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ON EELLS-SAMPSON'S EXISTENCE THEOREM FOR HARMONIC MAPS VIA EXPONENTIALLY HARMONIC MAPS

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Abstract. In this note, we introduce an approximation of harmonic maps via a sequence of exponentially harmonic maps. We then reestablish the existence theorem of harmonic maps due to Eells and Sampson.

§1. Introduction

Throughout this article, let (M,g) be an m-dimensional compact connected Riemannian manifold without boundary, and let (N,h) be an n-dimensional compact Riemannian manifold. A classical definition says that $u:(M,g)\to(N,h)$ is *harmonic* if it is a smooth critical point of the Dirichlet energy functional

$$E(u):=\int_M |du|^2\,d\mu_g,$$

where $|du|$ is the Hilbert-Schmidt norm of the differential du and where $d\mu_g$ is the Riemannian volume element on (M,g). A smooth map $u:M\to N$ is harmonic if and only if it satisfies the Euler-Lagrange equation

$$\tau(u)=\operatorname{div}_g(du)=0. \tag{1.1}$$

One of the most interesting and important subjects for harmonic maps is their existence. A typical existence problem can be formulated in the following manner:

Can a given map $f:M\to N$ be continuously deformed into a harmonic map $u:(M,g)\to(N,h)$?

In their famous paper, Eells and Sampson [4] first concerned themselves with such a problem in the general case and proved, under the assumption that (N,h) is nonpositively curved, that a given map $f:M\to N$ can

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be deformed into a harmonic map in its homotopy class. Their method is based on an analysis of the time-evolution problem corresponding to the harmonic map equation (1.1). They then proved, under the above curvature restriction, that such a time-evolution equation has a global regular solution, which converges to a harmonic map as time goes to infinity.

In this note, we consider a sequence $u_\varepsilon : (M, g) \to (N, h)$ of critical points of the parameterized exponential energy functional

$$\mathbb{E}_\varepsilon(u) := \int_M e^{\varepsilon|du|^2}\, d\mu_g$$

for $\varepsilon > 0$. The corresponding Euler-Lagrange equation is given by

$$\mathrm{div}_g(e^{\varepsilon|du|^2}\, du) = e^{\varepsilon|du|^2}\big\{\tau(u) + \varepsilon\langle\nabla|du|^2, du\rangle\big\} = 0,$$

where $\tau(u) = \mathrm{div}_g(du)$ is the tension field given in (1.1). This sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is then expected to approximate a harmonic map as $\varepsilon \to 0$. We actually have the following.

THEOREM 1.1. *Assume that the sectional curvature of* (N, h) *is nonpositive:* $\mathrm{Riem}^N \leq 0$. *Let* $\{u_\varepsilon\}_{\varepsilon>0}$ *be a sequence of smooth critical points of the functional* $\mathbb{E}_\varepsilon$ *for* $\varepsilon \to 0$ *satisfying the uniform boundedness condition of energy*

$$\int_M \frac{e^{\varepsilon|du_\varepsilon|^2} - 1}{\varepsilon}\, d\mu_g \leq E_0$$

with some constant $E_0 > 0$. *Then there exists a subsequence* $\{u_{\varepsilon(k)}\}_{k=1}^\infty \subseteq \{u_\varepsilon\}_{\varepsilon>0}$, $\varepsilon(k) \to 0$ *as* $k \to \infty$, *and a harmonic map* $u : (M, g) \to (N, h)$ *such that*

$$u_{\varepsilon(k)} \to u (k \to \infty) \quad \text{in } C^\infty(M, N).$$

This theorem will be found to give another approach to the Eells-Sampson existence theorem in [4]. That is to say, Theorem 1.1 implies the following.

COROLLARY 1.2. *Assume that* $\mathrm{Riem}^N \leq 0$. *Then any homotopy class of continuous maps from* M *to* N *admits a harmonic map.*

The organization of this article is as follows. Section 2 is devoted to some preliminary issues about exponentially harmonic maps needed in the sequel. Section 3 gives complete proofs of Theorem 1.1 and Corollary 1.2.

§2. Exponentially harmonic maps

This section provides some known results on exponentially harmonic maps, a part of which will be needed later. We start with the definition of exponentially harmonic maps, which was first introduced by Eells and Lemaire [3].

DEFINITION 2.1. An *exponentially harmonic map* $u : (M, g) \to (N, h)$ is a smooth critical point of the exponential energy functional

$$\mathbb{E}(u) := \int_M e^{|du|^2}\, d\mu_g.$$

The Euler-Lagrange equation of this problem can be written as

$$\mathrm{div}_g(e^{|du|^2}\, du) = e^{|du|^2}\big\{\tau(u) + \langle \nabla |du|^2, du\rangle\big\} = 0, \tag{2.2}$$

where $\tau(u) = \mathrm{div}_g(du)$ is the tension field of u.

One of the reasons why it is interesting to study the functional $\mathbb{E}$ is that the existence of its minima in a given homotopy class is always guaranteed without any special assumptions.

PROPOSITION 2.3 (see [3]). *For any homotopy class $\mathcal{H} \in [M, N]$ of continuous maps from M to N, there exists an $\mathbb{E}$-minimizer u in $\mathcal{H}$, which is necessarily α-Hölder-continuous for any exponent $0 < \alpha < 1$.*

This proposition follows essentially from the inequality

$$\frac{1}{k!}\int_M |du|^{2k}\, d\mu_g \le \int_M \sum_{k=0}^{\infty} \frac{1}{k!}|du|^{2k}\, d\mu_g = \mathbb{E}(u),$$

which guarantees that a minimizing sequence for $\mathbb{E}$ is uniformly bounded in each Sobolev space $W^{1,2k}(M, N)$. From the proof of this proposition in [3], however, it does not immediately follow that u is smooth, or even Lipschitz-continuous, or that it satisfies the Euler-Lagrange equation (2.2), even in a weak sense.

However, the faster the growth of a functional, the higher the regularity of its minima that we can expect. Indeed, in the case of $N = \mathbb{R}$, Duc and Eells [2] showed that an $\mathbb{E}$-minimizer $u : (M, g) \to \mathbb{R}$ of the Dirichlet problem is smooth in the interior of M, where (M, g) is a compact Riemannian manifold with boundary, and Lieberman [7] showed the global regularity for $u : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^m$ is an open subset. Also, for $n \geq 2$, Naito [8] showed that an $\mathbb{E}$-minimizer $u : \Omega \to \mathbb{R}^n$, where $\Omega \subseteq \mathbb{R}^m$ is a bounded domain, is smooth in the interior of Ω. Thereafter, Duc [1] at last showed the following strongest regularity theorem for $\mathbb{E}$-minimizers.

THEOREM 2.4 (see [1]). *Let $\mathcal{H} \in [M,N]$ be a given homotopy class. Then an $\mathbb{E}$-minimizer $u : (M,g) \to (N,h)$ in $\mathcal{H}$ is necessarily smooth.*

REMARK 2.5. (1) Combining this theorem with Proposition 2.3, we see that there always exists an exponentially harmonic map in a given homotopy class, which solves (2.2) in the classical sense.

(2) As mentioned in [1, Section 3], the Hölder norm $\|du\|_{C^\alpha}$ of the gradient of an exponentially harmonic map u is estimated by a constant depending only on (M,g), (N,h), $\mathbb{E}(u)$, and the Lipschitz constant $\|du\|_{L^\infty}$. Therefore, in order to verify Theorem 1.1, it suffices to show that $\|du_\varepsilon\|_{L^\infty}$ is uniformly bounded as $\varepsilon \to 0$.

Also, we need the following lemmas in the proof of Theorem 1.1. Their proofs are direct calculations, so we omit them.

LEMMA 2.6. *If $u : (M,g) \to (N,h)$ is an exponentially harmonic map, and if we consider a homothetic transformation $h \to \varepsilon^{-1}h$, for $\varepsilon > 0$, then $u : (M,g) \to (N,\varepsilon^{-1}h)$ is a critical point of the functional $\mathbb{E}_\varepsilon$.*

LEMMA 2.7. *An exponentially harmonic map $u : (M,g) \to (N,h)$ satisfies the following identity of Bochner-Weitzenböck type:*

$$\begin{aligned} S^{ij}\nabla_i\nabla_j e^{|du|^2} = 2e^{|du|^2}|\nabla\, du|^2 + 2e^{|du|^2}|\tau(u)|^2 \\ + 2e^{|du|^2}\sum_{i,j=1}^{m}\big\langle du\big(\mathrm{Ric}^M(e_i,e_j)e_j\big), du(e_i)\big\rangle \\ - 2e^{|du|^2}\sum_{i,j=1}^{m}\big\langle R^N\big(du(e_i),du(e_j)\big)\,du(e_j), du(e_i)\big\rangle, \end{aligned}$$

where Ric^M stands for the Ricci curvature of (M,g), R^N stands for the curvature tensor of (N,h), $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on M, and $S \in \Gamma(TM \otimes TM)$ is given by

$$S^{ij} := g^{ij} + 2\langle du(e_i), du(e_j)\rangle \quad (i,j = 1,2,\ldots,m). \tag{2.8}$$

We end this section by noting some basic properties similar to those of harmonic maps, which can be proved from the Bochner-Weitzenböck identity.

Corollary 2.9 (see [6]). *Let* $u : (M,g) \to (N,h)$ *be an exponentially harmonic map. If* $\mathrm{Ric}^M \geq 0$ *and* $\mathrm{Riem}^N \leq 0$*, then the following hold.*

(1) *u is totally geodesic.*
(2) *If* Ric^M *is positive at some point in* M*, then* u *is constant on* M*.*
(3) *If* $\mathrm{Riem}^N < 0$ *everywhere, then* u *is either a constant or a map onto a closed geodesic in* N*.*

§3. Proof of the main theorem

This section is devoted to the proof of Theorem 1.1. In what follows, by using the Nash isometric embedding $i : (N,h) \hookrightarrow \mathbb{R}^N$, we identify $i \circ u$ with u for a map $u : M \to N$. We mean by du the derivative of $u : M \to N$, and by ∇u the gradient of the function $u : M \to N \subseteq \mathbb{R}^N$.

Let $B_r \subseteq M$ be an open ball of radius $r > 0$ (centered at a fixed point of M). We need the Euler-Lagrange equation of the form

$$
(3.1) \quad 0 = \int_{B_r} \nabla_i u^A \nabla^i \varphi^A e^{|\nabla u|^2} \, d\mu_g + \int_{B_r} \nabla \, d\Pi^A(u)(\nabla u, \nabla u) \varphi^A e^{|\nabla u|^2} \, d\mu_g,
$$

$(A = 1, 2, \ldots, N)$, for any test function $\varphi \in C_0^\infty(B_r, \mathbb{R}^N)$, where $\Pi : U_\delta(N) \to N$ is the nearest projection from a tubular neighborhood $U_\delta(N)$ of N onto N. Note the relation $\nabla \, di(X,Y) = \nabla \, d\Pi(di(X), di(Y))$ for $X, Y \in \Gamma(TN)$.

Our first task is to show, under the assumption that $\mathrm{Riem}^N \leq 0$, that the gradient of an exponentially harmonic map is bounded by a constant depending only on (M,g) and its total energy and not on the target metric h. That is to say, we have the following.

Lemma 3.2. *Assume that* $\mathrm{Riem}^N \leq 0$*. Then for any exponentially harmonic map* $u : (M,g) \to (N,h)$*, there exists a constant* C_0 *depending only on* (M,g)*, the total energy* $\mathbb{E}(u)$*, and* not *on* h *such that*

$$
\sup_M |\nabla u|^2 \leq C_0 \int_M (e^{|\nabla u|^2} - 1) \, d\mu_g.
$$

Remark 3.3. Our proof of Lemma 3.2 is mainly due to the arguments in [8], which are for the case that (M,g) is a Euclidean domain Ω and that $u : \Omega \to \mathbb{R}^n$ is an exponentially harmonic function.

Proof of Lemma 3.2. We first consider the case of $m = \dim M \geq 3$. The proof has four steps.

STEP 1. There exists $\delta_0 = \delta_0(m) > 0$ such that

$$\Big((\sigma r)^{-m}\int_{B_{\sigma r}} e^{(1+\delta)|\nabla u|^2}\,d\mu_g\Big)^{(m-2)/m} \leq C_1 \frac{r^{-m}}{(1-\sigma)^2}\mathbb{E}(u) \tag{3.4}$$

for all $0 < \delta \leq \delta_0$ and $0 < \sigma < 1$, where $C_1 = C_1(M) > 0$.

As in the proof of [8, Proposition 2.10], choose $\gamma < 0$ so that $\gamma > -(2/m)$ and

$$\varphi^A = \nabla^k(w^{\gamma/2}\eta^2\nabla_k u^A) \tag{3.5}$$

as a test function in (3.1), where $w := e^{|\nabla u|^2}$ and where $\eta : B_r \to \mathbb{R}$ is a cutoff function satisfying

$$0 \leq \eta \leq 1, \qquad \eta = 1 \text{ on } B_{\sigma r}, \qquad \operatorname{supp}\eta \subseteq B_r, \qquad |\nabla\eta| \leq \frac{1}{(1-\sigma)r}.$$

First we note that it follows from the Ricci identity that

$$\begin{aligned}\nabla^i\varphi^A &= \nabla^i\nabla^k(w^{\gamma/2}\eta^2\nabla_k u^A)\\ &= \nabla^k\nabla^i(w^{\gamma/2}\eta^2\nabla_k u^A) - g^{ij}g^{kl}R^{M\,s}_{\;\;jlk}(w^{\gamma/2}\eta^2\nabla_s u^A),\end{aligned}$$

where $R^{M\,l}_{\;\;ijk}\partial_l = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k$ is the curvature tensor of (M,g). Then, after the integration by parts with respect to ∇^k, (3.1) becomes

$$\begin{aligned}0 &= \int_{B_r}(\nabla^k\nabla_i u^A + \nabla_i u^A\nabla^k|\nabla u|^2)\nabla^i(w^{\gamma/2}\eta^2\nabla_k u^A)e^{|\nabla u|^2}\,d\mu_g\\ &\qquad + \int_{B_r}\sum_{i,j=1}^m \big\langle du\big(\mathrm{Ric}^M(e_i,e_j)e_j\big), du(e_i)\big\rangle w^{(\gamma/2)+1}\eta^2\,d\mu_g\\ &\qquad - \int_{B_r}\nabla\, d\Pi^A(u)(\nabla u,\nabla u)\nabla^k(w^{\gamma/2}\eta^2\nabla_k u^A)e^{|\nabla u|^2}\,d\mu_g\\ &= \int_{B_r}(\nabla^k\nabla_i u^A + \nabla_i u^A\nabla^k|\nabla u|^2)\nabla^i\nabla_k u^A w^{(\gamma/2)+1}\eta^2\,d\mu_g\\ &\qquad + \frac{\gamma}{2}\int_{B_r}(\nabla^k\nabla_i u^A + \nabla_i u^A\nabla^k|\nabla u|^2)\nabla_k u^A\nabla^i|\nabla u|^2 w^{(\gamma/2)+1}\eta^2\,d\mu_g\\ &\qquad + 2\int_{B_r}(\nabla^k\nabla_i u^A + \nabla_i u^A\nabla^k|\nabla u|^2)\nabla_k u^A w^{(\gamma/2)+1}\eta\nabla^i\eta\,d\mu_g \qquad (3.6)\end{aligned}$$

$$+\int_{B_r}\sum_{i,j=1}^{m}\big\langle du\big(\mathrm{Ric}^M(e_i,e_j)e_j\big),du(e_i)\big\rangle w^{(\gamma/2)+1}\eta^2\,d\mu_g$$

$$-\int_{B_r}\nabla\,d\Pi^A(u)(\nabla u,\nabla u)\nabla^k(w^{\gamma/2}\eta^2\nabla_k u^A)e^{|\nabla u|^2}\,d\mu_g$$

for each $A=1,2,\dots,N$. Since $\nabla\,d\Pi(u)(\nabla u,\nabla u)$ is the vertical part of Δu to N, the last term becomes

$$-\int_{B_r}|\nabla\,d\Pi(u)(\nabla u,\nabla u)|^2 w^{(\gamma/2)+1}\eta^2\,d\mu_g$$

after taking the summation with respect to $A=1,2,\dots,N$. Also, by the Leibniz rule,

$$\begin{aligned}&|\nabla\nabla(i\circ u)|^2\\ &\quad=|\nabla\,du|^2+g^{ik}g^{jl}\langle\nabla\,d\Pi(u)(\nabla_i u,\nabla_j u),\nabla\,d\Pi(u)(\nabla_k u,\nabla_l u)\rangle,\end{aligned}\tag{3.7}$$

and by the Gauss formula,

$$\begin{aligned}&\sum_{i,j=1}^{m}\big\langle R^N\big(du(e_i),du(e_j)\big)\,du(e_j),du(e_i)\big\rangle\\ &\qquad=|\nabla\,d\Pi(u)(\nabla u,\nabla u)|^2\\ &\qquad\quad-g^{ik}g^{jl}\langle\nabla\,d\Pi(u)(\nabla_i u,\nabla_j u),\nabla\,d\Pi(u)(\nabla_k u,\nabla_l u)\rangle.\end{aligned}\tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6) after taking the summation then yields

$$\begin{aligned}0=&\int_{B_r}\Big\{|\nabla\,du|^2+\frac{\gamma}{2}\big|\langle\nabla|\nabla u|^2,\nabla u\rangle\big|^2\Big\}w^{(\gamma/2)+1}\eta^2\,d\mu_g\\ &+\frac{1}{2}\Big(\frac{\gamma}{2}+1\Big)\int_{B_r}\big|\nabla|\nabla u|^2\big|^2 w^{(\gamma/2)+1}\eta^2\,d\mu_g\\ &+\int_{B_r}\Big\{\langle\nabla|\nabla u|^2,\nabla\eta\rangle\\ &+2\sum_{A=1}^{N}\langle\nabla|\nabla u|^2,\nabla u^A\rangle\langle\nabla u^A,\nabla\eta\rangle\Big\}w^{(\gamma/2)+1}\eta\,d\mu_g\end{aligned}$$

$$+\int_{B_r}\sum_{i,j=1}^{m}\big\langle du\big(\mathrm{Ric}^M(e_i,e_j)e_j\big),du(e_i)\big\rangle w^{(\gamma/2)+1}\eta^2\,d\mu_g$$

$$-\int_{B_r}\sum_{i,j=1}^{m}\big\langle R^N\big(du(e_i),du(e_j)\big)\,du(e_j),du(e_i)\big\rangle w^{(\gamma/2)+1}\eta^2\,d\mu_g.$$

Here the first term is nonnegative by the choice of $\gamma>-(2/m)$ because u solves the Euler-Lagrange equation $\tau(u)+\langle\nabla|\nabla u|^2,\nabla u\rangle=0$. The last term is also nonnegative because (N,h) is assumed to be nonpositively curved, so that

$$\begin{aligned}
&\frac{1}{2}\Big(\frac{\gamma}{2}+1\Big)\int_{B_r}\big|\nabla|\nabla u|^2\big|^2 w^{(\gamma/2)+1}\eta^2\,d\mu_g\\
&\quad\le-\int_{B_r}\Big\{\langle\nabla|\nabla u|^2,\nabla\eta\rangle+2\sum_{A=1}^{N}\langle\nabla|\nabla u|^2,\nabla u^A\rangle\langle\nabla u^A,\nabla\eta\rangle\Big\}w^{(\gamma/2)+1}\eta\,d\mu_g\\
&\qquad-\int_{B_r}\sum_{i,j=1}^{m}\big\langle du\big(\mathrm{Ric}^M(e_i,e_j)e_j\big),du(e_i)\big\rangle w^{(\gamma/2)+1}\eta^2\,d\mu_g\\
&\quad\le C(m)\int_{B_r}\big|\nabla|\nabla u|^2\big|(1+|\nabla u|^2)w^{(\gamma/2)+1}|\nabla\eta|\eta\,d\mu_g\\
&\qquad+C(M)\int_{B_r}|\nabla u|^2 w^{(\gamma/2)+1}\eta^2\,d\mu_g\\
&\quad\le\frac{C(m)}{\delta}\int_{B_r}\big|\nabla|\nabla u|^2\big|w^{(\gamma/2)+1+\delta}|\nabla\eta|\eta\,d\mu_g+\frac{C(M)}{\delta}\int_{B_r}w^{(\gamma/2)+1+\delta}\eta^2\,d\mu_g\\
&\quad=\frac{C(m,\gamma)}{\delta}\int_{B_r}|\nabla(w^{((\gamma/2)+1)/2})|\eta\cdot w^{((\gamma/2)+1)/2+\delta}|\nabla\eta|\,d\mu_g\\
&\qquad+\frac{C(M)}{\delta}\int_{B_r}w^{(\gamma/2)+1+\delta}\eta^2\,d\mu_g,
\end{aligned}$$

where we used $xe^x\le(1/\delta)e^{(1+\delta)x}$ for all $\delta>0$ and $x\ge0$, and $\nabla(|\nabla u|^2)\cdot w=\nabla w$. If we choose $\delta=-(\gamma/4)>0$, then since $((\gamma/2)+1)/2+\delta=1/2$ and $(\gamma/2)+1+\delta<1$,

$$\begin{aligned}
(\mathrm{RHS})\le\frac{4C(m,\gamma)}{-\gamma}&\int_{B_r}|\nabla(w^{((\gamma/2)+1)/2})|\eta\cdot e^{(1/2)|\nabla u|^2}|\nabla\eta|\,d\mu_g\\
&+\frac{4C(M)}{-\gamma}\int_{B_r}e^{|\nabla u|^2}\eta^2\,d\mu_g.
\end{aligned}$$

On the other hand, the left-hand side can be written as

$$\frac{1}{2}\Big(\frac{\gamma}{2}+1\Big)\int_{B_r}\big|\nabla|\nabla u|^2\big|^2 w^{(\gamma/2)+1}\eta^2\,d\mu_g = C(\gamma)\int_{B_r}|\nabla(w^{((\gamma/2)+1)/2})|^2\eta^2\,d\mu_g.$$

Therefore, after using the Young inequality, we obtain

$$\int_{B_r}|\nabla(\eta w^{((\gamma/2)+1)/2})|^2\,d\mu_g \le \frac{C(M,\gamma)}{(1-\sigma)^2 r^2}\int_{B_r} e^{|\nabla u|^2}\,d\mu_g. \tag{3.9}$$

Applying the Sobolev embedding theorem to this yields

$$\begin{aligned}&\Big((\sigma r)^{-m}\int_{B_{\sigma r}} w^{((\gamma/2)+1)m/(m-2)}\,d\mu_g\Big)^{(m-2)/m}\\ &\qquad\le C(M,\gamma)\frac{r^{-m}}{(1-\sigma)^2}\int_{B_r} e^{|\nabla u|^2}\,d\mu_g,\end{aligned}$$

which proves (3.4) if we put $1+\delta_0 := ((\gamma/2)+1)m/(m-2) > 1$ because $\gamma > -(4/m)$.

Step 2. There exists $1 < p < m/(m-2)$ such that

$$\begin{aligned}&\Big((\sigma r)^{-m}\int_{B_{\sigma r}} e^{\alpha m/(m-2)|\nabla u|^2}\,d\mu_g\Big)^{(m-2)/m}\\ &\qquad\le C_2\Big(r^{-m}\int_{B_r} e^{\alpha p|\nabla u|^2}\,d\mu_g\Big)^{1/p}\end{aligned} \tag{3.10}$$

for all $\alpha \ge 1$ and $0 < \sigma < 1$, where

$$C_2 = \frac{C(M,\alpha)}{(1-\sigma)^2}\Big(r^{-m}\int_{B_r} e^{(1+\delta_0)|\nabla u|^2}\,d\mu_g\Big)^{1/q}$$

($(1/p)+(1/q)=1$), and $C(M,\alpha)$ is a constant depending only on m and $\|\mathrm{Ric}^M\|_{L^\infty}$ and admits at most polynomial growth in α.

As a test function, we choose (3.5) with $w = e^{|\nabla u|^2}$ and $\gamma \ge 0$. Then by a similar calculation,

$$\begin{aligned}&\int_{B_{\sigma r}}|\nabla(w^{((\gamma/2)+1)/2})|^2\,d\mu_g\\ &\qquad\le \frac{C(M,\gamma)}{\delta}\frac{1}{(1-\sigma)^2 r^2}\int_{B_r} w^{(\gamma/2)+1+\delta}\,d\mu_g\end{aligned}$$

for any $\delta > 0$, where $C(M,\gamma)$ is a constant which admits at most polynomial growth in γ. After putting $\alpha = (\gamma/2)+1 \geq 1$, we use the Sobolev embedding theorem to obtain

$$\begin{aligned}
&\Big((\sigma r)^{-m} \int_{B_{\sigma r}} w^{\alpha m/(m-2)}\, d\mu_g\Big)^{(m-2)/m} \\
&\quad \leq \frac{C(M,\alpha)}{\delta(1-\sigma)^2}\Big(r^{-m}\int_{B_r} w^{\alpha+\delta}\, d\mu_g\Big) \\
&\quad \leq \frac{C(M,\alpha)}{\delta(1-\sigma)^2}\Big(r^{-m}\int_{B_r} w^{\alpha p}\, d\mu_g\Big)^{1/p}\Big(r^{-m}\int_{B_r} w^{\delta q}\, d\mu_g\Big)^{1/q},
\end{aligned}$$

where $(1/p)+(1/q)=1$. If we choose p so that $1 < p < m/(m-2)$ and subsequently $\delta > 0$ so that $\delta q < 1+\delta_0$, then (3.10) is obtained.

Step 3 (Moser's iteration). There exists $C_3 = C_3(M, \mathbb{E}(u)) > 0$ such that

$$\sup_M |\nabla u| \leq C_3. \tag{3.11}$$

By Step 2, there exists $1 < p < m/(m-2)$ such that

$$\begin{aligned}
&\Big((\sigma r)^{-m} \int_{B_{\sigma r}} w^{\alpha m/(m-2)}\, d\mu_g\Big)^{(m-2)/m} \\
&\quad \leq \frac{C(M,\alpha,\mathbb{E}(u),r)}{(1-\sigma)^2}\Big(r^{-m}\int_{B_r} w^{\alpha p}\, d\mu_g\Big)^{1/p}
\end{aligned}$$

for all $\alpha \geq 1$ and $0 < \sigma < 1$, where $w = e^{|\nabla u|^2}$ and where $C(M,\alpha,\mathbb{E}(u),r)$ is a constant which admits at most polynomial growth in α. Now we set $r_0 := r$, and for every $k \in \mathbb{N}$, we set

$$r_k := r\prod_{j=1}^{k}\sigma_j, \qquad \sigma_j := \frac{1+2^{-j}}{1+2^{1-j}}, \qquad B_k := B_{r_k}, \qquad \alpha_k := \Big(\frac{1}{p}\cdot\frac{m}{m-2}\Big)^k.$$

Then by noting that $\alpha_k \geq 1$ and that $\alpha_k p = \alpha_{k-1} m/(m-2)$,

$$\begin{aligned}
&\Big(r_k^{-m}\int_{B_k} w^{\alpha_k m/(m-2)}\, d\mu_g\Big)^{\alpha_k^{-1}(m-2)/m} \\
&\quad \leq \Big(\frac{C(M,\alpha_k,\mathbb{E}(u),r_{k-1})}{(1-\sigma_k)^2}\Big)^{\alpha_k^{-1}} \\
&\qquad \times \Big(r_{k-1}^{-m}\int_{B_{k-1}} w^{\alpha_{k-1} m/(m-2)}\, d\mu_g\Big)^{\alpha_{k-1}^{-1}(m-2)/m}
\end{aligned} \tag{3.12}$$

$$\leq \Big\{ \prod_{j=1}^{k} \Big(\frac{C(M, \alpha_j, \mathbb{E}(u), r_{j-1})}{(1-\sigma_j)^2} \Big)^{\alpha_j^{-1}} \Big\}$$
$$\times \Big(r^{-m} \int_{B_r} w^{m/(m-2)} \, d\mu_g \Big)^{(m-2)/m}.$$

Claim. *The coefficient* $\prod_{j=1}^{k} \big((C(M, \alpha_j, \mathbb{E}(u), r_{j-1}))/(1-\sigma_j)^2 \big)^{\alpha_j^{-1}}$ *is bounded as* $k \to \infty$.

To this end, it suffices to prove that

$$\sum_{j=1}^{k} \frac{1}{\alpha_j} \log \Big[\frac{C(M, \alpha_j, \mathbb{E}(u), r_{j-1})}{(1-\sigma_j)^2} \Big]$$

is bounded as $k \to \infty$. Since $\alpha_j = s^j$, $s > 1$, while $C(M, \alpha_j, \mathbb{E}(u), r_{j-1})$ admits at most polynomial growth in α_j, it clearly follows that

$$\sum_{j=1}^{k} \frac{1}{\alpha_j} \log C\big(M, \alpha_j, \mathbb{E}(u), r_{j-1}\big)$$

is bounded as $k \to \infty$. Furthermore, by the choice of σ_j, we see that

$$\sum_{j=1}^{k} \frac{1}{\alpha_j} \log \frac{1}{(1-\sigma_j)^2} = \sum_{j=1}^{k} \frac{1}{s^j} \log \frac{(1+2^{-j})^2}{2^{-2j}} \leq \sum_{j=1}^{k} \frac{1}{s^j} \log(4^{j+1}),$$

which is clearly bounded as $k \to \infty$. This proves the claim.

Hence, we can take the limit $k \to \infty$ in (3.12) to obtain

$$\begin{aligned} \sup_{B_{r/2}} |\nabla u|^2 &\leq \sup_{B_{r/2}} w \\ &\leq C \Big(r^{-m} \int_{B_r} e^{m/(m-2)|\nabla u|^2} \, d\mu_g \Big)^{(m-2)/m} \\ &\leq C_3\big(M, \mathbb{E}(u)\big), \end{aligned}$$

proving (3.11).

Step 4. There exists a constant $C_0 = C_0(M, \mathbb{E}(u)) > 0$ such that

$$\sup_M |\nabla u|^2 \leq C_0 \int_M (e^{|\nabla u|^2} - 1) \, d\mu_g, \tag{3.13}$$

which proves Lemma 3.2.

Lemma 2.7 and (3.11), combined with the curvature assumption that $\mathrm{Riem}^N \le 0$, imply that

$$\begin{aligned} S^{ij}\nabla_i\nabla_j(e^{|\nabla u|^2}-1) &= S^{ij}\nabla_i\nabla_j e^{|\nabla u|^2} \\ &\ge 2e^{|\nabla u|^2}\sum_{i,j=1}^m \big\langle du\big(\mathrm{Ric}^M(e_i,e_j)e_j\big), du(e_i)\big\rangle \\ &\ge -C(m,\|\mathrm{Ric}^M\|_{L^\infty})e^{|\nabla u|^2}|\nabla u|^2 \\ &\ge -C(m,\|\mathrm{Ric}^M\|_{L^\infty}, e^{\|\nabla u\|^2_{L^\infty}})(e^{|\nabla u|^2}-1). \end{aligned}$$

In the fourth line, we have used the inequality $|\nabla u|^2 \le e^{|\nabla u|^2}-1$. Moreover, (3.11) then guarantees that the leading term S^{ij} of (2.8),

$$S^{ij} = g^{ij} + 2h_{\alpha\beta}\nabla_{e_i}u^\alpha\nabla_{e_j}u^\beta,$$

has the bounded eigenvalues both from above and from below by a constant depending only on (M,g) and $\mathbb{E}(u)$. This observation enables us to successfully apply the maximum principle (see [5, Theorem 9.20]) to acquire

$$|\nabla u|^2 \le e^{|\nabla u|^2}-1 \le C_0\big(M,\mathbb{E}(u)\big)\int_M (e^{|\nabla u|^2}-1)\,d\mu_g.$$

This proves (3.13), and we now complete the proof of Lemma 3.2 in the case of $m \ge 3$.

The proof in the case of $m=2$ is a slight modification of the above arguments.

Step 1. Fix $1<q_0<2$. Then there exists $\delta_0=\delta_0(q_0)>0$ such that

$$\Big((\sigma r)^{-2}\int_{B_{\sigma r}} e^{(1+\delta)|\nabla u|^2}\,d\mu_g\Big)^{1/q_0} \le C_1\frac{r^{-2}}{(1-\sigma)^2}\mathbb{E}(u) \tag{3.14}$$

for all $0<\delta\le\delta_0$ and $0<\sigma<1$, where $C_1=C_1(M)$.

To this end, taking $0>\gamma>2(1/q_0-1)$ and applying the Sobolev embedding theorem to (3.9), we obtain

$$\Big((\sigma r)^{-2}\int_{B_{\sigma r}} e^{((\gamma/2)+1)q_0|\nabla u|^2}\,d\mu_g\Big)^{1/q_0} \le C(M,\gamma)\frac{r^{-2}}{(1-\sigma)^2}\int_{B_r} e^{|\nabla u|^2}\,d\mu_g,$$

which proves (3.14) if we put $1+\delta_0=((\gamma/2)+1)q_0>1$.

STEP 2. There exists $1 < p < q_0$ such that

$$\left((\sigma r)^{-2}\int_{B_{\sigma r}} e^{\alpha q_0|\nabla u|^2}\,d\mu_g\right)^{1/q_0} \leq C_2\left(r^{-2}\int_{B_r} e^{\alpha p|\nabla u|^2}\,d\mu_g\right)^{1/p} \tag{3.15}$$

for all $\alpha \geq 1$ and $0 < \sigma < 1$, where

$$C_2 = \frac{C(M,\alpha)}{(1-\sigma)^2}\left(r^{-2}\int_{B_r} e^{(1+\delta_0)|\nabla u|^2}\,d\mu_g\right)^{1/q}$$

$((1/p)+(1/q)=1)$, and $C(M,\alpha)$ is a constant depending only on (M,g) and admits at most polynomial growth in α.

By using (3.15), we can apply Moser's iteration to obtain the bound (3.11) of the gradient of u, and the same argument as in Step 4 above is also valid in this case, which proves Lemma 3.2 in the case of $m=2$. ∎

Proof of Theorem 1.1. Let $u_\varepsilon : (M,g) \to (N,h)$ be a sequence of critical points of the functional $\mathbb{E}_\varepsilon$ as $\varepsilon \to 0$ satisfying

$$\int_M \frac{e^{\varepsilon|\nabla u_\varepsilon|^2}-1}{\varepsilon}\,d\mu_g \leq E_0. \tag{3.16}$$

As is mentioned in Remark 2.5(2), to complete the proof, it is enough to show that $\|\nabla u_\varepsilon\|_{L^\infty}$ is uniformly bounded as $\varepsilon \to 0$.

If we consider the homothetic transformation $h \to h_\varepsilon := \varepsilon h$, then by Lemma 2.6, each $u_\varepsilon : (M,g) \to (N,h_\varepsilon)$ is an exponentially harmonic map. Then as a consequence of Lemma 3.2, we have

$$\begin{aligned}\varepsilon|\nabla u_\varepsilon|_h^2 = |\nabla u_\varepsilon|_{h_\varepsilon}^2 &\leq C(M,E_0)\int_M (e^{|\nabla u_\varepsilon|_{h_\varepsilon}^2}-1)\,d\mu_g\\ &= C(M,E_0)\int_M (e^{\varepsilon|\nabla u_\varepsilon|_h^2}-1)\,d\mu_g.\end{aligned} \tag{3.17}$$

(Note that (3.16) implies that the total energy $\mathbb{E}(u_\varepsilon)$ with respect to h_ε, which is equal to $\mathbb{E}_\varepsilon(u_\varepsilon)$ with respect to h, is bounded by E_0. Also, note that the curvature assumption that $\mathrm{Riem}^N \leq 0$ does not change under the homothetic transformation.)

Dividing (3.17) by ε yields

$$|\nabla u_\varepsilon|_h^2 \leq C(M,E_0)\int_M \frac{e^{\varepsilon|\nabla u_\varepsilon|_h^2}-1}{\varepsilon}\,d\mu_g \leq C(M,E_0)E_0$$

for all $\varepsilon > 0$, which proves Theorem 1.1. ∎

Proof of Corollary 1.2. Let $\varphi \in \mathcal{H}$ be any smooth map. Theorem 2.4 then implies that there exists, for each $\varepsilon > 0$, a smooth map $u_\varepsilon : (M, g) \to (N, h)$ which minimizes $\mathbb{E}_\varepsilon$ in $\mathcal{H}$. Since the resulting sequence $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies

$$\int_M \frac{e^{\varepsilon|du_\varepsilon|^2} - 1}{\varepsilon} \, d\mu_g \leq \int_M \frac{e^{\varepsilon|d\varphi|^2} - 1}{\varepsilon} \, d\mu_g,$$

taking the limit as $\varepsilon \to 0$ yields

$$\limsup_{\varepsilon\to 0} \int_M \frac{e^{\varepsilon|du_\varepsilon|^2} - 1}{\varepsilon} \, d\mu_g \leq \limsup_{\varepsilon\to 0} \int_M \frac{e^{\varepsilon|d\varphi|^2} - 1}{\varepsilon} \, d\mu_g = \int_M |d\varphi|^2 \, d\mu_g.$$

This implies that some subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies the uniform boundedness condition of energy in Theorem 1.1, so that it moreover contains a subsequence which converges uniformly to some harmonic map $u : (M, g) \to (N, h)$. The obtained harmonic map u represents the homotopy class $\mathcal{H}$. $\square$

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