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Ramsey's Theorem for Pairs and Provably Recursive Functions

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Abstract This paper addresses the strength of Ramsey's theorem for pairs (RT^2_2) over a weak base theory from the perspective of 'proof mining'. Let RT^{2-}_2 denote Ramsey's theorem for pairs where the coloring is given by an explicit term involving only numeric variables. We add this principle to a weak base theory that includes weak König's Lemma and a substantial amount of Σ^0_1-induction (enough to prove the totality of all primitive recursive functions but not of all primitive recursive functionals). In the resulting theory we show the extractability of primitive recursive programs and uniform bounds from proofs of $\forall\exists$-theorems.

There are two components of this work. The first component is a general proof-theoretic result, due to the second author, that establishes conservation results for restricted principles of choice and comprehension over primitive recursive arithmetic PRA as well as a method for the extraction of primitive recursive bounds from proofs based on such principles. The second component is the main novelty of the paper: it is shown that a proof of Ramsey's theorem due to Erdős and Rado can be formalized using these restricted principles.

So from the perspective of proof unwinding the computational content of concrete proofs based on RT^2_2 the computational complexity will, in most practical cases, not go beyond primitive recursive complexity. This even is the case when the theorem to be proved has function parameters f and the proof uses instances of RT^2_2 that are primitive recursive in f.

1 Introduction

Ramsey's theorem for pairs and two colors RT^2_2 has been at the center of a lot of research in computability theory and reverse mathematics aiming at determining

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the complexity of the homogeneous sets in RT^2_2 and the contribution to the provably recursive functions of RT^2_2 when added to theories such as RCA_0 from reverse mathematics (see, e.g., Specker [22]; Jockusch [10]; Hirst [9]; Seetapun and Slaman [20]; Cholak et al. [3]; Hirschfeldt and Shore [8]; Simpson [21]; Hirschfeldt et al. [7]). One of the main open questions (see [3]) is whether the provably recursive functions of $\mathrm{RCA}_0+\mathrm{RT}^2_2$ are the primitive recursive ones or whether the totality of the Ackermann function can be established in this system. From the perspective of applied proof theory (proof mining) this question is of relevance for determining what type of bounds one can expect to be extractable from concrete mathematical proofs of—say—Π^0_2-sentences $\forall m \in \mathbb{N} \exists n \in \mathbb{N}\, A_{qf}(m,n)$ or sentences $\forall f \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N}\, A_{qf}(f,n)$ (with A_{qf} quantifier-free) that are based on RT^2_2. Experience from the logical analysis of many proofs in different areas of mathematics indicates that, typically, proofs of theorems $\forall f \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N}\, A_{qf}(f,n)$ that make use of second-order principles $\forall g\, P(g)$ such as RT^2_2 that state that for all functions g or sets of a certain type some property (here for all colorings c a property $\mathrm{RT}^2_2(c)$) holds only need explicit instances $\psi(f)$ for g, respectively, c, that are effectively definable in the parameter f by some closed term ψ of the underlying system $\mathcal{T}$; that is,

$$\mathcal{T} \vdash \forall f \in \mathbb{N}^{\mathbb{N}} \left(\mathrm{RT}^2_2(\psi(f)) \to \exists n \in \mathbb{N}\, A_{qf}(f,n) \right).$$

In this paper we show that, in such a situation and for sufficiently weak systems $\mathcal{T}$, the extractability of a primitive recursive functional Φ (in the ordinary sense, see Kleene [11]; i.e., no higher type recursion in the sense of Gödel's System T; see Gödel [5]) with

$$\forall f\, A_{qf}(f, \Phi(f))$$

is guaranteed. Moreover, the proof theoretic method used provides an extraction algorithm for Φ from a given proof.

We work in a setting based on fragments of (extensional) arithmetic formulated in the language of functionals of all finite types. In [12] (see also [17]), the second author introduced a hierarchy $\text{E-G}_n\text{A}^\omega$ of such fragments containing functionals corresponding to the nth level of the Grzegorczyk hierarchy and quantifier-free induction.

As usual in proof mining, universal axioms do not matter and so arbitrary true (in the sense of the full set-theoretic type structure over $\mathbb{N}$; see [17]) universal sentences can always be added to the theories used in our paper.[1]

The union of all these systems is denoted by $\text{E-G}_\infty\text{A}^\omega$ and contains terms for all primitive recursive functions but not for all primitive recursive functionals (in the sense of Kleene) of type level 2 (e.g., not $\Phi_{it}(f,x,y) := f^{(x)}(y)$). This distinguishes the system from $\widehat{\text{E-PA}}^\omega\!\upharpoonright$ in [12; 17] (sometimes also denoted by PRA^ω).

As the theory $\mathcal{T}$ in the result above we may take

$$\mathcal{T} := \text{E-G}_\infty\text{A}^\omega + \text{QF-AC} + \text{WKL},$$

where QF-AC is the union of the schemata of quantifier-free choice from functions to numbers

$$\forall f \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N}\, A_{qf}(f,n) \to \exists F \forall f \in \mathbb{N}^{\mathbb{N}}\, A_{qf}(f, F(f))$$

and quantifier-free choice from numbers to functions

$$\forall n \in \mathbb{N} \exists f \in \mathbb{N}^{\mathbb{N}} A_{qf}(n,f) \to \exists F \forall n \in \mathbb{N}\, A_{qf}(n, F(n)) \quad (A_{qf} \text{ quantifier-free})$$

and WKL is the weak König's Lemma (i.e., König's Lemma for 0/1-trees; see [21; 17]).

Let WKL_0^* be the theory consisting of WKL_0 with Σ_1^0-induction replaced by quantifier-free induction plus the exponential function so that sequence coding still can be defined; see [21, X.4]. The system WKL_0^* can be viewed as a second-order version of Kalmar elementary arithmetic augmented with WKL. It is clear that $\mathcal{T}$ contains WKL_0^* via the usual embedding.

For this system, the second author has shown in [15; 16] that the addition of the use of fixed instances $\Pi_1^0\text{-CA}(\varphi(f))$ of Π_1^0-comprehension

$$\Pi_1^0\text{-CA}(f) :\equiv \exists g \in \mathbb{N}^{\mathbb{N}} \forall x \in \mathbb{N}\left(g(x) = 0 \leftrightarrow \forall y \in \mathbb{N}(f(x,y) = 0)\right)$$

only causes primitive recursive provably recursive functions. More precisely, by the proof of Corollaries 4.4 and 4.5 in [16] (for $k := 1$), we have the following.

Proposition 1.1 ([16]) *Let $\mathcal{T} := \mathrm{E\text{-}G_\infty A^\omega{+}QF\text{-}AC{+}WKL}$ and $\forall f \exists n\, A_{qf}(f,n)$ a sentence as above. Furthermore, let φ be a closed term of $\mathcal{T}$ (of suitable type). Then the following rule holds:*

$$\begin{cases} \mathcal{T} \vdash \forall f \in \mathbb{N}^{\mathbb{N}}\left(\Pi_1^0\text{-CA}(\varphi(f)) \rightarrow \exists n \in \mathbb{N}\, A_{qf}(f,n)\right) \\ \Rightarrow \textit{there exists a primitive recursive (in the sense Kleene) functional } \Phi \textit{ such that} \\ \widehat{\text{E-PA}}^{\omega}\!\restriction \vdash \forall f \in \mathbb{N}^{\mathbb{N}}\, A_{qf}(f, \Phi(f)). \end{cases}$$

In this rule, we may add an arbitrary set of true purely universal sentences $\mathcal{P}$ as additional axioms to both $\mathcal{T}$ and $\widehat{\text{E-PA}}^{\omega}\!\restriction$.

The main technical result in this paper establishes that over $\mathcal{T}$ one can prove $\mathrm{RT}_2^2(c)$ (i.e., Ramsey's theorem for pairs and a 2-coloring c) from a suitable instance Π_1^0-CA$(\tilde{\varphi}(c))$ of Π_1^0-CA.

Theorem 1.2 (see Theorem 4.5 below)

$$\mathcal{T} \vdash \forall c : [\mathbb{N}]^2 \rightarrow \mathbf{2}\left(\Pi_1^0\text{-CA}(\tilde{\varphi}(c)) \rightarrow \mathrm{RT}_2^2(c)\right).$$

Instead of RT_2^2 we may have also RT_n^2 for any fixed *number $n \geq 2$ of colors, where then $c : [\mathbb{N}]^2 \rightarrow \mathbf{n}$.*

Here $[\mathbb{N}]^2$ denotes the set of unordered pairs in $\mathbb{N}$ and $\mathbf{n}$ the set $\{0,\ldots,n-1\}$.

Combined with the previous result (and the fact that finitely many and even sequences of instances of Π_1^0-CA can be encoded into a single instance) we obtain the following theorem.

Theorem 1.3 (see Theorem 5.1 below) *Let φ, ψ be closed terms of $\mathcal{T}$ (of suitable type). Then the following rule holds:*

$$\begin{cases} \mathcal{T} \vdash \forall f \in \mathbb{N}^{\mathbb{N}}\left(\Pi_1^0\text{-CA}(\varphi(f)) \wedge \forall k \in \mathbb{N}\,(\mathrm{RT}_2^2(\psi(f,k))) \rightarrow \exists n \in \mathbb{N}\, A_{qf}(f,n)\right) \\ \Rightarrow \textit{there exists a primitive recursive (in the sense Kleene) functional } \Phi \textit{ such that} \\ \widehat{\text{E-PA}}^{\omega}\!\restriction \vdash \forall f \in \mathbb{N}^{\mathbb{N}}\, A_{qf}(f, \Phi(f)). \end{cases}$$

Instead of RT_2^2 we, again, may have RT_n^2 for any fixed number n of colors.

We, furthermore, may add arbitrary true universal sentences as axioms to the theories in question.

Note that we cannot replace $\mathcal{T}$ by $\widehat{\text{E-PA}}^{\omega}\restriction$ or any other system containing either Σ^0_1-induction (with function parameters) or the functional Φ_{it} as in such a system even Proposition 1.1 would be wrong; see [15].

For Π^0_2-sentences $\forall m \in \mathbb{N} \exists n \in \mathbb{N}\, A_{qf}(m, n)$ one gets with Theorem 1.3—using the well-known fact that $\widehat{\text{E-PA}}^{\omega}\restriction$ is Π^0_2-conservative over primitive recursive arithmetic (with quantifiers) PRA—as conclusion

$$\text{PRA} \vdash \forall m \in \mathbb{N}\, A_{qf}(m, \varphi(m)).$$

Let (for fixed n) RT^{2-}_n and $\Pi^0_1\text{-CA}^-$ be (the universal closures) of all instances $\mathrm{RT}^2_n(s)$ and $\Pi^0_1\text{-CA}(t)$ for terms s, t containing only number parameters. Then we get the following corollary.

Corollary 1.4

$$\mathcal{T} + \Pi^0_1\text{-CA}^- + \mathrm{RT}^{2-}_n$$

is Π^0_2*-conservative over* PRA.

Combined with further results from [16], it also follows $\mathcal{T} + \Pi^0_1\text{-CA}^- + \mathrm{RT}^{2-}_n$ is Π^0_3-conservative over PRA$+\Sigma^0_1$-IA.

The system in Corollary 1.4 contains arbitrary primitive recursively defined sequences of instances of Π^0_1-comprehension and RT^2_2. However, these principles cannot be applied in a nested way where the result of the use of an instance of these principles is used as a function parameter in forming another instance of one of these principles. Theorem 1.3 is slightly more general as it includes the use of sequence of instances of Π^0_1-comprehension and RT^2_n that are primitive recursive *in the function parameter* f of the theorem to be proved. In our case, this parameter is usually the coloring (sequence of colorings).

Officially every variable in our system has a type (e.g., 0 for a natural number and 1 for a function $\mathbb{N} \to \mathbb{N}$; for details see [17]), but for simplicity of notation in the following we will denote by b, c, f, g, h, q number-theoretic functions of suitable arity and by $x, y, z, k, l, m, n, u, v$, natural numbers.

At a first look, it seems that the framework provided by $\mathcal{T}$ is very restricted as only quantifier-free induction QF-IA (with parameters of arbitrary types) is included. However, from $\Pi^0_1\text{-CA}(\varphi(f))$ (for suitable φ) combined with QF-IA one obtains fixed sequences of instances

$$\Sigma^0_1\text{-IA}(f) :\equiv \left\{ \begin{array}{l} \forall l\, \big(\exists y\, (f(0, y, l) = 0) \wedge \forall x\, (\exists y\, (f(x, y, l) = 0) \to \exists y\, (f(x+1, y, l) = 0)) \\ \qquad\quad \to \forall x \exists y\, (f(x, y, l) = 0)\big) \end{array} \right.$$

of Σ^0_1-induction. Hence the theorem above also holds with

$$\Pi^0_1\text{-CA}(\varphi(f)) \wedge \forall k \in \mathbb{N}\, (\mathrm{RT}^2_2(\psi(f, k)))$$

being replaced by

$$\Pi^0_1\text{-CA}(\varphi(f)) \wedge \forall k \in \mathbb{N}\, (\mathrm{RT}^2_2(\psi(f, k))) \wedge \Sigma^0_1\text{-IA}(\chi(f)).$$

So, in particular, any sequence of instances of the schema of Σ^0_1-IA given by a Σ^0_1-formula that only has free number variables is allowed (short $\Sigma^0_1\text{-IA}^-$). What is not possible is that the *result* of the application of $\Pi^0_1\text{-CA}(\varphi(f))$ (i.e., the comprehension function) or of $\mathrm{RT}^2_2(\psi(f,k))$ (i.e., the monochromatic set given by its characteristic function) is used in a Σ^0_1-instance of induction featuring as a function argument. However, we can freely apply WKL, QF-IA and QF-AC to arbitrary function variables.

The results can be extended to even allow the instances of those principles to depend on the results of instances of WKL; see Remark 2.2 below.

In the following $\text{QF-AC}^{\mathbb{N},\mathbb{N}}$ denotes the special case of QF-AC where both variables (n,f) are natural numbers.

One can use $\Pi^0_1\text{-CA}(f)$ combined with $\text{QF-AC}^{\mathbb{N},\mathbb{N}}$ even to obtain every instance of Δ^0_2-comprehension as well as Π^0_1-countable choice for numbers. As a consequence one can also obtain every fixed sequence of instances of Δ^0_2-induction and Π^0_1-bounded collection, where the latter is defined as

$$\begin{aligned}&\Pi^0_1\text{-CP}(f) :\equiv\\ &\forall k,l\,\big(\forall x<l\exists y\forall z\,(f(k,x,y,z)=0)\\ &\qquad\qquad\to \exists y^*\forall x<l\exists y<y^*\forall z\,(f(k,x,y,z)=0)\big);\end{aligned}$$

(see [15]). Finally, we note that relative to $\mathcal{T}$ fixed sequences of instances of the Bolzano-Weierstraß principle and even the Ascoli lemma can be proven from Π^0_1-CA(ξ) for a suitable ξ (see [13]).

What all this indicates is that from the perspective of unwinding the computational content of concrete proofs based on RT^2_2 (and even RT^2_n for fixed n) the computational complexity of that content will in most practical cases not go beyond primitive recursive complexity.

Let Σ^0_1-WKL be König's Lemma for $0/1$-trees which are given by a Σ^0_1-formula. Theorem 1.2 is established by a careful analysis of the proof of Ramsey's theorem for pairs due to Erdős and Rado [4]. This first yields that relative to suitable (sequences of) instances of Π^0_1-induction with the coloring c as the only free function variable (so that these instances can be covered as discussed above)

$$\Sigma^0_1\text{-WKL}(\varphi(c)) \;\to\; \mathrm{RT}^2_2(c)$$

for a suitable elementary functional φ. $\Sigma^0_1\text{-WKL}(\varphi(c))$ (as well as the inductions needed) is then reduced using $\Pi^0_1\text{-CA}(\tilde{\varphi}(c))$ (for a suitable functional $\tilde{\varphi}$) to WKL and quantifier-free induction which both are available in $\mathcal{T}$.

2 Elimination of Monotone Skolem Functions

In [15; 16], the second author developed a technique for the elimination of monotone Skolem functions that allows one to calibrate the arithmetical strength of fixed (sequences of) instances of various comprehension and choice principles over systems such as $\text{E-G}_\infty\text{A}^\omega$. In this section we collect the results of this type that will be used later.

The next result immediately follows (as special case for $k:=1$) from the proofs of Corollaries 4.4 and 4.5 in [16]. It only differs from Proposition 1.1 by stating the existence of a bound that is independent from bounded function parameters.

Proposition 2.1 ([16]) *Let $A_{qf}(f,g,n)\in\mathcal{L}(\text{E-G}_\infty\text{A}^\omega)$ be a quantifier-free formula which contains only the function variables f, g and the number variable n free. Furthermore, let φ, ψ be functionals (of suitable type) that are definable in* $\text{E-G}_\infty\text{A}^\omega$.

Then the following rule holds:

$$\begin{cases} \text{E-G}_\infty\text{A}^\omega + \text{QF-AC} + \text{WKL} \vdash \forall f\, \forall g \le \varphi(f)\, \big(\Pi^0_1\text{-CA}(\psi(f,g)) \to \exists n\, A_{qf}(f,g,n)\big) \\ \textit{then one can extract a closed term } \Phi \textit{ of } \widehat{\text{E-PA}}^\omega\!\upharpoonright \textit{ such that} \\ \widehat{\text{E-PA}}^\omega\!\upharpoonright \vdash \forall f\, \forall g \le \varphi(f)\, \exists n \le \Phi(f)\, A_{qf}(f,g,n). \end{cases}$$

Here '$g \le h$' for functions g, h is defined pointwise; that is, $\forall x\, g(x) \le h(x)$.

Proof As in the proof of Corollary 4.5 in [16], we can replace WKL by the principle F^- and then use elimination of extensionality (see, e.g., [17], the restrictions on the types in QF-AC are made precisely to allow for this) to obtain

$$(\text{G}_\infty\text{A}^\omega + \text{QF-AC}) \oplus F^- \vdash \forall f\, \forall g \le \varphi(f)\, \big(\Pi^0_1\text{-CA}(\psi(f,g)) \to \exists n A_{qf}(f,g,n)\big).$$

Then apply Corollary 4.4 (for $\Delta := \varnothing$) and note that for $k := 1$ the conclusion can be verified in (even the weakly extensional and intuitionistic version of) $\widehat{\text{E-PA}}^\omega\!\upharpoonright$. □

In the following in expressions like '$b \le 1$' by '1' we denote the constant-1 function.

Remark 2.2 The instance of Π^0_1-comprehension in Proposition 2.1 may also depend on the results of instances of WKL: WKL$(\tau(f))$ is implied by $\exists b \le 1\, \forall x (\tilde{\tau}(f)(\bar{b}x) = 0)$ for a suitable term $\tilde{\tau}$ in E-G$_\infty$A$^\omega$, with

$$\text{E-G}_\infty\text{A}^\omega \vdash \forall f, x^*\, \exists b \le 1\, \forall x \le x^*\ (\tilde{\tau}(f)(\bar{b}x) = 0);$$

see [17, Proposition 9.18] (note that the g in the proof of this proposition is definable in E-G$_\infty$A$^\omega$). Suppose now that E-G$_\infty$A$^\omega$+ QF-AC + WKL proves

$$\forall f\, \forall g \le \varphi(f)$$
$$\forall b \le 1\, \Big(\forall x\, \tilde{\tau}(f)(\bar{b}x) = 0 \to \Big(\Pi^0_1\text{-CA}(\xi(f,b)) \to \exists n\, A_{qf}(f,n)\Big)\Big),$$

which is equivalent to

$$\forall f\, \forall g \le \varphi(f)$$
$$\forall b \le 1\, \Big(\Pi^0_1\text{-CA}(\xi(f,b)) \to \exists n, x\, \big(\tilde{\tau}(f)(\bar{b}x) = 0 \to A_{qf}(f,g,n)\big)\Big).$$

Applying Proposition 2.1 yields bounds $x^* := \chi(f)$ and $n^* := \Phi(f)$ on x and n depending only on f; that is,

$$\widehat{\text{E-PA}}^\omega\!\upharpoonright \vdash \forall f\, \forall g \le \varphi(f)\, (\exists b \le 1\, \forall x \le \chi(f)\, (\tilde{\tau}(f)(\bar{b}x) = 0)$$
$$\to \exists n \le \Phi(f)\, A_{qf}(f,g,n))$$

and so, finally,

$$\widehat{\text{E-PA}}^\omega\!\upharpoonright \vdash \forall f\, \forall g \le \varphi(f)\, \exists n \le \Phi(f)\, A_{qf}(f,g,n).$$

Instead of fixed instances of Π^0_1-CA also fixed sequences of such instances, that is, fixed instances of

$$\Pi^0_1\text{-CA}^*(f) :\equiv \forall l \exists g \forall x \big(g(x) = 0 \leftrightarrow \forall y (f(l,x,y) = 0)\big)$$

are covered since (provably in E-G$_\infty$A$^\omega$)

$$\Pi^0_1\text{-CA}(\varphi(f)) \to \Pi^0_1\text{-CA}^*(f),$$

where $\varphi(f) := f(j_1 x, j_2 x, y)$ for some unpairing functions j_1, j_2.

We now consider sequences of Π^0_1*-instances of countable choice for numbers:*

$$\Pi^0_1\text{-AC}(f) :\equiv \forall l \big(\forall x \exists y \forall z (f(l,x,y,z) = 0) \to \exists g \forall x, z (f(l,x,g(x),z) = 0)\big).$$

Π^0_1-AC(f) can be reduced to Π^0_1-CA(g) uniformly by the following.

Proposition 2.3 ([15])

$$\text{E-G}_\infty\text{A}^\omega + \text{QF-AC}^{\mathbb{N},\mathbb{N}} \vdash \forall f \, \big(\Pi^0_1\text{-CA}(\varphi(f)) \to \Pi^0_1\text{-AC}(f)\big)$$

for a suitable elementary functional φ*.*

Similarly, one has the following.

Proposition 2.4 ([16])

$$\text{E-G}_\infty\text{A}^\omega + \text{QF-AC}^{\mathbb{N},\mathbb{N}} \vdash \forall f, g \, \big(\Pi^0_1\text{-CA}(\varphi(f,g)) \to \Delta^0_2\text{-CA}(f,g)\big)$$

for a suitable φ*, where*

$$\Delta^0_2\text{-CA}(f,g) :\equiv \begin{cases} \forall l \, \big(\forall x ([\forall u \exists v \, (f(l,x,u,v) = 0) \\ \qquad \leftrightarrow \exists m \forall n (g(l,x,m,n) = 0)]) \\ \to \exists h \forall x (h(x) = 0 \leftrightarrow \forall u \exists v (f(l,x,u,v) = 0))\big). \end{cases}$$

As a consequence of Propositions 2.3 and 2.4 we obtain the following.

Proposition 2.5 ([16]) *Proposition 2.1 also holds with* Π^0_1-AC$(\chi(f,g))$ *and* Δ^0_2-CA$(\zeta_1(f,g), \zeta_2(f,g))$ *in addition to* Π^0_1-CA$(\psi(f,g))$ *(and likewise for sequences of instances of* Δ^0_2-IA *and* Π^0_1-CP*).*

With the restriction P^- of second-order principles P to instances with at most number parameters as discussed in Section 1 we can formulate the next proposition which follows (as special case for $k := 1$) from Corollaries 4.8 and 4.10 in [16].

Proposition 2.6 ([16]) E-G$_\infty$A$^\omega$+QF-AC+WKL+Δ^0_2-CA$^-$+Π^0_1-AC$^-$ *is* Π^0_3*-conservative over* PRA+Σ^0_1-IA *and* Π^0_4*-conservative over* PRA+Π^0_1-CP.

3 Trees and König's Lemma

Definition 3.1 (Tree)

1. A partial order on the natural numbers $\prec$ is called *tree* if for every $x \in \mathbb{N}$ the set of all predecessors $pd(x) := \{y \in \mathbb{N} \mid y \prec x\}$ is well-ordered.
2. A maximal linear order in $\prec$ is called *branch*.
3. A tree $\prec$ is called *finitely branching* if for all $x \in \mathbb{N}$ the set of all immediate successors $\text{succ}(x) := \{y \in \mathbb{N}, x \prec y \wedge (\neg \exists z \, (x \prec z \wedge z \prec y)\}$ is finite.
   A tree is called *n-branching* if $|\text{succ}(x)| \leq n$ for all $x \in \mathbb{N}$.

Definition 3.2 (König's Lemma) König's Lemma is the statement that every infinite, finitely branching tree contains an infinite branch.

3.1 Fragments of König's Lemma and formalizations We formalize trees as characteristic functions of finite, initial segments of branches in a tree; that is, a tree $\prec$ is described by f if

$$\begin{aligned} f(\langle\rangle) &= 0 \\ f(\langle x \rangle) &= 0 \quad \text{iff} \quad x \text{ is } \prec\text{-minimal} \\ f(\langle n_1, \ldots, n_k, x \rangle) &= 0 \quad \text{iff} \quad f(\langle n_1, \ldots, n_k \rangle) = 0 \text{ and } x \in \text{succ}_\prec(n_k). \end{aligned}$$

We define $*$, $\langle\,\rangle$, $\bar{b}$ using a suitable surjective sequence coding; for details see [17].

Definition 3.3 (Weak König's Lemma $\mathrm{WKL}(\varphi)$**)**

$$\mathrm{WKL}(\varphi)\colon\ T(\varphi) \wedge \forall x^0\, \exists s^0\ (lth(s) = x \wedge \varphi(s)) \to \exists b \le 1\, \forall x\, \varphi(\bar{b}x),$$

where T asserts that φ describes a 0, 1-tree with respect to the prefix relation $\sqsubseteq$

$$T(\varphi) :\equiv \forall s, r\ (\varphi(s * r) \to \varphi(s)) \wedge \forall s, x\ (\varphi(s * \langle x\rangle) \to x \le 1)\,.$$

Definition 3.4 (Bounded König's Lemma $(\mathrm{WKL}^*(\varphi, h))$**)**

$$\mathrm{WKL}^*(\varphi, h)\colon\ T^*(\varphi, h) \wedge \forall x^0\, \exists s^0\ (lth(s) = x \wedge \varphi(s)) \to \exists b \le h\, \forall x\, \varphi(\bar{b}x),$$

where T^* asserts that φ describes a tree bounded by h

$$T^*(\varphi, h) :\equiv \forall s, r\ (\varphi(s * r) \to \varphi(s)) \wedge \forall s, x\ (\varphi(s * \langle x\rangle) \to x \le h(lth(s)))\,.$$

We denote by Σ^0_1-$\mathrm{WKL}(f)$, respectively, Σ^0_1-$\mathrm{WKL}^*(f)$ weak/bounded König's Lemma with $\varphi(s) \equiv \exists z\, f(z, s) = 0$ and Σ^0_1-$\mathrm{WKL}^{(*)} :\equiv \forall f\ \Sigma^0_1$-$\mathrm{WKL}^{(*)}(f)$.

Proposition 3.5 *In* $\text{E-G}_\infty\text{A}^\omega$ *every instance of bounded König's Lemma is equivalent to an instance of weak König's Lemma* (WKL). *Moreover, every instance of* Σ^0_1-WKL^* *can be proven from an instance of* Σ^0_1-WKL.

Proof Simpson proves this equivalence in the system RCA_0 in [21, IV.1.3]. This proof can be carried out in $\text{E-G}_\infty\text{A}^\omega$. For $\varphi \in \Sigma^0_1$, this property is preserved. □

Remark 3.6

$$\text{E-G}_\infty\text{A}^\omega + \text{QF-AC}^{\mathbb{N},\mathbb{N}} \vdash \Sigma^0_1\text{-WKL}(\xi(f)) \to \Pi^0_1\text{-CA}(f),$$

since Σ^0_1-$\mathrm{WKL}(\sigma(g))$ implies Π^0_2-$\mathrm{WKL}(g)$ for a suitable term σ; see [14, Proposition 3.3] or [21, proof of Lemma IV.4.4] and note that $\text{E-G}_\infty\text{A}^\omega + \text{QF-AC}^{\mathbb{N},\mathbb{N}}$ proves Σ^0_1-CP. Π^0_2-$\mathrm{WKL}(\tau f)$ implies Π^0_1-$\mathrm{CA}(f)$; see Troelstra [23, §5].

Combined with the discussion at the end of Section 1, it follows that over $\text{E-G}_\infty\text{A}^\omega + \text{QF-AC}^{\mathbb{N},\mathbb{N}} + \text{WKL}$ each instance of Σ^0_1-WKL is equivalent to an instance of Π^0_1-CA and vice versa.

4 Ramsey's Theorem

Now we turn to Ramsey's Theorem for pairs. In this section we will present two proofs of it. The first proof is the standard textbook proof (see Graham et al. [6]); the second is due to Erdős and Rado [4, 10.2].

Definition 4.1

1. $[X]^k := \{Y \subseteq X \mid |Y| = k\}$.
2. An n*-coloring* c of $[X]^k$ is a map $c\colon [X]^k \to \mathbf{n}$.
3. A set $H \subseteq X$ is called *monochromatic* under c if c is constant on $[H]^k$.
4. Let $(X, \prec)$ be a partial order and c an n-coloring of $[X]^2$. A set $H \subseteq X$ is called *min-monochromatic* under c if for all $i \in H$ the map $c_i(x) := c(\{i, x\})$ is constant on $\{x \in H : i \prec x\}$.

Definition 4.2 (Ramsey's Theorem [19]) For all k, n and every n-coloring c of $[\mathbb{N}]^k$ exists an infinite set $H \subseteq \mathbb{N}$ such that H is monochromatic under c. RT^k_n denotes Ramsey's Theorem for n-colorings of $[\mathbb{N}]^k$ and, $\mathrm{RT}^k_{<\infty}$ is defined as $\forall n\, \mathrm{RT}^k_n$.

The proofs we are going to present share the same structure. First an infinite min-monochromatic set is constructed. Then using RT^1_n one finds an infinite monochromatic set.

The textbook proof is simpler and seemingly elementary, but it cannot even be formalized in ACA_0; see [21, p. 123]. Therefore, this proof is unusable for a detailed analysis of the proof-theoretic strength of RT^2_n.

Erdős' and Rado's proof can be formalized in ACA_0 (see [21, Lemma III.7.4]). It uses König's Lemma, which is open for a detailed analysis in this case.

Textbook proof Fix an n-coloring $c\colon [\mathbb{N}]^2 \to \mathbf{n}$. We construct an enumeration $(x_j)_{j\in\mathbb{N}}$ of an infinite min-monochromatic set. Define $c_y(x) := c(\{y,x\})$.

- Set $x_0 := 0$.
- Using RT^1_n we find an infinite set $X_1 \subseteq \mathbb{N} \setminus \{x_0\}$ such that X_1 is monochromatic under c_0. Set $x_1 := \min X_1$.
- Similarly, we find an infinite set $X_2 \subseteq X_1 \setminus \{x_1\}$ such that X_2 is monochromatic under c_{x_1}. Set $x_2 := \min X_2$.
- $\vdots$

Iterating this process gives a sequence $(x_j)_{j\in\mathbb{N}}$. By construction $X := \{x_0, x_1, \dots\}$ is min-monochromatic under c.

Define $c' : X \to n$ with $c'(x_j) := c(\{x_j, x_{j+1}\})$. c' is well defined since the sequence $(x_j)_j$ is injective. Using RT^1_n we find an infinite $H \subseteq X$ such that H is monochromatic under c'. Since H is min-monochromatic under c, we get for all $x, y \in H$, $x < y$

$$c(\{x,y\}) = c'(x) = c'(H).$$

In other words, H is monochromatic under c. □

Erdős' and Rado's Proof[2]

Fix an n-coloring $c : [\mathbb{N}]^2 \to \mathbf{n}$. Let $c_k\colon \mathbf{k} \to \mathbf{n}$ be defined as $x \mapsto c(\{x,k\})$. Now define recursively a partial order $\prec$ on $\mathbb{N}$:

- $0 \prec 1$
- If $\prec$ is already defined on $\mathbf{m}$, then let

$$P_k := \left\{x \in \mathbf{m} \mid x \prec k\right\} \quad \text{for } k \in \mathbf{m}.$$

Now, to extend $\prec$ to $\mathbf{m+1}$, for $k \in \mathbf{m}$ set

$$k \prec m \quad \text{iff} \quad c_k|_{P_k} = c_m|_{P_k}.$$

Claim

(i) $\prec \subseteq <_{\mathbb{N}}$, in particular, $P_k = pd(k)$.
(ii) $0 \prec x$ for all $x \in \mathbb{N} \setminus \{0\}$.
(iii) $\prec$ is transitive.
(iv) On $pd(m)$ the relations $<_{\mathbb{N}}$ and $\prec$ describe the same order; that is, for $x, y \in pd(m)$

$$x < y \quad \text{iff} \quad x \prec y.$$

(i) By definition, $k \prec m$ implies $k < m$. So $P_k = \{x \in \mathbf{m} \mid x \prec k\}$ is independent from the choice of m as long $k \in \mathbf{m}$ and also $P_k = \{x \in \mathbb{N} \mid x \prec k\} = pd(k)$.

(ii) Follows immediately from the definition of $\prec$.

(iii) We prove the statement ($x \prec y$ and $y \prec z$) $\Rightarrow x \prec z$ by induction on z. The base case $z = 0$ is trivial because of (i). Assume that transitivity holds for all $z' < z$. Then

$$\begin{aligned} x \prec y \text{ and } y \prec z &\Rightarrow c_x|_{P_x} = c_y|_{P_x}, c_y|_{P_y} = c_z|_{P_y} \\ &\qquad \text{and } P_x \subseteq P_y \qquad \text{(induction hypothesis for } y < z) \\ &\Rightarrow c_x|_{P_x} = c_y|_{P_x} = c_z|_{P_x} \\ &\Rightarrow x \prec z. \end{aligned}$$

(iv) $\Leftarrow$: follows from (i).

$\Rightarrow$: By (ii) the case $x = 0$ is trivial. Let $x \neq 0$. Proof by induction on m:

- $m = 0$ is obvious.
- Let $m > 0$, $x, y \in pd(m)$ with $x < y$ and assume the statement holds for all $m' < m$. Let i be the $<$-maximal natural number such that $i \prec x$ and $i \prec y$ (such an i exists because of $0 \prec x, y$ by (ii)). Let p be an immediate $\prec$-successor of i comparable with m (such a p exists because of $i \prec x \prec m$). From $i \prec y \prec m$ and $i \prec p \prec m$ we get

$$c_y(i) = c_m(i) = c_p(i).$$

  Using the induction hypothesis for $m' = p$, we deduce that all $i' \prec p$ are comparable with i, in particular, $p \in \text{succ}(i)$ and

$$P_p = P_i \cup \{i\}.$$

  Since $i \prec y$ and $c_y(i) = c_p(i)$, this shows $p \prec y$ (the case $p = y$ is impossible). Analogously, it follows that $p \prec x$ or $p = x$. The maximality of i renders the case $p \prec x$ impossible, so $p = x$ and, in particular, $x \prec y$.

By (iv) the relation $\prec$ defines a tree on $\mathbb{N}$. By definition, every branch of $\prec$ is min-monochromatic under c. The tree is n-branching (in particular, finitely branching) since for all $x, y \in \text{succ}(i)$ such that $x < y$ the induced colorings c_x and c_y must differ at i. Otherwise, $x \prec y$ since $c_x|_{P_i} = c_y|_{P_i}$ and $P_x = P_i \cup \{i\}$.

By König's Lemma we find an infinite min-monochromatic branch B. As in Ramsey's proof, we construct using RT^1_n an infinite monochromatic set H under c. □

Note that we cannot simply reduce the application of König's Lemma in this proof (or in Simpson's proof [21, Lemma III.4.7]) to WKL using an instance of Π^0_1-AC since we need Σ^0_1-IA depending on the result of a Π^0_1-AC application to prove that there is a bounding function on the labels of the tree needed to apply WKL (respectively, WKL*). Such a bounding function on the labels of the tree can be constructed using only fixed instances of Σ^0_1-IA, Π^0_1-IA, and Π^0_1-AC, but this construction depends crucially on the special structure of the Erdős-Rado-tree (see [18]). However, we will follow here a slightly different approach.

4.1 Formalized proof of RT^2_n In the following we formalize the proof of Erdős and Rado and show Theorem 1.2. We prove this theorem for every fixed number $n \geq 2$ of colors, since the usual proof of the equivalence between RT^2_2 and RT^2_n needs nested applications of RT^2_2 and therefore cannot be formalized using only RT^2_2.

In E-$G_\infty A^\omega$ we represent an n-coloring $c\colon [\mathbb{N}]^2 \to \mathbf{n}$ using a mapping $\hat{c}\colon \mathbb{N}\times\mathbb{N} \to \mathbf{n}$ such that $\hat{c}(x,y) = \hat{c}(y,x) = c(\{x,y\})$. We formalize RT^2_n as follows.

$$(\mathrm{RT}^2_n)\colon \forall c\colon \mathbb{N}\times\mathbb{N} \to \mathbf{n}\,\exists f \le 1\,\exists i < n\,\big(\forall k\,\exists x > k\; f(x) = 0 \\ \wedge \forall x, y\,(x \ne y \wedge f(x) = 0 \wedge f(y) = 0 \to \hat{c}(x,y) = i)\big),$$

where $\hat{c}(x,y) = \begin{cases} c(x,y) & x \le y, \\ c(y,x) & x > y. \end{cases}$

RT^2_n expresses that f is the characteristic function of an infinite set in which every (unordered) pair $\{x,y\}$ is mapped to the color i.

$\mathrm{RT}^2_n(t)$ denotes RT^2_n for a fixed coloring t, RT^{2-}_n denotes the set of all instances of $\mathrm{RT}^2_n(t)$, where the only free variables of t are of degree 0, that is, number variables. We omit n when no confusion can arise.

We now formalize the claims (i)–(iv) from Erdős' and Rado's proof.

Lemma 4.3 *For every coloring $c : \mathbb{N}\times\mathbb{N} \to \mathbf{n}$ the partial order $\prec$ as in the proof of Erdős and Rado can be defined in* E-$G_\infty A^\omega$. E-$G_\infty A^\omega$ *proves that $\prec$-chains are min-monochromatic and the properties (i)–(iv); that is,*

(i) $\forall x, y\ (x \prec y \to x < y)$,
(ii) $\forall x > 0\ (0 \prec x)$,
(iii) $\forall x, y, z\ (x \prec y \wedge y \prec z \to x \prec z)$,
(iv) $\forall m, x, y\ (y \prec m \to (x \prec y \leftrightarrow x \prec m \wedge x < y))$.

Proof We may assume $c(x,y) = c(y,x)$. Define

$$\begin{aligned} \tilde{q}(0) &:= \langle\rangle, \\ \tilde{q}(1) &:= \langle 0\rangle, \\ \tilde{q}(m+1) &:= \langle q_0^{m+1}, \ldots, q_m^{m+1}\rangle, \end{aligned}$$

where $q_k^{m+1} := \begin{cases} 0, & \text{if } \forall x < k\ ((\tilde{q}(k))_x = 0 \to c(k,x) = c(m+1,x))\,, \\ 1, & \text{otherwise} \end{cases}$

for $k := 0, \ldots, m$. By definition $\tilde{q}(m) \le \overline{1}(m)$, where '$\overline{1}m$' is the code of the initial segment of the constant-1 function of length m.

The mapping

$$q(x,y) := \begin{cases} \tilde{q}(y)_x & x < y \\ 1 & x \ge y \end{cases}$$

is the characteristic function of $\prec$. Hence the relation $\prec$ can be defined with elementary recursion and so, in particular, in E-$G_\infty A^\omega$. Set

$$x \prec y :\equiv q(x,y) = 0.$$

(i), (ii) immediately follow from the definition of $\prec$, respectively, the mapping q.

(iii) is (using (i)) equivalent to

$$\forall z\,\forall y < z, x < y\ (x \prec y \wedge y \prec z \to x \prec z)\,.$$

We prove this statement using quantifier-free course-of-value induction on z. The base case is trivial. Assume that the statement holds for $z' < z$.

$$
\begin{aligned}
& x \prec y \wedge y \prec z \\
& \to \big[(\forall i < x\,(i \prec x \to c(x,i) = c(y,i))) \wedge (\forall i < y\,(i \prec y \to c(y,i) = c(z,i)))\big]
\end{aligned}
$$

using induction hypothesis for $y < z$

$$
\begin{aligned}
& \to \big[(\forall i < x\,(i \prec x \to c(x,i) = c(y,i))) \wedge (\forall i < y\,(i \prec x \to c(y,i) = c(z,i)))\big] \\
& \to [\forall i < x\,(i \prec x \to c(x,i) = c(z,i))] \\
& \to x \prec z.
\end{aligned}
$$

(iv) The $\to$-direction follows from (i) and (iii).

The $\leftarrow$-direction is (using (i)) equivalent to

$$\forall m\, \forall x < m, y < m\ (x \prec m \wedge y \prec m \wedge x < y \to x \prec y)\,.$$

We prove this statement using quantifier-free course-of-value induction on m. The base step is trivial. Assume that the statement holds for all $m' < m$. For $x = 0$ the statement is obvious. Hence we assume $x \neq 0$. Let $x \prec m$, $y \prec m$, and $x < y$.

$$
\begin{aligned}
& x \neq 0 \\
& \xrightarrow{\text{(ii)}} \exists i < x\,(i \prec x \wedge i \prec y) \qquad \text{(e.g., } i = 0) \\
& \xrightarrow{\mu_b} \exists i < x\,\big(i \prec x \wedge i \prec y \wedge \underbrace{\forall i' < x\,((i' \prec x \wedge i' \prec y) \to i' \le i)}_{\equiv: i \text{ maximal}}\big) \\
& \to \exists i < x\,\big(i \prec x \wedge i \prec y \wedge i \text{ maximal} \wedge \exists p < m\,(p \prec m \wedge i \prec p)\big) \text{ (e.g., } p = x) \\
& \xrightarrow{\mu_b} \exists i < x\,\big(i \prec x \wedge i \prec y \wedge i \text{ maximal} \\
& \qquad \wedge \exists p < m\,(\underbrace{p \prec m \wedge i \prec p \wedge \forall p' < m\,(p' \prec m \wedge i \prec p' \to p' \ge p)}_{p \text{ minimal with } i \prec p \prec m})\big).
\end{aligned}
$$

Using (iii), we deduce $i \prec m$. Since $y \prec m$ and $i \prec y$, this gives $c(y,i) = c(m,i)$. From $p \prec m$ and $i \prec p$ it follows $c(p,i) = c(m,i)$. Therefore,

$$c(y,i) = c(p,i). \tag{1}$$

From the induction hypothesis (for p) and (i) we obtain

$$\forall j, j'\ \big((j \prec p \wedge j' \prec p) \to (j \prec j' \leftrightarrow j < j')\big)\,. \tag{2}$$

We claim that p is an immediate successor of i. In other words, no i' exists such that $i \prec i' \prec p$. Suppose such an i' exists. Then (iii) gives $i \prec i' \prec m$. As p is minimal with this property we get $i' \ge p$. This contradicts (together with (i)) the assumption $i' \prec p$.

Combining this with (2), we see

$$\forall i'\ \big(i' \prec p \to (i' = i \vee i' \prec i)\big)\,. \tag{3}$$

Since $i \prec y, p$ we get $c(p,i') = c(i,i') = c(y,i')$ for all $i' \prec i$. This, (1), and (3) shows $c(p,i') = c(y,i')$ for all $i' \prec p$ and, in particular, $p \prec y$ ($p = y$ is impossible because of $p \le x < y$). This implies

$$\exists i < x\,\big(i \prec x \wedge i \prec y \wedge i \text{ maximal} \wedge \exists p < m\,(p \prec y \wedge p \in \mathrm{succ}(i))\big).$$

Analogously, we deduce $p = x$. Here the maximality of i renders the case $p \prec x$ impossible. Put together, we obtain

$$\exists i < x\,(i \prec x \wedge i \prec y \wedge i \text{ maximal} \wedge \exists p < m\,(p \prec y \wedge p = x)$$

and so $x \prec y$. □

We now proceed by formalizing the construction of the infinite min-monochromatic set through König's Lemma.

Lemma 4.4 *For every fixed $n \geq 2$ there are closed terms ξ_1 and ξ_2 such that*

$$\begin{aligned}\text{E-G}_\infty\text{A}^\omega + \text{QF-AC}^{\mathbb{N},\mathbb{N}} \vdash \forall c \colon \mathbb{N} \times \mathbb{N} \to \mathbf{n}\, \big(&\Pi^0_1\text{-CA}(\xi_1 c) \wedge \Sigma^0_1\text{-WKL}(\xi_2 c) \\ &\to \exists b\,(b0 = 0 \wedge \forall i\, b(i+1) \in \text{succ}(bi))\big).\end{aligned}$$

Proof We prove the existence of b using two instances of Π^0_1-induction ζ_1, ζ_2, and Σ^0_1-WKL$(\xi_2 c)$. These instances of Π^0_1-induction follow then from corresponding instances of Π^0_1-comprehension, which can be coded together into Π^0_1-CA$(\xi_1 c)$ for a suitable ξ_1. Notation as in the proof of the preceding lemma.

Define

$$\begin{aligned} h(0, q, r) &:= \langle\rangle = 0 \\ h(m+1, q, r) &:= h(m, q, r) * \begin{cases} \langle (r)_{m+1} \rangle & \text{if } (q)_{m+1} = 0 \\ \langle\rangle = 0 & \text{else} \end{cases} \\ h(k, q, r) &\leq r \\ g(m) &:= h(m, \Phi_{\langle\rangle}(\lambda x.q(x, m), m), \Phi_{\langle\rangle}(\lambda x.c(x, m), m)). \end{aligned}$$

Here for a binary function $f(x, y)$, $\lambda x.f(x, y)$ denotes the function in x with fixed y.

The function h deletes the entries i in c, where $(q)_i \neq 0$ holds. Hence $g(m) = \langle c(m, i_0), \ldots, c(m, i_k)\rangle$, where $i_0 \prec i_1 \prec \cdots \prec i_k$ are the predecessors of m ordered by $\prec$. Note h and g can be defined in E-G$_\infty$A$^\omega$.

By definition of g,

$$(g(x))_i < n, \tag{4}$$
$$x \prec y \to g(x) \sqsubset g(y). \tag{5}$$

We deduce

$$\begin{aligned} & g(z) = m * \langle x \rangle \\ \xrightarrow{\mu_b} & \exists v < z\,\big(g(z) = m * \langle x \rangle \wedge \underbrace{v \prec z \wedge \forall v' < z\,(v' \prec z \to v' \leq v)}_{v \text{ maximal with } v \prec z}\big) \\ \to & \exists v < z\,\big(g(z) = m * \langle x \rangle \wedge z \in \text{succ}(v)\big) \\ \xrightarrow{\text{(iv)}} & \exists v < z\,\big(g(z) = m * \langle x \rangle \wedge z \in \text{succ}(v) \wedge \forall x < v\,(x \prec v \leftrightarrow x \prec z)\big) \\ \to & \exists v < z\,\big(g(z) = m * \langle x \rangle \wedge z \in \text{succ}(v) \wedge \tilde{q}(v) \sqsubset \tilde{q}(z)\big) \end{aligned}$$

since v is maximal with $v \prec z$, (i) yields $(\tilde{q}(z))_i = 0$ for all $i \in \{v+1, \ldots, z-1\}$. This gives us

$$\exists v < z\,\big(g(z) = m * \langle x \rangle \wedge z \in \text{succ}(v) \wedge g(v) = m\big)$$

and, in particular,

$$\exists v \ (g(v) = m \wedge z \in \mathrm{succ}(v)) .$$

We conclude

$$\forall n, x, z \left(g(z) = m * \langle x \rangle \to \exists v < z \left(g(v) = m \wedge z \in \mathrm{succ}(v)\right)\right). \tag{6}$$

We proceed to prove that g is injective by showing

$$\forall l \, \forall x, y \ (x \neq y \wedge lth(g(x)) = l \to g(x) \neq g(y)) \tag{7}$$

using Π^0_1-induction on l. Note that the induction formula can be written as Π^0_1-IA$(\zeta_1 c)$ for a suitable ζ_1.

The base case is an immediate consequence of (ii) and the definition of g. Assume that (7) holds for l.

$$\exists x, y \, (x \neq y \wedge lth(g(x)) = l + 1 \wedge g(x) = g(y))$$

$$\xrightarrow{(6)} \exists x, y \, \exists x', y' \, (x \neq y \wedge lth(g(x)) = l + 1 \wedge x \in \mathrm{succ}(x') \wedge y \in \mathrm{succ}(y') \wedge$$
$$g(x) = g(y) \wedge g(x') \sqsubset g(x) \wedge lth(g(x')) = l \wedge g(y') \sqsubset g(y) \wedge lth(g(y')) = l)$$

$$\xrightarrow{\mathrm{IH}} \exists x, y \, \exists x' \, (x \neq y \wedge lth(g(x) = l + 1 \wedge x, y \in \mathrm{succ}(x') \wedge g(x) = g(y)).$$

Since x and y are immediate successors of x' and $c(x, x') = (g(x))_l = (g(y))_l = c(y, x')$, either x, y are equal or comparable. The former case contradicts our assumption, the latter together with (5) the fact that $g(x) = g(y)$. This finishes the proof of the injectivity of g.

The injectivity of g together with (6) yields

$$\forall z, v \ (g(z) = m * \langle x \rangle \wedge g(v) = m \to v \prec z) .$$

Using Π^0_1-induction and Lemma 4.3(iii), we conclude

$$\forall l \, \forall x, y \, (lth(g(y)) = l \wedge g(x) \sqsubset g(y) \to x \prec y).$$

Since g is definable in terms of E-G$_\infty$A$^\omega$ and c, the induction formula can be written as Π^0_1-IA$(\zeta_2 c)$ for a suitable term ζ_2. Together with (5), this gives us

$$x \prec y \leftrightarrow g(x) \sqsubset g(y). \tag{8}$$

Using (6), it is clear that

$$\xi_2(c, x, s) :\equiv g(x) = s$$

defines a Σ^0_1-tree bounded by the constant-n function. By definition the tree is the image of g. As g is an injection from the natural numbers the tree is infinite. Applying Σ^0_1-WKL$(\xi_2 c)$ yields a branch b' with

$$\forall i \, \exists x \, g(x) = \bar{b}'i,$$

$$\xrightarrow{\text{QF-AC}} \exists b \, \forall i \, g(bi) = \bar{b}'i.$$

This and (8) establishes the lemma. □

Theorem 4.5 *For each fixed $n \geq 2$ there exists a closed term ξ such that*

$$\text{E-G}_\infty\text{A}^\omega + \text{QF-AC}^{\mathbb{N},\mathbb{N}} + \text{WKL} \vdash \forall c \colon \mathbb{N} \times \mathbb{N} \to \mathbf{n} \left(\Pi^0_1\text{-CA}(\xi c) \to \text{RT}^2_n(c)\right).$$

Proof Clearly, in $\text{E-G}_\infty\text{A}^\omega + \text{WKL}$, for every closed term ξ there is a closed term ξ' such that

$$\Pi^0_1\text{-CA}(\xi' c) \to \Sigma^0_1\text{-WKL}(\xi c).$$

By coding two instances of comprehension into a single instance we obtain a closed term ξ satisfying for the terms ξ_1 and ξ_2 from Lemma 4.4,

$$\Pi^0_1\text{-CA}(\xi c) \to \left(\Pi^0_1\text{-CA}(\xi_1 c) \wedge \Sigma^0_1\text{-WKL}(\xi_2 c)\right).$$

Lemma 4.4 gives us now an infinite branch b of the Erdős-Rado tree $\prec$. By definition of $\prec$, $b(\mathbb{N})$ is min-monochromatic under c and

$$\forall x, y\ (x < y \leftrightarrow x \prec y).$$

Define $c'(x) := c(bx, b(x+1))$. Since $b(\mathbb{N})$ is min-monochromatic, we get

$$\forall x\, \forall y > x\ \left(c(bx, by) = c'x\right).$$

By RT^1_n there exists a color i occurring infinitely often. The set $H := \{bx \mid c'x = i\}$ is infinite and monochromatic under c, so $b^*k := t_{\exists x \leq k\, bx = k \wedge c'k = i}[k]$ forms a solution of RT^2_n. □

Remark 4.6 Using the tuple coding from Section 2 it is obvious that, for a suitable ξ, $\Pi^0_1\text{-CA}(\xi((c_k)_k))$ proves a sequence of instances $(\text{RT}^2_n(c_k))_{k\in\mathbb{N}}$.

Note that the number of colors in such a sequence of instances of RT^2_n has to be bounded. For an unbounded number of colors we would need $\text{RT}^1_{<\infty}$ in the proof of Theorem 4.5. But as $\text{RT}^1_{<\infty}$ is equivalent to $\Pi^0_1\text{-CP}$ (see [9]) it is not provable in $\text{E-G}_\infty\text{A}^\omega$.

5 Results

Using Theorem 4.5 we can extend the theorems of Section 2 by adding RT^2_n.

Theorem 5.1 *Let $A_{qf}(f, g, k) \in \mathcal{L}(\text{E-G}_\infty\text{A}^\omega)$ be a quantifier-free formula which contains only the variables f, g, k free. Furthermore, let φ, ψ, χ be functionals (of suitable type) that are definable in $\text{E-G}_\infty\text{A}^\omega$. Then for every fixed $n \geq 2$ the following rule holds:*

$$\begin{cases} \text{E-G}_\infty\text{A}^\omega + \text{QF-AC} + \text{WKL} \\ \quad \vdash \forall f \forall g \leq \varphi(f)\big(\Pi^0_1\text{-CA}(\psi(f,g)) \wedge \forall l\, \text{RT}^2_n(\chi(f,g,l)) \to \exists k A_{qf}(f,g,k)\big) \\ \textit{then one can extract a closed term } \Phi \textit{ of } \widehat{\text{E-PA}}^\omega{\restriction} \textit{ such that} \\ \widehat{\text{E-PA}}^\omega{\restriction} \vdash \forall f\, \forall g \leq \varphi(f)\, \exists k \leq \Phi(f)\, A_{qf}(f,g,k). \end{cases}$$

Proof Proposition 2.1, Theorem 4.5, and Remark 4.6. □

Theorem 5.2 *Let $\mathcal{S} := \text{E-G}_\infty\text{A}^\omega + \text{QF-AC} + \text{WKL} + \Delta^0_2\text{-CA}^- + \Pi^0_1\text{-AC}^-$ and n fixed. Then the following hold:*

(i) $\mathcal{S} + \text{RT}^{2-}_n$ *is Π^0_2-conservative over* PRA,

(ii) $\mathcal{S} + \text{RT}^{2-}_n$ *is Π^0_3-conservative over* $\text{PRA} + \Sigma^0_1\text{-IA}$,

(iii) $\mathcal{S} + \text{RT}^{2-}_n$ *is Π^0_4-conservative over* $\text{PRA} + \Pi^0_1\text{-CP}$.

Proof (i) follows from Corollary 1.4 and Theorem 4.5. (ii) and (iii) follow from Proposition 2.6 and Theorem 4.5. □

The bound in (ii) is sharp. Avigad constructed in [1] a Σ^0_3-sentence provable from Π^0_1-CP$^-$ and hence from Π^0_1-AC$^-$ that is not provable in PRA + Σ^0_1-IA. These theorems cannot be extended to $\mathrm{RT}^2_{<\infty}$.

Proposition 5.3

$$\text{E-G}_\infty\text{A}^\omega + \text{QF-AC} + \text{WKL} + \Delta^0_2\text{-CA}^- + \Pi^0_1\text{-AC}^- \nvdash \mathrm{RT}^{2-}_{<\infty}.$$

Proof A sequence of instances of $\mathrm{RT}^2_{<\infty}$ with unbounded number of colors is sufficient to prove the totality of (a version of) the Ackermann function; see [9, 6.12]. All instances of $\mathrm{RT}^2_{<\infty}$ in the proof of this theorem are of the form $\mathrm{RT}^{2-}_{<\infty}$. Since the diagonal of the Ackermann Function cannot be primitive recursively bounded, the theorem follows from Theorem 1.3 and Propositions 2.3 and 2.4. □

Remark 5.4 Our formalization of the proof of RT^2_n also can be used to analyze the complexity of RT^2_n relative to the comprehension used (in our case Σ^0_1-WKL) like Bellin did in [2] using Ramsey's proof. The proof of Lemma 4.4 yields the concrete instance of the comprehension needed as an elementary functional in the coloring c (namely, the term ξ_3 derived from the construction of the Erdős-Rado-tree). As we are not using Ramsey's proof in our case, a weaker instance of comprehension suffices. It should be noted, though, that the main concern in [2] is to derive a parametric version of Ramsey's theorem that displays the common structural features of the (proofs of the) (infinite) Ramsey theorem, the finite Ramsey theorem, and the Paris-Harrington theorem.

Notes

1. In [12] we officially added all true universal sentences as axioms. As in the convention made in Chapter 13 of [17], we in this paper instead only add universal sentences that are provable in $\widehat{\text{E-PA}}^\omega\!\upharpoonright$ (see below) which covers, in particular, the schema of quantifier-free induction. In this way we can state various conservation results over primitive recursive arithmetic PRA but still can add further universal axioms as might be useful in concrete proofs.

2. The notation of this proof follows Farah's lecture notes "Set theory and its applications," York University, 2008.

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