On Deligne's conjecture for Hilbert motives over totally real number fields

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Abstract In this article we prove that if Deligne's conjecture holds for motives associated to Hilbert modular forms of weight at least 3, then Deligne's conjecture holds for arbitrary base change to totally real number fields of motives associated to Hilbert modular forms of weight at least 3.

1. Introduction

Let M be a motive defined over a number field F with coefficients in a number field E. One can associate to M an L-function $\mathbb{L}(M,s)$ having values in $E\otimes_{\mathbb{Q}}\mathbb{C}$. From the proprieties of the restriction of scalars, one knows that $\mathbb{L}(M,s)=\mathbb{L}(\mathrm{Res}_{F/\mathbb{Q}}M,s)$. When M is critical, one has the $+$-period defined by Deligne $c^{+}(\mathrm{Res}_{F/\mathbb{Q}}M)\in E\otimes_{\mathbb{Q}}\mathbb{C}$. Then Deligne's conjecture states the following.

CONJECTURE 1.1
If M is a critical motive defined over F with coefficients in E, then

$$\mathbb{L}(M,0)/c^{+}(\mathrm{Res}_{F/\mathbb{Q}}M)\in E\otimes 1\subset E\otimes_{\mathbb{Q}}\mathbb{C}.$$

This conjecture is known to be true for rank 1 motives if F is either totally real or a CM field (see [B]) and for motives associated to classical modular forms of $\mathrm{GL}(2)/\mathbb{Q}$ (see [D]).

In this article, we prove the following result. (We remark that in the proof of this theorem, we assume the Tate conjecture for motives; see §4 for details.)

THEOREM 1.2
Let F be a totally real number field, let I_F be the set of infinite places of F, let f be a Hilbert cusp form of weight $k=(k_\tau)_{\tau\in I_F}$ of $\mathrm{GL}(2)/F$, where all k_τ have the same parity and all $k_\tau\geq 3$. Let $M(f)(j)$ be the j-Tate twist of the motive $M(f)$ associated to f, where j is an integer such that $(k_0+1)/2\leq j<(k_0+k^0)/2$, where $k_0=\max\{k_\tau\mid\tau\in I_F\}$ and $k^0=\min\{k_\tau\mid\tau\in I_F\}$. Assume that Conjecture 1.1 is true for all the motives of the form $M(g)(j)$, where g is an arbitrary modular form of weight k of $\mathrm{GL}(2)/L$, and L is an arbitrary totally real finite extension

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of F. Then Conjecture 1.1 is true for all the motives of the form $M(f)(j)/F'$, where F' is an arbitrary totally real finite extension of F.

Note the following point: We don't know that the motive $M(f)(j)/F'$ corresponds to a Hilbert modular form since arbitrary totally real base change is not yet established.

2. Periods for motives

Consider a motive M defined over a number field F with coefficients in a number field E. Denote by Γ_F the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$. We recall now the definition of the L-function $\mathbb{L}(M,s)$ of M. Consider the étale cohomology $H_\lambda(M)$ for each prime ideal λ of E. It is conjectured that the Galois representation $\rho_\lambda : \Gamma_F \to \mathrm{GL}(H_\lambda(M))$ is unramified outside the residual characteristic l of λ and a finite set S of primes of F independent of λ. Denote by $V := H_\lambda(M)$ the representation space of ρ_λ. If $\wp$ is a prime ideal of F prime to l, we choose an inertia group $I_\wp$ at $\wp$ and a geometric Frobenius $\mathrm{Frob}_\wp$. It is conjectured that the characteristic polynomial $Z_\wp(M,X) = \det(1 - \rho_\lambda(\mathrm{Frob}_\wp)|_{V^{I_\wp}} X)$ has coefficients in E and is independent of λ. Assume all these conjectures. Denote by I_E the set of infinite places of E. For $\tau \in I_E$, put

$$L_\wp(\tau,M,s) = \tau Z_\wp\big(M, N(\wp)^{-s}\big)^{-1}$$

and

$$L(\tau,M,s) = \prod_\wp L_\wp(\tau,M,s).$$

One has the isomorphism $E \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^{I_E}$ given by $e \otimes z \to (z \cdot \tau(e))_{\tau \in I_E}$. One can define a function $\mathbb{L}(M,s)$ taking values in $E \otimes_{\mathbb{Q}} \mathbb{C}$ by arranging $L(\tau,M,s)$.

Let $\mathbb{L}_\infty(M,s)$ be the infinite part of the L-function of M which is a product of Γ-functions. If one puts $\Lambda(M,s) = \mathbb{L}(M,s)\mathbb{L}_\infty(M,s)$, then the conjectural functional equation has the following form:

$$\Lambda(M,s) = \epsilon(M,s)\Lambda(M^\vee, 1-s),$$

where $\epsilon(M,s)$ is a multiple of an exponential function of s with values in $E \otimes_{\mathbb{Q}} \mathbb{C}$ and $M^\vee$ is the dual of M. We say that an integer n is critical for M if neither $\mathbb{L}_\infty(M,s)$ nor $\mathbb{L}_\infty(M^\vee, 1-s)$ has a pole at $s = n$. We call M *critical* if M is critical at zero.

Consider now a motive M defined over $\mathbb{Q}$ with coefficients in E. Let $H_B(M)$ denote the Betti realization of M. Then $H_B(M)$ is a finite-dimensional vector space over E. The complex conjugation F_∞ acts on $H_B(M)$, and one gets a decomposition

$$H_B(M) = H_B^+(M) \oplus H_B^-(M),$$

where $H_B^\pm(M)$ denote the eigenspaces of $H_B(M)$ with eigenvalues ± 1.

Assume that the motive M is homogeneous of weight w. Then one has the Hodge decomposition

$$H_B(M)\otimes_{\mathbb{Q}}\mathbb{C}=\bigoplus_{p+q=w}H^{pq}(M),$$

where $H^{pq}(M)$ is a free $E\otimes_{\mathbb{Q}}\mathbb{C}$-module.

Let $H_{DR}(M)$ denote the de Rham realization of M. Then $H_{DR}(M)$ is a finite-dimensional vector space over E. One has the comparison isomorphism

$$I: H_B(M)\otimes_{\mathbb{Q}}\mathbb{C}\cong H_{DR}(M)\otimes_{\mathbb{Q}}\mathbb{C}$$

as $(E\otimes_{\mathbb{Q}}\mathbb{C})$-modules.

Define the Hodge filtration $\{F^m\}$ on $H_{DR}(M)$ by

$$I^{-1}\big(F^m(H_{DR}(M))\otimes_{\mathbb{Q}}\mathbb{C}\big)=\bigoplus_{p\geq m}H^{pq}(M).$$

For M a motive of odd weight $w=2p+1$, define $F^{\pm}(M)=: F^{p+1}(H_{DR}(M))$ (for a motive M of even weight $w=2p$, one can define in a similar way $F^{\pm}(M)$, see [Y, §2]). If one defines $H^{\pm}_{DR}(M)=H_{DR}(M)/F^{\mp}(M)$, then one has the comparison isomorphisms

$$I^{\pm}: H^{\pm}_B(M)\otimes_{\mathbb{Q}}\mathbb{C}\cong H^{\pm}_{DR}(M)\otimes_{\mathbb{Q}}\mathbb{C}. \tag{2.1}$$

Let $c^{\pm}(M)=\det(I^{\pm})$ be the determinants calculated using E-rational basis. Hence $c^{\pm}(M)\in(E\otimes_{\mathbb{Q}}\mathbb{C})^{\times}$ are determined up to multiplication by elements of E.

If M is a motive defined over F with coefficients in E, let I_F be the set of infinite places of F. Then $H_{DR}(M)$ is a free $E\otimes_{\mathbb{Q}}F$-module of some rank $d(M)$, and for each $\sigma\in I_F$, one has the Betti realization $H_B(M^{\sigma})$ which is a vector space of dimension $d(M)$ over E. The number $d(M)$ is called the rank of M.

We recall now the definition of the restriction of scalars $\mathrm{Res}_{F/F'}(M)$ of M to a subfield F' of F. For the de Rham side one forgets the F-vector space structure and put $H_{DR}(\mathrm{Res}_{F/F'}(M))=H_{DR}(M)$ as an F'-vector space. For the Betti side, one sets $H_B(\mathrm{Res}_{F/F'}(M)^{\sigma})=\bigoplus_{\tau|_{F'}=\sigma}H_B(M^{\tau})$. Hence $\mathrm{Res}_{F/F'}(M)$ is a motive over F' of rank $[F:F']d(M)$ with coefficients in E.

3. L-functions

Let F be a totally real number field, and let I_F be the set of infinite places of F. If π is an automorphic representation of weight $k=(k_{\tau})_{\tau\in I_F}$ of $\mathrm{GL}(2)/F$, where all k_{τ} have the same parity and all $k_{\tau}\geq 2$, then there exists (see [T]) a λ-adic representation

$$\rho_{\pi,\lambda}:\Gamma_F\to \mathrm{GL}_2(O_{\lambda})\hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_l),$$

which satisfies $L(\rho_{\pi,\lambda},s)=L(f,s)$ and is unramified outside the primes dividing $\mathbf{n}l$. Here O is the coefficients ring of π and λ is a prime ideal of O above some prime number l, $\mathbf{n}$ is the level of π, and f is the modular form of $\mathrm{GL}(2)/F$ of weight k corresponding to π. In order to simplify the notation, we denote by ρ_{π} the representation $\rho_{\pi,\lambda}$. (By fixing an isomorphism $i:\bar{\mathbb{Q}}_l\to\mathbb{C}$, we can regard always ρ_Π as a complex-valued representation.) We know the following result (see [V, Theorem 1.1].)

THEOREM 3.1

If π is a cuspidal automorphic representation of weight k as above of $\mathrm{GL}(2)/F$ *for some totally real number field F and F' is a totally real extension of F, then there exists a totally real finite Galois extension F'' of $\mathbb{Q}$ containing F' and a prime λ of the field coefficients of π, such that $\rho_{\pi,\lambda}|_{\Gamma_{F''}}$ is modular; that is, there exists an automorphic representation π'' of weight k of* $\mathrm{GL}(2)/F''$ *and a prime β of the field of coefficients of π'' such that $\rho_{\pi,\lambda}|_{\Gamma_{F''}} \cong \rho_{\pi'',\beta}$.*

Fix a cuspidal automorphic representation π as in the Theorem 3.1. Let F'/F be a totally real finite extension. Then one can find a totally real finite Galois extension F'' of $\mathbb{Q}$ containing F', a prime λ of the field coefficients of π and an automorphic representation π'' of $\mathrm{GL}(2)/F''$ and a prime β of the field of coefficients of π'' such that $\rho_{\pi,\lambda}|_{\Gamma_{F''}} \cong \rho_{\pi'',\beta}$.

By Brauer's theorem (see [Se, Theorems 16, 19]), one can find some subfields $F_i \subset F''$ such that $\mathrm{Gal}(F''/F_i)$ are solvable, some characters $\varphi_i : \mathrm{Gal}(F''/F_i) \to \bar{\mathbb{Q}}^\times$, and some integers m_i, such that the trivial representation

$$1 : \mathrm{Gal}(F''/F') \to \bar{\mathbb{Q}}^\times$$

can be written as $1 = \sum_{i=1}^{i=u} m_i \operatorname{Ind}_{\mathrm{Gal}(F''/F_i)}^{\mathrm{Gal}(F''/F')} \varphi_i$ (a virtual sum). Then

$$\begin{aligned} L(\rho_\pi|_{\Gamma_{F'}}, s) &= \prod_{i=1}^{i=u} L(\rho_\pi|_{\Gamma_{F'}} \otimes \operatorname{Ind}_{\Gamma_{F_i}}^{\Gamma_{F'}} \varphi_i, s)^{m_i} \\ &= \prod_{i=1}^{i=u} L\big(\operatorname{Ind}_{\Gamma_{F_i}}^{\Gamma_{F'}} (\rho_\pi|_{\Gamma_{F_i}} \otimes \varphi_i), s\big)^{m_i} \\ &= \prod_{i=1}^{i=u} L(\rho_\pi|_{\Gamma_{F_i}} \otimes \varphi_i, s)^{m_i}. \end{aligned}$$

Since $\rho_\pi|_{\Gamma_{F''}}$ is modular and $\mathrm{Gal}(F''/F_i)$ is solvable, from Langland's base change for solvable extensions one can deduce easily that the representation $\rho_\pi|_{\Gamma_{F_i}}$ is modular, and thus there exists an automorphic representation π_i of weight k such that $\rho_\pi|_{\Gamma_{F_i}} \cong \rho_{\pi_i}$. Denote by f_i the modular form corresponding to π_i. Then

$$L(\rho_\pi|_{\Gamma_{F'}}, s) = \prod_{i=1}^{i=u} L(\rho_{\pi_i} \otimes \varphi_i, s)^{m_i} = \prod_{i=1}^{i=u} L(f_i, \varphi_i, s)^{m_i}, \tag{3.1}$$

where $L(f_i, \varphi_i, s)$ are defined in §4.

4. Deligne's conjecture for $M(f)(j)/_{F'}$

Let F be a totally real number field, and let f be a modular form of weight k as in §3 of $\mathrm{GL}(2)/F$. Let θ be a Hecke character of F of finite order. For $\mathbf{n}$ an ideal of the ring of integers O_F of F, define $a(\mathbf{n})$ by $T(\mathbf{n})f = a(\mathbf{n})f$, where $T(\mathbf{n})$ is the Hecke operator of level $\mathbf{n}$. The field of coefficients of f is by definition the field $\mathbb{Q}_f$ generated by the values $a(\mathbf{n})$ over $\mathbb{Q}$. It is well known that $\mathbb{Q}_f$ is a finite extension of $\mathbb{Q}$. We consider a number field E which contains $\mathbb{Q}_f$ and the field of coefficients $\mathbb{Q}(\theta)$ of θ. Put

$$L(f,\theta,s) = \sum_{\mathbf{n}} a\big(T(\mathbf{n})\big)\theta(\mathbf{n})N(\mathbf{n})^{-s}.$$

For $\tau \in I_E$, we define

$$L(\tau,f,\theta,s) = \sum_{\mathbf{n}} a\big(T(\mathbf{n})\big)^{\tau}\theta(\mathbf{n})^{\tau}N(\mathbf{n})^{-s}.$$

Using the isomorphism $E \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^{I_E}$, one gets an $(E \otimes_{\mathbb{Q}} \mathbb{C})$-valued L-function $\mathbb{L}(f,\theta,s)$ by arranging the factors $L(\tau,f,\theta,s)$. In the same way, one can define the L-function $\mathbb{L}(f,s)$.

Let f be a modular form of weight k of $\mathrm{GL}(2)/F$, and let $M(f)$ be the motive conjecturally corresponding to f. Then $M(f)$ is a motive of rank 2 over F with coefficients in $\mathbb{Q}_f$. By the definition of $M(f)$, we have $\mathbb{L}(M(f),s) = \mathbb{L}(f,s)$. Since the modular form f has weight k, if we define $k_0 = \max\{k_\tau \mid \tau \in I_F\}$ and $k^0 = \min\{k_\tau \mid \tau \in I_F\}$, then any integer $(k_0 - k^0)/2 < j < (k_0 + k^0)/2$ is a critical value for $M(f)$.

Let $m \in \mathbb{Z}$, and let $T(m)$ be the Tate motive over F. Put $M(f)(m) = M(f) \otimes T(m)$. One has

$$\mathbb{L}\big(M(f)(m),s\big) = \mathbb{L}\big(M(f),m+s\big).$$

Hence, from the fact that $M(f)$ is critical at j for $(k_0 - k^0)/2 < j < (k_0 + k^0)/2$, one gets that $M(f)(j)$ is critical at zero. If θ is a finite-order character of a number field, then we denote by $M(\theta)$ the motive corresponding to θ. Then $M(\theta)$ satisfies $L(\theta,s) = L(M(\theta),s)$.

Now we prove Theorem 1.2.

Proof

Thus we assume from now on that $k_\tau \geq 3$ for all $\tau \in I_F$. Using the same notation as in §3, we assume that f is the cuspform corresponding to the cuspidal automorphic representation π which appears in Theorem 3.1. Since $k^0 \geq 3$, we know from [S, Proposition 4.16] that for each integer j such that $(k_0+1)/2 \leq j < (k_0+k^0)/2$, we have $L(f_i,\varphi_i,j) \neq 0$. Thus for such a j, from formula (3.1) above we obtain the identity

$$L\big(M(f)_{/F'},j\big) = \prod_{i=1}^{i=u} L(f_i,\varphi_i,j)^{m_i}.$$

Define $E_1 := \mathbb{Q}_f \bigcup_{i=1}^{i=u} \mathbb{Q}(\varphi_i)$, where $\mathbb{Q}(\varphi_i)$ is the field of coefficients of φ_i. By extending their fields of coefficients, we regard the functions $\mathbb{L}(M(f)_{/F'},s)$ and

$\mathbb{L}(f_i, \varphi_i, s)$ as having values in $E_1 \otimes_{\mathbb{Q}} \mathbb{C}$. Hence we get

$$\mathbb{L}\big(M(f)_{/F'}, j\big) = \prod_{i=1}^{i=u} \mathbb{L}(f_i, \varphi_i, j)^{m_i} \in E_1 \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.1}$$

Since $1 = \sum_{i=1}^{i=u} m_i \operatorname{Ind}_{\operatorname{Gal}(F''/F_i)}^{\operatorname{Gal}(F''/F')} \varphi_i$, we get the equality of motives (by assuming the Tate conjecture for motives)

$$\operatorname{Res}_{F'/\mathbb{Q}}\big(M(f)_{/F'}(j)\big) = \bigoplus_{i=1}^{i=u} \big(\operatorname{Res}_{F_i/\mathbb{Q}}(M(f)_{/F_i} \otimes M(\varphi_i))(j)\big)^{m_i},$$

from which, by looking at the E_1 rational basis (see (2.1)), we obtain trivially

$$c^+\big(\operatorname{Res}_{F'/\mathbb{Q}}(M(f)_{/F'}(j))\big) = \prod_{i=1}^{i=u} c^+\big(\operatorname{Res}_{F_i/\mathbb{Q}}(M(f)_{/F_i} \otimes M(\varphi_i))(j)\big)^{m_i}. \tag{4.2}$$

Under the assumptions of Theorem 1.2, we have

$$\frac{\mathbb{L}(M(g)(j), 0)}{c^+(\operatorname{Res}_{L/\mathbb{Q}}(M(g)(j)))} \in \mathbb{Q}_g \otimes 1$$

for any modular form g of weight k of $\mathrm{GL}(2)/L$, for L totally real number field.

Thus we get

$$\frac{\mathbb{L}(M(f_i) \otimes M(\varphi_i)(j), 0)}{c^+(\operatorname{Res}_{F_i/\mathbb{Q}}(M(f_i) \otimes M(\varphi_i))(j))} \in E_1 \otimes 1$$

because $f_i \otimes \varphi_i$ is a modular form. From (4.1) and (4.2), we deduce Theorem 1.2:

$$\frac{\mathbb{L}(M(f)(j)_{/F'}, 0)}{c^+(\operatorname{Res}_{F'/\mathbb{Q}}(M(f)(j)_{/F'}))} \in E_1 \otimes 1.$$

Actually, in this last result one can replace E_1 by $\mathbb{Q}_f$ since $M(f)(j)_{/F'}$ has coefficients in $\mathbb{Q}_f$. □

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