L be any algebraically closed exponential field satisfying Schanuel's conjecture (such as the pseudo-exponential field $\mathcal{K}$). Then there are continuum many nonisomorphic (as fields) countable real closed exponential subfields of L, each with an order-preserving exponential map which is surjective onto the nonnegative elements.*

We prove this theorem in two steps, first constructing real closed exponential fields where the exponential map is not surjective and then showing how to construct them so that every positive element is in the image of the exponential map. It is easier to see how this construction works in two steps, rather than one, and the results of the first construction are more examples of real closed exponential subfields of $\mathcal{K}$.

We will define real closed E-field and algebraically closed E-field in the next section.

We will need the following definitions.

Definition 4 A field F is *formally real* if either one of the following equivalent conditions holds:

- -1 is not a sum of squares in F;
- F admits a field ordering.

Similarly, we will say that a ring R is *formally real* if it is an integral domain and its field of fractions is formally real or, equivalently, it admits a ring ordering.

We say that a field F is *real closed* if it is formally real and no proper algebraic extension is formally real.

We will use a number of classic facts about formally real fields (for a full exposition see Lam [1], Lang [2]).

Throughout this paper, we make use of the following notation and conventions.

- We use the tuple notation to denote a finite subset. That is, $\bar{t} \subset T$ is some finite set $t_1, \ldots, t_n$ in T.
- For any set A, we write $\langle A \rangle$ for the $\mathbb{Q}$-additively linear span of A.
- For any set A, we write $[A]$ to mean the subring of $\mathcal{K}$ generated by A.
- For any integral domain R, R^{alg} is the field-theoretic algebraic closure of R. Throughout this paper, the term *algebraic* refers to the field-theoretic notion.
- For a set A, $\mathbb{Q}$-l.d.(A) is the $\mathbb{Q}$-linear dimension of $\langle A \rangle$.
- For any set A, we write $\exp(A)$ for the set $\{\exp(a) : a \in A\}$.

- $\mathbb{Q}^{\mathrm{rc}}$ is the real closure of the rational numbers or, equivalently, the real algebraic numbers.
- For R an ordered ring, we write $R^{>0}$ for $\{r \in R : r > 0\}$.
- We say that $b_1, \ldots, b_n$ are $\mathbb{Q}$-*linearly dependent* over X if $\exists q_1, \ldots, q_n \in \mathbb{Q}$, not all zero, such that $q_1 b_1 + \cdots + q_n b_n \in \langle X \rangle$. We say that $b_1, \ldots, b_n$ are $\mathbb{Q}$-*multiplicatively dependent* over X if $\exists q_1, \ldots, q_n \in \mathbb{Q}$, not all zero, such that $b_1^{q_1} \cdots b_n^{q_n}$ is in the multiplicative span of X. Unless we specify that we are referring to a multiplicative linear space, the word linear will mean additively linear.
- For a finite set $\bar{s}$, we write $\mathrm{td}(\bar{s})$ to mean $\mathrm{td}(\mathbb{Q}(\bar{s})/\mathbb{Q})$.

We also make use of the following elementary facts about exponential functions.

- If $b \in \langle X \rangle$, then $\exp(b)$ is algebraic over $\exp(X)$.
- Suppose that $b_1, \ldots, b_n$ are $\mathbb{Q}$-linearly dependent over X. Then $\exp(b_1, \ldots, b_n)$ is $\mathbb{Q}$-multiplicatively dependent over $\exp(X)$.

2 Free Extensions and Formally Real Fields

We begin with the following definitions.

Definition 5 In this paper, a *(total) E-ring* is a $\mathbb{Q}$-algebra R with no zero divisors, together with a homomorphism $\exp : \langle R, + \rangle \to \langle R^*, \cdot \rangle$.

A *partial E-ring* is a $\mathbb{Q}$-algebra R with no zero divisors, together with a $\mathbb{Q}$-linear subspace $A(R)$ of R and a homomorphism $\exp_R : \langle A(R), + \rangle \to \langle R^*, \cdot \rangle$. The subspace $A(R)$ is then the domain of $\exp_R$ and an E-ring satisfies $A(R) = R$.

An *E-field* is an E-ring which is a field.

An *algebraically closed E-field* is an E-field whose underlying field is algebraically closed. A *formally real E-ring* is an E-ring whose underlying ring is formally real. A *real closed E-field* is an E-field whose underlying field is real closed.

Definition 6 We say that S is a *partial E-ring extension of* R if R and S are partial E-rings, $R \subseteq S$, $A(R) \subseteq A(S)$, and for all $r \in A(R)$, $\exp_S(r) = \exp_R(r)$.

When there is no ambiguity, we drop the subscript.

The following example is an important subtlety with regard to the definition of a partial E-ring extension.

Example 7 Let S be a partial E-ring. If one considers $R = S$ and $A(R) \subsetneq A(S)$ a $\mathbb{Q}$-subspace of $A(S)$, then S is a (proper) partial E-ring extension of R.

Definition 8 Let R be a partial E-ring. We say that $R' \supseteq R$ is a *free partial E-ring extension of* R if

- R' is a partial E-ring extension of R;
- the domain of $\exp_{R'}$ contains R;
- if $\{a_1, \ldots, a_n\} \subset R$ is $\mathbb{Q}$-linearly independent over $A(R)$, then $\{\exp(a_1), \ldots, \exp(a_n)\} \subset R'$ is algebraically independent over R;
- there is no proper partial E-subring of R' satisfying these conditions.

It is worth noting at this point that the fourth condition implies that $A(R') = R$. The next lemma easily follows from equivalent constructions of van den Dries [6] and Macintyre [3].

Lemma 9 *Given any partial E-ring R, there is a free partial E-ring extension R' of R. Furthermore, if R' and S' are free partial E-ring extensions of R, then $R' \cong S'$.*

Proof Let F be a large algebraically closed field extension of R. Let $\{b_i \mid i \in I\}$ be a $\mathbb{Q}$-basis of R over $A(R)$, and for each $i \in I$ and $q \in \mathbb{Q}$, choose $d_{i,q} \in F$ such that $\{d_{i,1} \mid i \in I\}$ is algebraically independent over R, and for all $s \in \mathbb{Z}$, $d_{i,q}^s = d_{i,qs}$. Let R' be the subring of F generated by R and all the $d_{i,q}$. Extend $\exp_R$ to $\exp_{R'}$ by defining $\exp_{R'}(qb_i) = d_{i,q}$ for $q \in \mathbb{Q}$ and $i \in I$, and extending additively. It is an easy exercise to see that this map is well defined. R' is a free partial E-ring extension of R.

Let S' be a different free partial E-ring extension of R. Consider $\widehat{d}_{i,q} \in S'$ where $\exp_{S'}(qb_i) = d_{i,q}$. At this point note that the set $\{\widehat{d}_{i,1} : i \in I\}$ is algebraically independent over R since S' is a free partial E-ring extension of R. The subring of S' generated by R and $\widehat{d}_{i,q}$ is also a partial E-ring extension of R. By minimality of free partial E-ring extensions, S' must be generated as a ring by R and $\widehat{d}_{i,q}$. Our claim is that S' is isomorphic to R'. Consider the ring homomorphism $\varphi : R' \to S'$ defined by

$$\varphi(qb_i) = qb_i \qquad \text{and} \qquad \varphi(d_{i,q}) = \widehat{d}_{i,q}.$$

Consider an algebraically closed field containing both S' and R'. Then, there is an automorphism of this algebraically closed field which fixes the algebraic closure of R, sends $d_{i,1}$ to $\widehat{d}_{i,1}$, and sends any coherent system of roots of $d_{i,1}$ to any coherent system of roots of $\widehat{d}_{i,1}$. Thus φ extends to an automorphism of this algebraically closed field. If we restrict this automorphism to R', the image is S'. It is easy to check that φ preserves the exponential map. Thus, S' is isomorphic to R' as a partial E-ring. □

For any given partial E-ring, we use the prime notation to denote the free extension; that is, if R is a partial E-ring, R' is the free partial E-ring extension of R. We now connect free extensions to formally real fields via this next lemma.

Lemma 10 *Suppose that R is a formally real partial E-ring. Then, R' is formally real.*

Proof Let $\{d_{i,q} : i \in I\}$ be as in the proof of Lemma 9. Consider $R[\{d_{i,1} : i \in I\}]$, the ring extension of R generated by $\{d_{i,1} : i \in I\}$. This is a purely transcendental extension of R and is thus formally real. If we extend an ordering on R such that $d_{i,1}$ is positive for all $i \in I$, then any real closure of $R[\{d_{i,1} : i \in I\}]$ in a large algebraically closed field extension of $R[\{d_{i,1} : i \in I\}]$ will contain a consistent system of positive nth roots $\{d_{i,\frac{1}{n}} : i \in I, n \in \mathbb{N}\}$. Thus, R' is a subring of a real closed field and is thus formally real. □

3 Countable Real Closed Exponential Fields

The goal of the next two sections is to prove Theorem 3. Let L be any algebraically closed E-field satisfying Schanuel's conjecture. In this section, we will prove that we can construct a chain of subfields of L,

$$R_0 \hookrightarrow R_1 \hookrightarrow \cdots,$$

where each R_i is a real closed partial E-field and the union is closed under the exponential map. We will do this by considering a more general chain of subrings of L and then demonstrate the consequences of Schanuel's conjecture on these subrings.

Consider a chain of subrings of L of the following form:

$$Q_0 \hookrightarrow Q_1 \hookrightarrow Q_2 \hookrightarrow \cdots$$

where $Q_0 \subseteq \mathbb{Q}^{\text{alg}}$ and $[Q_i \cup \exp(Q_i)] \subseteq Q_{i+1} \subseteq [Q_i \cup \exp(Q_i)]^{\text{alg}}$. Let $\widetilde{Q} = \bigcup Q_i$.

Definition 11 Let $A \subseteq \widetilde{Q}$ be finite. We say that a set $D \subseteq \widetilde{Q}$ is an *E-source of A* if for all $a \in A$,

1. $a \in (Q_0 \cup \exp(D))^{\text{alg}}$,
2. $\forall d \in D, d \in (Q_0 \cup \exp(D))^{\text{alg}}$,
3. D is minimal such.

By the definition of $\widetilde{Q}$, E-sources always exist and can be chosen to be finite. For the purposes of this paper, all E-sources are assumed to be finite. Furthermore, if $A \subseteq Q_i$ and D is an E-source of A, then we may and do assume that $D \subseteq Q_{i-1}$.

Lemma 12 *E-sources are $\mathbb{Q}$-linearly independent.*

Proof Suppose that D is an E-source of A and D is not $\mathbb{Q}$-linearly independent. Then there is $d \in D$ such that $d \in \langle D \cup \{d\}\rangle$ and $\exp(d)$ is algebraic over $\exp(D - \{d\})$. If $c \in L$ is such that $c \in [Q_0 \cup \exp(D)]^{\text{alg}}$, then in fact, $c \in [Q_0 \cup \exp(D - \{d\})]^{\text{alg}}$. Thus, $D - \{d\}$ contains an E-source of A which contradicts minimality. □

Lemma 13 $[Q_i \cup \exp(Q_i)] \cong Q_i'$.

Proof The statement of this lemma is a priori puzzling, as it is not clear that Q_i satisfies the domain condition of a partial E-subring of L. However, in the following argument we prove that if $\{r_1, \dots, r_n\} \subset Q_i$ is $\mathbb{Q}$-linearly independent over Q_{i-1}, then $\{\exp(r_1, \dots, r_n)\}$ is algebraically independent over Q_i. In particular, since each Q_j is a $\mathbb{Q}$-vector space, if $r \notin Q_{i-1}$, then $\exp(r) \notin Q_i$. This implies that the domain of $\exp_{Q_i}$ is exactly Q_{i-1}, and indeed, Q_i is a partial E-ring extension of Q_{i-1}. Since $[Q_{i-1} \cup \exp(Q_{i-1})]$ is the smallest subring of Q_i to satisfy this, we have proven the lemma.

Let $\bar{r} \subset Q_i$ be $\mathbb{Q}$-linearly independent over Q_{i-1}. Suppose that $\exp(\bar{r})$ is algebraically dependent over Q_i. Then there is $\bar{s} \subset Q_i$ such that $\exp(\bar{r})$ is algebraically dependent over $\bar{s}$ and $\bar{r} \subset \bar{s}$.

Let $\bar{q} \subset Q_0, \bar{t} \in Q_{i-1}$ be such that $\{\bar{q}, \bar{t}\}$ is an E-source of $\bar{s}$. So each element of $\bar{t}$ is algebraic over $\{\bar{q}, \exp(\bar{t})\}$, and each element of $\bar{s}$ as well as each element of $\bar{r}$ is algebraic over $\{\bar{q}, \exp(\bar{t})\}$. Then $\exp(\bar{r})$ is algebraically dependent over $\{\bar{q}, \exp(\bar{t})\}$. Thus

$$\begin{aligned}
&\operatorname{td}\big(\bar{q}, \bar{t}, \bar{r}, \exp(\bar{q}), \exp(\bar{t}), \exp(\bar{r})\big) \\
&\quad = \operatorname{td}\big(\bar{q}, \exp(\bar{q}), \exp(\bar{t})\big) + \operatorname{td}\big(\bar{t}, \bar{r}, \exp(\bar{r}) / \big(\bar{q}, \exp(\bar{q}), \exp(\bar{t})\big)\big) \\
&\quad \lneq |\bar{q}| + |\bar{t}| + |\bar{r}|.
\end{aligned}$$

Since L satisfies Schanuel's conjecture, we conclude that $\{\bar{q}, \bar{t}, \bar{r}\}$ is $\mathbb{Q}$-linearly dependent. Since $\{\bar{q}, \bar{t}\}$ is an E-source and thus $\mathbb{Q}$-linearly independent and a subset of Q_{i-1}, this implies that $\bar{r}$ is $\mathbb{Q}$-linearly dependent over Q_{i-1}. □

Corollary 14 *If Q_i is formally real, then $[Q_i \cup \exp(Q_i)]$ is formally real.*

Corollary 15 *Consider the chain*

$$Q_0 \hookrightarrow Q_1 \hookrightarrow Q_2 \hookrightarrow \cdots$$

where $Q_0 = \mathbb{Q}^{\mathrm{rc}}$ and Q_{i+1} is a real closure of $[Q_i \cup \exp(Q_i)]$. Then the union $\widetilde{Q}$ is a real closed exponential subfield of L.

In order to define a specific real closure of a formally real ring R, the order must be fixed. We have shown that if R is formally real, then R' will be formally real and an element in R' transcendental over R can satisfy any positive cut over $\mathbb{Q}$ which is transcendental over R. Notice that these positive transcendental cuts are actually types over the empty set. Thus, isomorphic real closed exponential fields must satisfy the same cuts over $\mathbb{Q}$, and every positive transcendental cut is satisfiable in some construction of a real closed exponential $\widetilde{Q}$. Since a given $\widetilde{Q}$ is countable and can only satisfy countably many types, there are uncountably many nonisomorphic constructions of a countably real closed exponential field $\widetilde{Q}$.

Proposition 16 $\exp_{\widetilde{Q}}$ *is injective.*

Proof Consider first the exponential map restricted to $\mathbb{Q}^{\mathrm{rc}}$. Schanuel's conjecture implies that the kernel is trivial. Now consider an element $q \in Q_i$ where $q \notin Q_{i-1}$. We have shown that $\exp_L(q)$ is transcendental over Q_i. Thus, the kernel of the exponential map restricted to $\widetilde{Q}$ is trivial. □

Corollary 17 *There are uncountably many nonisomorphic constructions of $\widetilde{Q}$ where the exponential map is order preserving.*

Proof At each stage we have shown the extension to be free, and then we took the real closure. It is enough to show that if at stage n we require that $\{d_{i,1}\}$ from the construction of the free extension has the same order type over the image of $\exp_{Q_{n-1}}$ as $\{b_i \mid i \in I\}$ has over $A(Q_{n-1})$, then the exponential map will be order preserving, and since $\{d_{i,1}\}$ are algebraically independent over Q_{n-1}, we can do this. It will be important that we keep the image of the exponential map positive, and from the proof below it will be clear that this can certainly be done. Then, if you notice that even when constructing Q_1 any positive cut can be satisfied, there are still continuum many real closed exponential subfields of L each with an order-preserving exponential map.

We will do this by induction on n and use the fact that at each stage it is a free construction.

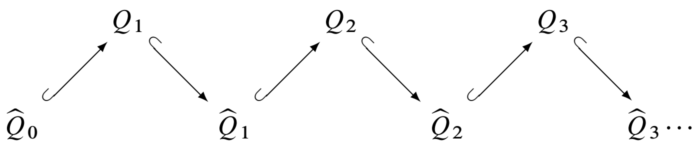
Let $n = 0$. Since the domain of $Q_0 = \{0\}$, $\exp_{Q_0}$ will be order preserving. Now suppose that $\exp_{Q_{n-1}}$ is order preserving and that the image is strictly positive. We will now proceed with the free construction of Q_n and use the notation. Let $\{d_{i,1}\}$ satisfy the same positive order type over image of $\exp_{Q_{n-1}}$ as $\{b_i\}$ satisfy over $A(Q_{n-1})$. We know that the domain of $\exp_{Q_n}$ is the $\mathbb{Q}$-span of $A(Q_{n-1} \cup \{b_i : i \in I\}$. We also know that for $a \in A(Q_{n-1})$, $d_{i,1} < \exp(a) \iff b_i < a$. So we know that $d_{i,q} < \exp(a) \iff qb_i < a$ and $\prod d_{i,q_i} < \exp(a) \iff \sum q_i b_i < a$. Therefore, as we have defined it,

$$\sum_i q_i b_i + a_1 < \sum_j q'_j b_j + a_2 \iff x \sum_i q_i b_i - \sum_j q'_j b_j < a_2 - a_1$$
$$\iff \prod_i d_{i,q_i} \prod_j d_{j,-q'_j} < \frac{\exp(a_2)}{\exp(a_1)},$$

and since we are requiring that the cuts we choose be positive, we know that this is if and only if $\exp(a_1) \prod_i d_{i,q_i} < \exp(a_2) \prod_j d_{j,q'_j}$. □

4 Adding Logs

In this section, we will prove by induction that we can construct the following chain of partial exponential rings:



where $\widehat{Q}_0 = \mathbb{Q}^{\mathrm{rc}}$,

$$Q_{i+1} = \left[\widehat{Q}_i \cup \exp(\widehat{Q}_i)\right]^{\mathrm{rc}},$$

and

$$\widehat{Q}_{i+1} = \left[Q_{i+1} \cup \log(Q_{i+1}^{>0})\right]^{\mathrm{rc}}.$$

Let $\widetilde{Q} = \bigcup Q_i$.

Similarly to the proofs we did earlier in this paper, the proof that at each stage of this construction the rings we are considering are formally real will rely on showing that we are essentially dealing with purely transcendental extensions. In order to understand the construction of $\widetilde{Q}$, it is useful to know what the expected domain and image of the exponential map are at each stage and to keep track of notation. We will show the following:

- $[\widehat{Q}_i \cup \exp(\widehat{Q}_i)] \cong \widehat{Q}_i[E_{i+1}^{\mathbb{Q}}]$, the free extension of $\widehat{Q}_i$. Here, we are denoting the algebraically independent set $\{d_{i,1} \mid i \in I\}$ from the construction of the free extension E_{i+1}, and the set $\{d_{i,q} : i \in I, q \in \mathbb{Q}\}$ from this stage of the construction we denote $E_{i+1}^{\mathbb{Q}}$.
- $[Q_{i+1} \cup \log(Q_{i+1}^{>0})] \cong Q_{i+1}[L_{i+1}]$ where L_{i+1} is a set which is algebraically independent over Q_{i+1}.

The domain and image of the map are as small as possible at each stage; that is,

- $\mathrm{dom}(\exp_{\widehat{Q}_i})$ is the $\mathbb{Q}$-additively linear span of $\widehat{Q}_{i-1} \cup L_i$;
- $\mathrm{img}(\exp_{Q_i})$ is the $\mathbb{Q}$-multiplicative span of $Q_{i-1}^{>0} \cup E_i$.

Definition 18 Let $\bar{s} \subset \widehat{Q}_n$. We say that $\{E, L\} := \{\bar{e}_1, \ldots, \bar{e}_n, \bar{l}_1, \ldots, \bar{l}_n : \bar{e}_i \subset E_i, \bar{l}_i \subset L_i\}$ is an *LE-source of* $\bar{s}$ if

- for all $s \in \bar{s}$, s is algebraic over $\{E, L\}$;
- for all $e \in \bar{e}_i$ for $i = 1, \ldots, n$, $\log(e)$ is algebraic over $\{E, L\}$;
- for all $l \in \bar{l}_i$ for $i = 1, \ldots, n-1$, $\exp(l)$ is algebraic over $\{E, L\}$;
- $\{E, L\}$ is minimal such.

If $\bar{s} \subset Q_n$, then we use the same definition but note that $\{E, L\} := \{\bar{e}_1, \ldots, \bar{e}_n, \bar{l}_1, \ldots, \bar{l}_{n-1} : \bar{e}_i \subset E_i, \bar{l}_i \subset L_i\}$, since we have not yet added the logs at the nth stage. This will be key in the proofs below.

Notice now that Q_1 and T_1 exist by the proofs done at the end of Section 3. Thus, we have a base case for the induction, and we will assume for purposes of induction that we have carried out the construction though Q_n or $\widehat{Q}_n$ and chosen an ordering at each stage so that all elements of $E_i^{\mathbb{Q}}$ are positive. Also notice that for $l \in L_i$, $\exp(l) \in Q_i$. Similarly, for $e \in E_i$, $\log(e) \in \widehat{Q}_{i-1}$. By the induction assumption that we have carried out the construction up to and including Q_n or $\widehat{Q}_n$, LE-sources exist, are finite as defined, and minimality guarantees that they are algebraically independent as sets. We will need the following claim about LE-sources.

Claim 19 *Let $\{E, L\} := \{\bar{e}_1, \ldots, \bar{e}_n, \bar{l}_1, \ldots, \bar{l}_n : \bar{e}_i \subset E_i, \bar{l}_i \subset L_i\}$ be an LE-source for some finite subset of $\widehat{Q}_n$. Let $\bar{q} = \log(\bar{e}_1) \subset \mathbb{Q}^{\mathrm{rc}}$. Then, the set $\{\bar{q}, \bar{l}_1, \log(\bar{e}_2), \bar{l}_2, \ldots, \bar{l}_{n-1}, \log(\bar{e}_n), \bar{l}_n\}$ is $\mathbb{Q}$-linearly independent.*

Proof Suppose $n = 1$. Then, by induction, since E_1 is algebraically independent and $\bar{e}_1 \subset E_1$, $\log(\bar{e}_1)$ must be $\mathbb{Q}$-linearly independent. Since $\bar{l}_1$ is algebraically independent over Q_1, and thus $\mathbb{Q}$-linearly independent over Q_1, $\{\bar{q}, \bar{l}_1\}$ is $\mathbb{Q}$-linearly independent. Similarly, if $\{\bar{q}, \bar{l}_1, \ldots, \log(\bar{e}_i)\}$ is linearly independent, then since this set is in $\widehat{Q}_{i-1}$ and L_i is algebraically independent over Q_i and thus over $\widehat{Q}_{i-1}$, the set $\{\bar{q}, \bar{l}_1, \ldots, \log(\bar{e}_i), \bar{l}_i\}$ is $\mathbb{Q}$-linearly independent.

Now suppose that the set up to $\bar{l}_i$ is $\mathbb{Q}$-linearly independent. Then, since E_{i+1} is algebraically independent over $\widehat{Q}_i$ and $\bar{e}_{i+1} \subset E_{i+1}$, we know that $\log(\bar{e}_{i+1}) \subset \widehat{Q}_i$ is Q-linearly independent over the domain of the exponential map in $\widehat{Q}_i$. Since the domain contains $\widehat{Q}_{i-1} \cup L_i$ and the set up to $\bar{l}_i$ is contained in $\widehat{Q}_{i-1} \cup L_i$, we have that the set up to $\log(\bar{e}_{i+1})$ is $\mathbb{Q}$-linearly independent. □

We are now ready to prove that the construction can be extended from Q_n to $\widehat{Q}_n$.

Lemma 20 *Notice that $Q_n^{>0}$ is a $\mathbb{Q}$-multiplicatively linear space since Q_n is real closed and every positive element has a unique positive nth root. Suppose that $\bar{a} \in Q_n^{>0}$ is $\mathbb{Q}$-multiplicatively independent over $Q_{n-1}^{>0} \cup E_n$. Then $\log(\bar{a})$ is algebraically independent over Q_n.*

This lemma will guarantee that L_n exists as described and that $[Q_n \cup \log(Q_n^{>0})] \cong Q_n[L_n]$.

Proof Suppose that $\log(\bar{a})$ is algebraically dependent over Q_n. Then there is $\bar{s} \subset Q_n$ such that $\log(\bar{a})$ is algebraically dependent over $\bar{s}$ and without loss of generality, we may assume that each $a \in \bar{a}$ is algebraic over $\bar{s}$. Let $\{E, L\} = \{\bar{e}_1, \ldots, \bar{e}_n, \bar{l}_1, \ldots, \bar{l}_{n-1}\}$ be an LE-source of $\bar{s}$. Consider

$$\{\bar{q}, \bar{l}_1, \log(\bar{e}_2), \bar{l}_2, \ldots, \bar{l}_{n-1}, \log(\bar{e}_n), \log(\bar{a}), \bar{e}_1, \exp(\bar{l}_1), \ldots, \bar{e}_n, \bar{a}\}$$

where $\bar{q} = \log(\bar{e}_1) \subset \mathbb{Q}^{\mathrm{rc}}$ and thus the second half of the set we are considering is the exponential image of the first half. By definition of an LE-source, we compute

$$\begin{aligned} &\mathrm{td}\big(\bar{q}, \bar{l}_1, \log(\bar{e}_2), \bar{l}_2, \ldots, \bar{l}_{n-1}, \log(\bar{e}_n), \log(\bar{a}), \bar{e}_1, \exp(\bar{l}_1), \ldots, \bar{e}_n, \bar{a}\big) \\ &\quad < \big|\{E, L\}\big| + |\bar{a}|. \end{aligned}$$

So, by Schanuel's conjecture, we have that $\{\bar{q}, \bar{l}_1, \log(\bar{e}_2), \bar{l}_2, \ldots, \bar{l}_{n-1}, \log(\bar{e}_n), \log(\bar{a})\}$ is $\mathbb{Q}$-linearly dependent. By the claim, we know that $\{\bar{q}, \bar{l}_1, \log(\bar{e}_2), \bar{l}_2, \ldots, \bar{l}_{n-1}, \log(\bar{e}_n)\}$ is $\mathbb{Q}$-linearly independent. Thus, $\log(\bar{a})$ is $\mathbb{Q}$-linearly dependent over

$\{\bar q, \bar l_1, \log(\bar e_2), \bar l_2, \ldots, \bar l_{n-1}, \log(\bar e_n)\}$. So $\bar a$ is $\mathbb{Q}$-multiplicatively dependent over $\{\bar e_1, \exp(\bar l_1), \ldots, \exp(\bar l_{n-1}), \bar e_n\} \subset Q_{n-1}^{>0} \cup E_n$. □

Thus, if Q_n is formally real, then so is the purely transcendental extension $Q_n[L_n]$ and we can take the real closure as $\widehat{Q}_n$.

As in Section 3, the following lemma will guarantee that $[\widehat{Q}_n \cup \exp(\widehat{Q}_n)]$ is indeed the free extension of $\widehat{Q}_n$ and that the domain of the exponential map is precisely what we described above.

Lemma 21 *Suppose that $\bar a \subset \widehat{Q}_n$ is $\mathbb{Q}$-linearly independent over $\widehat{Q}_{n-1} \cup L_n$. Then $\exp(\bar a)$ is algebraically independent over $\widehat{Q}_n$.*

Proof Suppose that $\exp(\bar a)$ is algebraically dependent over $\widehat{Q}_n$. Then there is $\bar s \subset \widehat{Q}_n$ such that $\exp(\bar a)$ is algebraically independent over $\bar s$, and we may assume without loss of generality that each $a \in \bar a$ is algebraic over $\bar s$. Let $\{E, L\} = \{\bar e_1, \ldots, \bar e_n, \bar l_1, \ldots, \bar l_n\}$ be an LE-source of $\bar s$. Now, where $\bar q = \log(\bar t_1) \subset \mathbb{Q}^{\mathrm{rc}}$, we have

$$\operatorname{td}\big(\bar q, \bar l_1, \log(\bar e_2), \bar l_2, \ldots, \log(\bar e_n), \bar l_n, \bar a, \bar e_1, \exp(\bar l_1), \ldots, \bar e_n, \exp(\bar l_n), \exp(\bar a)\big), \\ \big|\{E, L\}\big| + |\bar a|.$$

So, by Schanuel's conjecture, $\bar a$ is $\mathbb{Q}$-linearly dependent over $\{\bar q, \bar l_1, \log(\bar e_2), \bar l_2, \ldots, \log(\bar e_n), \bar l_n\} \subset \widehat{Q}_{n-1} \cup L_n$. □

Thus, if $\widehat{Q}_n$ is formally real, then so is the free extension $[\widehat{Q}_n \cup \exp(\widehat{Q}_n)]$, and we can take the real closure to get Q_{n+1}. This completes the proof that the chain exists as described and that at each stage the domain and image of the exponential map are precisely the minimal possible set.

To finish the proof of the theorem, notice that at each stage we are adding transcendental elements. If we make the L_i satisfy the same order type over the previous domain as their exponential image satisfies over the previous image, and make the E_i satisfy the same order type over the previous image as their preimage satisfies over the previous domain, the exponential map will be order preserving. As there are clearly continuum many positive cuts that can be satisfied when constructing Q_1 and only countably many are satisfied in any one construction, we have proven the theorem.

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