Regarding the justification of the usual "sharp" axioms, a convenient reference is Cox (1946), a paper that was unjustly reviewed only by title in *Mathematical Reviews*, and was overlooked by most probabilists until it was essentially reprinted in Cox (1961, pages 1–24).

I'd like to comment concerning Fishburn's discussion of transitivity. It seems intuitively clear to me that if you prefer A to B and B to C, then you *should* rationally prefer A to C. The example concerning Sue's intransitivity seems to me to show that she simply made a mistake, one that, if pointed out to her, should make her reconsider her judgments unless either she is obstinate or, owing to shortage of time, she prefers to live with inconsistency. It may be useful for theoretical psychology, and practical economics, to find axioms that describe actual behavior, but my interest has been in a normative theory.

The example where $A \sim C$, $C \sim B$, and $A > B$ requires more discussion. It is analogous to a situation where A, B, and C are three points on a line, A and C being too close to distinguish, and similarly C and B; but A just far enough from B to be distinguished. The situation is like the one discussed by Good and Tideman (1981). A man in a restaurant can't at first decide between steak and chicken. He then thinks to himself that he would prefer steak to lobster (which wasn't in fact on the menu), but would not be able to *perceive* that chicken is better than lobster. From this he deduces that the utility of chicken lies between those of lobster and steak. Symbolically $U(\text{steak}) > U(\text{chicken}) > U(\text{lobster})$. We described this situation by saying that steak is *discernibly* better than chicken but not *perceptibly* better.

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Comment

Patrick Suppes

Peter Fishburn has provided an excellent, well-documented survey of the substantial literature on the axiomatic foundations of subjective probability. In my comments it is my purpose not to criticize Fishburn's survey but rather to raise some conceptual questions about the literature itself. I hope thereby to stress the importance of some problems that have received little emphasis in the literature, and consequently are scarcely mentioned in Fishburn's survey, but that are fundamental from a foundational standpoint.

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PROBLEM OF UNIQUENESS

In Section 2 of his article, Fishburn brings out the following well-known fact. The known necessary and sufficient conditions for the existence of a probability measure agreeing with the qualitative ordering, given that the algebra of events is finite, do not establish uniqueness of the measure, if no extensions to some additional sort of infinite structure are provided. The point I want to emphasize is what seems to be the fundamental character of the results here. For a given finite algebra and a given ordering, it is of course possible to write down conditions that are necessary and sufficient for existence of a unique measure, but there do not seem to be any very interesting general results available. Fishburn does give a useful, necessary, and sufficient uniqueness condition in terms of linear independence, conditional on the existence of a measure strictly positive for all atoms. My point is to emphasize that the qualitative theory of probability for finite collection of events does not seem to be a really suitable object for extensive study. As Fishburn points out, when the measure is not unique in such situations, conditional probabilities are not uniquely defined and in a similar way independence is not uniquely defined if we use the standard quantitative definition for independence: events A and B are independent if and only if $P(A \cap B) = P(A)P(B)$.

It is of course easy to give sufficient but not necessary conditions that make the measure unique in the finite case. A simple such condition, for example, is that all the atoms have equivalent qualitative probability. It is clear, at the same time, that much weaker structures will still yield sufficient conditions for uniqueness. It seems to me very likely that even if we could state in general form qualitative necessary and sufficient conditions for the existence of a unique probability measure in the finite case we might well find the conditions uninteresting. I am of course not trying to give a knockdown argument that what I am saying is correct. It just seems to me that there are persuasive arguments for thinking that finite algebras of events with a qualitative ordering are not really the right objects for proper development of the subjective theory of probability.

CONCEPT OF EXCHANGEABILITY

It is an unfortunate and paradoxical fact that the concept of exchangeability, which many people consider perhaps the most important single concept of the subjective theory of probability, has played almost no role in the formulation of the qualitative axioms of subjective probability. Fishburn has really nothing to say about exchangeability but this is not surprising, for the many qualitative axiom systems he surveys do not use exchangeability in their formulation. Now it may be said that this is a natural omission, for the objective of qualitative foundations is to give the simplest axioms of a qualitative character on probability judgments that are sufficient to guarantee the existence of a quantitative probability measure. However, this argument is not really a sound one. If the axioms that have been given were always concerned with necessary and sufficient axioms, then it could be argued that since exchangeability is not necessary for the existence of a probability measure it should be ignored. It is apparent enough, however, from Fishburn's extensive survey that many of the axiom systems that have played a prominent role give only sufficient and not necessary conditions. Once various kinds of structural axioms are sought to make the axioms simple and at the same time sufficient but not necessary, it is natural to ask about a place for the central concept of exchangeability.

To focus my remarks, I restrict them to two topics that are of central importance in the foundations of probability, namely, the concepts of independence and randomness. I want to say something about how each of these central concepts may be related to exchangeability. Before doing that, it seems to me unnecessary to give a definition of exchangeability in technical notation, for the intuitive idea is simple and easily understandable. Let us suppose that we have a sequence of n events, for example, the occurrence of rain on n successive days. We then say that the n possible events are exchangeable if and only if the probability of exactly h of them occurring (for $h = 0, 1, 2, \ldots, n$) is the same no matter in what order the h might occur. Thus, for example, if we were asking if ten flips of a coin, by a device which might not guarantee independence, are exchangeable, we would mean that, for example, the occurrence of three heads in the ten flips would have the same probability no matter on what trials the three heads occurred. A standard example of an exchangeable process in which the events are not independent is drawing balls from an urn without replacement.

Exchangeability and Independence

At least two of the works cited by Fishburn, namely, Domotor (1969) and Luce and Narens (1978), introduce a qualitative binary relation of independence for events as part of their axiomatization of subjective probability. Luce and Narens axiomatize the repetition of an experiment where the repetitions are independent and, in principle, unlimited in number. This is a natural and useful concept in many statistical contexts. However, there is a central criticism conceptually of this approach from the standpoint of subjective probability, a criticism made prominent many years ago by de Finetti (1937). If a sequence of possible events is judged independent, then there is, as de Finetti puts it, no learning from experience. If independence is assumed from the beginning, then knowledge of occurrence of events up to the present time will not affect the judgment of the probability of events yet to occur. Exchangeability, on the other hand, does not impose such a strong condition of independence and is, as has been argued by de Finetti and others on many occasions, the appropriate subjective replacement for independence.

Exchangeability and Randomness

What to do about randomness is a continuing puzzle for Bayesians or subjectivists. It is not appropriate to canvass the many subtle issues involved, but there is one central point relevant to the present discussion. In many contexts a finite sequence of trials being exchangeable is the Bayesian surrogate for randomness. In fact, it is possible to establish a close relation between exchangeable sequences and sequences that are random in the sense of complexity (see, e.g., Fine, 1973). My point is that qualitative axioms of exchangeability should also be introduced to provide at least one natural subjective approach to randomness, a concept that needs explicit treatment in a fully developed subjective theory of probability.

It is, of course, possible for someone to respond to these remarks about exchangeability by denying the need for a *qualitative* approach. On this view the simplest least constrained qualitative axioms that lead to the existence of a probability measure should be adopted. Additional constraints should be stated in terms of a given probability measure. It can still be argued that it is of foundational interest to understand the role that can be played by qualitative axioms of exchangeability, much in the spirit of earlier studies of qualitative independence.

EXPECTATION OR PROBABILITY

A constant issue in the theory of subjective probability is whether expectation or probability is the more fundamental concept. In the standard textbook treatments of probability, one begins with the Kolmogorov quantitative axioms for probability spaces and then defines random variables as real valued measurable functions on the probability space. Expectation is introduced as the expectation of a random variable. So, in this standard line of development, expectation is defined in terms of probability.

There is much about subjective theory that argues for a reversal of this dependency of expectation on probability. First of all, there is already good support in the early history of probability theory. It is to be found in the early writings of Huygens and of Bayes. Fishburn quotes the definition that Bayes gives of probability in terms of expectation in his well-known essay of 1763:

> The *probability of any event* is the ratio between the value at which an expectation depending on the happening of the event ought to be computed, and the value of the thing expected upon it's [sic] happening (page 376).

Bayes's meaning in this brief statment is not entirely clear, but if we think in terms of gambles it is easy to give examples. Bayes himself, in the first proposition following the definitions, gives a rather clear example that it will be useful to quote.

> When several events are inconsistent the probability of the happening of one or other of them is the sum of the probabilities of each of them.
>
> Suppose there be three such events, and which ever of them happens I am to receive N, and that the probability of the 1st, 2d, and 3d are, respectively, a/N, b/N, c/N. Then (by the definition of probability) the value of my expectation from the 1st will be a, from the 2d b, and from the 3d c. Wherefore the value of my expectations from all three will be $a + b + c$. But the sum of my expectations from all three is in this case an expectation of receiving N upon the happening of one or other of them. Wherefore (by definition 5) the probability of one or other of them is $(a + b + c)$/N or $(a/\mathrm{N}) + (b/\mathrm{N}) + (c/\mathrm{N})$. The sum of the probabilities of each of them (pages 376–377).

Finally it is worth quoting from proposition 2, which has a very decided modern flavor.

> If a person has an expectation depending on the happening of an event, the probability of the event is to the probability of its failure as his loss if it fails to his gain if it happens (page 377).

de Finetti as well, in his coherence argument, certainly takes random variables or, as he prefers, random quantities as primitive.

It may be argued that in ordinary experience expectation rather than probability is the more widely used concept. As I have put it elsewhere, the argument for this is evident from a practical standpoint. Once we leave events and talk about what correspond to random variables it is natural in ordinary talk to want to know only the expectation and not the full probability distribution. Thus we talk about the expectation of walking at least 10 km in the next 3 days, the expectation of at least 3 cm of rain in the next 8 hours, or the expectation that the rate of inflation in the next year will be about 9%. In all of these cases, we are dealing in a natural way with a quantitative variable, but we are not prepared to give, and are really not interested in giving, the full probability distribution of that variable or even, usually, the variance.

Although Fishburn mentions all the main references, e.g., Savage (1954), he does not really sharply distinguish those axiomatizations that begin with expectation from those that begin with probability, and make no mention in the axioms of expectation. Savage takes as primitive the preference relation between possible acts or decisions; the one that is preferred should have the greater expected utility. This is also the case for the earlier work of Ramsey (1931), but not, e.g., of Koopman (1940a). On the other hand, what is taken as primitive can be deceiving. Fishburn gives a good account of the partition axiom Savage actually uses, whose content is more naturally stated in terms of probability (Savage's Axiom P6′), even though the stronger formulation (Axiom P6) is in terms of expectation.

There is a more important point to be made about whether expectation or probability is to be taken as primitive. When one begins with expectation, it may not be possible to get to probability, but not conversely, for expectation is in some sense the weaker and more general concept. Here is an example from Krantz et al. (1971). Axioms are given for conditional expected utility. Acts are now conditional acts defined only for a certain set of states, i.e., acts are now partial functions. Thus, f_A is the act f restricted to the set A of possible states of nature. Let $u(f_A)$ be the expected utility of f_A. Then the axioms are such that for all conditional acts f_A and g_B:

(i) $u(f_A) \geq u(g_B)$ iff $f_A \geq g_B$;
(ii) if $A \cap B = \varnothing$, then

$$u(f_A \cup g_B) = u(f_A)P(A \mid A \cup B) + u(g_B)P(B \mid A \cup B).$$

From a formal standpoint, the expectation $u(f_A)$ is not further analyzed into utility and probability in the standard fashion, i.e.,

$$u(f_A) = \sum_{s \in A} p(s)u(f(s)), \tag{1}$$

where $p(s)$ is the probability of the state s and $u(f(s))$ is the pure utility of consequence $f(s)$. Additional structural axioms are required to prove that the representation given by equation (1) is possible.

We can go a good deal further than this example suggests in developing a very general and therefore weak theory of expectation that involves no sample space, no algebra of events, and no probability explicitly. This development, which I sketch, might be regarded as the qualitative axiomatic theory of de Finetti's random quantities (1937), which also assume no underlying sample space or algebra of events. After Fishburn's extensive and detailed survey of many systems of axioms, it does not seem appropriate to offer additional systems as my closing remarks. I do want to sketch how axioms on expectations of the sort indicated can be developed. As an example, consider a person or group selecting possible acts to execute from a set, say A. Let a and b be members of A, e.g., the set of acts of buying individual commodities in a supermarket. Then $a \cdot b$ is the composite act of executing a and also b. Moreover, as usual we order acts, including composite ones, according to their qualitative expectation. Then, various familiar axioms for extensive quantities can be stated for this setup (see, e.g., Krantz et al., 1971). The representation, using E for numerical expectation, would look like this:

(i) $E(a) \geq E(b)$ iff $a \succcurlyeq b$;
(ii) $E(a \cdot b) = E(a) + E(b)$.

Much more complicated axioms are required to get the standard expected utility representation as expressed in equation (1). The weak theory of expectation expressed in (i) and (ii) is, in contrast, very simple.

An alternative axiomatic formulation of expectation in terms of additive conjoint measurement can also be given. Here n-tuples of acts are considered, and the representation is:

$$\sum_{i=1}^{n} E(a_i) \geq \sum_{i=1}^{n} E(b_i) \quad \text{iff} \quad (a_1, \ldots, a_n) \succcurlyeq (b_1, \ldots, b_n)$$

(For axioms of conjoint measurement, see Krantz et al., 1971).

It may properly be claimed that much use of the concept of expectation is missed by the sort of interpretation in terms of acts I have just given. Expectation of the values of properties are parallel to the subjective probability of events rather than the expectation of acts. The same axioms for extensive measurement can be used, without, again, necessarily being able to prove the existence of an underlying probabilistic representation. Empirical tests of what axioms in fact are satisfied by subjective expectations as a generalization of subjective beliefs have, as far as I know, as yet scarcely begun.