The Annals of Statistics
1987, Vol. 15, No. 2, 830–844

A NONLINEAR, NONTRANSITIVE AND ADDITIVE-PROBABILITY MODEL FOR DECISIONS UNDER UNCERTAINTY

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Let $\succ$ denote a preference relation on a set F of lottery acts. Each f in F maps a state space S into a set P of lotteries on decision outcomes. The paper discusses axioms for $\succ$ on F which imply the existence of an SSB (skew-symmetric bilinear) functional ϕ on $P \times P$ and a finitely additive probability measure π on 2^S such that, for all f and g in F,

$$f \succ g \Leftrightarrow \int_S \phi(f(s), g(s))\, d\pi(s) > 0.$$

This S^3B (states SSB) model generalizes the traditional Ramsey–Savage model in which ϕ decomposes as $\phi(p, q) = u(p) - u(q)$, where u is a linear functional on P. The S^3B model preserves the probability structure of the Ramsey–Savage model while weakening their assumptions of transitivity and independence.

1. Introduction. Let P be a convex set of probability distributions (one-stage lotteries) on a set of decision outcomes, and let F denote the set of all functions (acts) from a set of S of states of the world into P. This paper presents axioms for a preference relation $\succ$ on F that imply the existence of an SSB (skew-symmetric bilinear) functional ϕ on $P \times P$ and a finitely additive probability measure π on the algebra 2^S of all subsets of S such that, for all f and g in F,

$$f \succ g \Leftrightarrow \int_S \phi(f(s), g(s))\, d\pi(s) > 0.$$

We refer to the integral as an S^3B *utility functional* since it extends by expectation over states the SSB utility functional ϕ from $P \times P$ to $F \times F$. The representation itself for the *is preferred to* relation $\succ$ on F will be referred to as the S^3B *model.*

The S^3B model was introduced by Loomes and Sugden (1982) and Bell (1982) for finite S. Fishburn (1984) axiomatized the model for finite S, and we are currently exploring its implications for Bayesian decision theory. Our purpose here is to extend the finite-S axiomatization to infinite state spaces by adding axioms that are inapplicable or superfluous when S is finite. At the same time, Fishburn's finite-S axioms are modified to accommodate the present context. We interpret our model normatively, as done by Savage (1954) and the authors cited above for their models.

Received March 1986; revised September 1986.
AMS 1980 *subject classifications.* Primary 90A05, 90A06; secondary 90A10, 06A99.
Key words and phrases. Decision theory, SSB utility, lottery acts, preference axioms.

Section 2 presents Fishburn's axioms and their principal implications for arbitrary nonempty S. Our new axioms and their consequences appear in Section 3, where S is presumed to be infinite. To facilitate mathematical simplicity without giving up much in the way of applicability, it is assumed that for each positive integer n, there is an n-part partition of S each event in which is nonnull. This is much weaker than Savage's (1954) partition axiom P6 and axioms used by others that for any n imply that S can be partitioned into n equally-likely events. However, it should be noted that all probabilities between 0 and 1 are used in the formation of the lottery set P, which is not used by Savage as a primitive.

The other new axioms in Section 3 are a weak and a strong version of states-dominance axioms. The weak version says that if $f(s) \succsim g(s)$ for all s in S, then $f \succsim g$, where $\succsim$ denotes the union of $\succ$ and its symmetric complement $\sim$. The strong version entails the weak version and apparently more: See Section 3, where a difficulty in the formulation of the strong version is discussed. It is noted among other things that the weak states-dominance axiom implies that ϕ is bounded and that the S^3B model holds when at least one of f and g is simple, i.e., assigns at most a finite number of lotteries in P to the states in S. The strong states-dominance axiom yields the S^3B model for all f and g in F.

Section 4 concludes the paper with proofs of assertions not proved previously.

Other generalizations of the familiar Ramsey (1931)–Savage (1954) model

$$f \succ g \Leftrightarrow \int u(f(s))\, d\pi(s) > \int u(g(s))\, d\pi(s)$$

are discussed by Schmeidler (1984), Gilboa (1985), Luce and Narens (1985) and Fishburn (1984, 1985). The first three of these assume that $\succ$ is a weak order, retain the utility separation between f and g of the Ramsey–Savage model and replace π by a nonadditive measure or measures. Fishburn's models do not assume that either $\succ$ or $\sim$ is transitive and interconnect the utility evaluations for f and g through ϕ. In addition, Fishburn (1985) generalizes the additivity property for subjective probability. The S^3B model is the only generalization among these that preserves the additivity feature of the Ramsey–Savage model.

2. Formulation and basic axioms. It is assumed henceforth that S is a nonempty set, P is a nonempty convex set of probability distributions on an outcome set and F is the set of all functions from S into P. We refer to elements in S as *states*, subsets of S as *events* and functions in F as *acts*. Convexity for P means that $\lambda p + (1 - \lambda)q \in P$ whenever $p, q \in P$ and $0 \leq \lambda \leq 1$. Given $f, g \in F$ and $0 \leq \lambda \leq 1$, $\lambda f + (1 - \lambda)g$ denotes the act that assigns $\lambda f(s) + (1 - \lambda)g(s)$ from P to state s. F is convex since P is convex. We refer to act f as *simple* if $\{f(s)\colon s \in S\}$ is finite.

Let $\succ$ be a binary relation on F. Our axioms imply that $\succ$ is asymmetric, so $f \succ g \Rightarrow$ not $(g \succ f)$, but they do not imply that $\succ$ is transitive. We define $\sim$

(indifference) and $\succsim$ (preferred or indifferent to) on F by

$$f \sim g, \quad \text{if not } (f \succ g) \text{ and not } (g \succ f),$$

$$f \succsim g, \quad \text{if } f \succ g \text{ or } f \sim g.$$

Given $p \in P$, let p^* denote the *constant act* in F that assigns p to every state. We define $p \succ (\sim, \succsim) q$ to mean that $p^* \succ (\sim, \succsim) q^*$. Similarly, $f \succ p$ means $f \succ p^*$, and so forth.

For any event A in 2^S, fAg denotes the act that assigns $f(s)$ to each $s \in A$ and $g(s)$ to each $s \in A^c$, the *complement* of A. Similarly, fAp is the act that assigns $f(s)$ to $s \in A$ and $p \in P$ to all $s \in A^c$, and pAq assigns p to all $s \in A$ and q to all $s \in A^c$. For example, if you "do" pAq, then you get p if A obtains and you get q otherwise.

Event A is *null* if, for all $p, q, r \in P$, $pAr \sim qAr$. We let $\mathcal{N}$ denote the set of null events in 2^S.

The following six axioms apply to all $p, q, r \in P$, all $A, B \in 2^S$, all $f, g, h \in F$, all $0 < \lambda < 1$ and all $0 \leq \mu \leq 1$.

(A.1) $f \succ g \succ h \Rightarrow g \sim \alpha f + (1-\alpha)h$ for some $0 < \alpha < 1$.

(A.2) $[f \succ g,\ f \succsim h] \Rightarrow f \succ \lambda g + (1-\lambda)h$; $[g \succ f,\ h \succsim f] \Rightarrow \lambda g + (1-\lambda)h \succ f$; $[f \sim g,\ f \sim h] \Rightarrow f \sim \lambda g + (1-\lambda)h$.

(A.3) $[f \succ g,\ g \succ h,\ f \succ h,\ g \sim \frac{1}{2}f + \frac{1}{2}h] \Rightarrow [\lambda f + (1-\lambda)h \sim \frac{1}{2}f + \frac{1}{2}g \Leftrightarrow \lambda h + (1-\lambda)f \sim \frac{1}{2}h + \frac{1}{2}g]$.

(B.1) $S \notin \mathcal{N}$.

(B.2) $[A \notin \mathcal{N},\ B \notin \mathcal{N}] \Rightarrow [pAr \succ qAr \Leftrightarrow pBr \succ qBr]$.

(B.3) $[A \cap B = \varnothing,\ \{f_1, f_2, f_3\} = \{fAp, gBp, p^*\}] \Rightarrow [f_1 \sim \mu f_2 + (1-\mu)f_3 \Rightarrow \frac{1}{2}f_1 + \frac{1}{2}f_3 \sim \frac{1}{2}(\mu f_2 + (1-\mu)f_3) + \frac{1}{2}f_3]$.

Axioms (A.1)–(A.3), which make no explicit use of S, are the basic SSB axioms from Fishburn (1982). They entail neither transitivity nor the usual independence or substitution axioms of expected utility. (A.1) is a continuity or intermediate-value assumption: If f is preferred to g, and g is preferred to h, then g is indifferent to a nontrivial convex combination of f and h. (A.2) is a convexity-dominance axiom. Its final part ($\sim$) says that the subset of F whose acts are indifferent to f is convex. Its first two parts imply that the subsets of acts less preferred than f and more preferred than f are both convex. (A.3) is a special case of the symmetry condition, which says that if g lies midway in preference between f and h in the sense that $f \succ g \succ h$ and $g \sim \frac{1}{2}f + \frac{1}{2}h$, then an $\sim$ statement involving only f, g and h remains at $\sim$ when f and h are interchanged throughout.

Axioms (B.1)–(B.3) are modifications of axioms in Fishburn (1984) for finite S. (B.1) asserts that the universal event is not null. (B.2) is an independence assumption which says that nonnull events have similar preference implications for special comparisons of two-lottery simple acts. It is similar to the P3 part of Savage's (1954) sure-thing principle. Finally, (B.3) is another type of independence axiom that brings 50–50 mixtures into play for acts based on two disjoint events. It is a restriction of the much more powerful independence axiom of Herstein and Milnor (1953) which says that, for all $f, g, h \in F$, $f \sim g \Leftrightarrow \frac{1}{2}f + \frac{1}{2}h \sim \frac{1}{2}g + \frac{1}{2}h$. As noted in Fishburn (1984, page 267), if (B.3) were replaced by the Herstein–Milnor axiom, then our S^3B model would simplify to the Ramsey–Savage model.

The following two theorems summarize most implications of (A.1)–(A.3) and then (A.1)–(B.3). We recall that a functional ϕ on $F \times F$ is SSB if it is skew-symmetric $[\phi(g, f) = -\phi(f, g)]$ and linear separately in each argument. Thus, for the first argument,

$$\phi(\lambda f + (1 - \lambda)g, h) = \lambda\phi(f, h) + (1 - \lambda)\phi(g, h),$$

when $0 \leq \lambda \leq 1$, and similarly for the second argument. Our first result is the main theorem in Fishburn (1982).

THEOREM 1. (A.1), (A.2) *and* (A.3) *hold if and only if there is an SSB functional* ϕ *on* $F \times F$ *such that, for all* $f, g \in F$,

$$f \succ g \Leftrightarrow \phi(f, g) > 0. \tag{1}$$

Given (1) *for all* $f, g \in F$, (1) *holds for an SSB functional* ϕ' *on* $F \times F$ *in place of* ϕ *if and only if* $\phi' = c\phi$ *for some real number* $c > 0$.

The latter part of Theorem 1 is abbreviated by saying that ϕ is unique up to similarity transformations. Given ϕ on $F \times F$, we *always* define ϕ on $P \times P$ by

$$\phi(p, q) = \phi(p^*, q^*).$$

THEOREM 2. *Suppose* (A.1)–(A.3) *and* (B.1)–(B.3) *hold. Then there is an SSB functional* ϕ *on* $F \times F$ *and a unique finitely additive probability measure* π *on* 2^S *such that, for all* $f, g \in F$, *all* $A \in 2^S$ *and all* $p_i, q_j \in P$

(a) (1) *holds*;

(b) $A \in \mathcal{N} \Leftrightarrow \pi(A) = 0$;

(c) *if* $\{A_1, \ldots, A_n\}$ *and* $\{B_1, \ldots, B_m\}$ *are partitions of* S *with* $f(s) = p_i$ *for all* $s \in A_i$, $1 \leq i \leq n$, *and* $g(s) = q_j$ *for all* $s \in B_j$, $1 \leq j \leq m$, *then*

$$\phi(f, g) = \sum_{i=1}^{n} \sum_{j=1}^{m} \pi(A_i \cap B_j)\phi(p_i, q_j).$$

This extends Theorem 5 in Fishburn (1984) when S is infinite. That theorem implies that if $T = \{A_1, \ldots, A_n\}$ is any finite partition of S, then there is a unique probability measure π_T on the algebra with atoms A_1–A_n such that (with ϕ as in Theorems 1 and 2)

$$\phi(f, g) = \sum_{i=1}^{n} \pi_T(A_i)\phi(p_i, q_i),$$

when $f = p_i$ and $g = q_i$ on A_i for $i = 1, \ldots, n$. For Theorem 2, it remains only to show that $\pi_T(A) = \pi_V(A)$ whenever finite partitions T and V both have A as an "event." When they do, $f = pAr$ and $g = qAr$ imply $\phi(f, g) = \pi_T(A)\phi(p, q) = \pi_V(A)\phi(p, q)$. If $A \in \mathcal{N}$, then $\pi_T(A) = \pi_V(A) = 0$; if $A \notin \mathcal{N}$, then, since (B.1) assures $\phi(p, q) \neq 0$ for some $p, q \in P$, we get $\pi_T(A) = \pi_V(A) > 0$. We can therefore drop the subscripts on π to complete the proof of Theorem 2.

Theorem 2 is not stated in the "if and only if" form because one of its axioms, (B.3), is not wholly necessary for its conclusions although (B.3) *is* necessary for the general S^3B model. [If (B.3) is weakened by replacing f and g in its statement by lotteries in P, then (A.1)–(B.3) are necessary and sufficient for the conclusions of Theorem 2.] The following lemma, which is needed later, shows a further implication of (B.3).

LEMMA 1. *Suppose* (A.1)–(A.3) *and* (B.3) *hold with* ϕ *as in Theorem* 1. *Then, for all* $f, g \in F$, *all* $p \in P$ *and every partition* $\{A_1, \dots, A_n\}$ *of* S,

$$\phi(f, g) = \sum_{i=1}^{n} \phi(fA_i p, gA_i p).$$

PROOF. First, let A be an event with $A \notin \{\varnothing, S\}$. For (B.3), let $B = A^c$ with $\{f_1, f_2, f_3\} = \{fAp, gA^cp, p^*\}$. We show that

$$\phi(f_1, f_2) + \phi(f_2, f_3) + \phi(f_3, f_1) = 0. \tag{2}$$

Clearly, there is some $\mu \in [0,1]$ and some permutation f_1, f_2 and f_3 of fAp, gA^cp and p^* such that $f_1 \sim \mu f_2 + (1 - \mu) f_3$. By Theorem 1, $\phi(f_1, \mu f_2 + (1 - \mu) f_3) = 0$, so linearity in the second argument yields

$$\mu[\phi(f_1, f_2) - \phi(f_1, f_3)] = -\phi(f_1, f_3).$$

Also, by (B.3), $\frac{1}{2}f_1 + \frac{1}{2}f_3 \sim \frac{1}{2}(\mu f_2 + (1 - \mu) f_3) + \frac{1}{2}f_3$, or $\phi(\frac{1}{2}f_1 + \frac{1}{2}f_3, \frac{1}{2}(\mu f_2 + (1 - \mu) f_3) + \frac{1}{2}f_3) = 0$. Bilinearity and skew-symmetry then give

$$\mu\phi(f_2, f_3) = \phi(f_1, f_3).$$

When this is added to the preceding equation, we get $\mu[\phi(f_1, f_2) + \phi(f_2, f_3) + \phi(f_3, f_1)] = 0$, so (2) holds if $\mu > 0$. If $\mu = 0$, simply interchange f_2 and f_3 throughout the preceding derivation to obtain (2).

Continuing with $A \notin \{\varnothing, S\}$, observe that

$$\tfrac{1}{3}f + \tfrac{1}{3}(gAp) + \tfrac{1}{3}(gA^cp) = \tfrac{1}{3}g + \tfrac{1}{3}(fAp) + \tfrac{1}{3}(fA^cp),$$
$$\tfrac{1}{2}f + \tfrac{1}{2}p^* = \tfrac{1}{2}(fAp) + \tfrac{1}{2}(fA^cp),$$
$$\tfrac{1}{2}g + \tfrac{1}{2}p^* = \tfrac{1}{2}(gAp) + \tfrac{1}{2}(gA^cp).$$

When ϕ(left-hand side, right-hand side) = 0 is expanded bilinearly for each of these and substitutions are made in the first $\phi = 0$ from the other two, we get

$$\phi(f, g) + \phi(fAp, p^*) + \phi(fA^cp, p^*) + \phi(p^*, gAp) + \phi(p^*, gA^cp)$$
$$+\phi(gAp, fAp) + \phi(gA^cp, fA^cp) + \phi(gAp, fA^cp) + \phi(gA^cp, fAp) = 0.$$

But, by (2),

$$\phi(fAp, p^*) + \phi(p^*, gA^cp) + \phi(gA^cp, fAp) = 0$$

and

$$\phi(fA^c p, p^*) + \phi(p^*, gAp) + \phi(gAp, fA^c p) = 0.$$

Therefore,

$$\phi(f, g) = \phi(fAp, gAp) + \phi(fA^c p, gA^c p).$$

The conclusion of Lemma 1 follows directly from this as follows:

$$\begin{aligned}\phi(f, g) &= \phi(fA_1 p, gA_1 p) + \phi(fA_1^c p, gA_1^c p)\\ &= \phi(fA_1 p, gA_1 p) + \phi(fA_2 p, gA_2 p)\\ &\quad + \phi(f(A_1 \cup A_2)^c p, g(A_1 \cup A_2)^c p)\\ &\;\;\vdots\\ &= \sum \phi(fA_i p, gA_i p).\end{aligned}$$

□

Theorem 2 yields the S^3B model for all simple f and g in F. The next section adds axioms sufficient for the model for all pairs of acts in F.

3. Extension. It is assumed henceforth that (A.1)–(B.3) hold along with ϕ and π as in Theorems 1 and 2. Since there is nothing more to say beyond Theorem 2 if S is finite, we consider infinite S in what follows.

(B.1)* *For every positive integer n there is an n-part partition of S each part of which is not null.*

Although unnecessary for the S^3B model, (B.1)* is generous in the types of π measures it allows. For example, some $\{s\}$ could have positive probabilities, or we could have $\pi(A) = 0$ for all finite A in 2^S. Our main use of (B.1)* is its implication, in view of Theorem 2(b), that there is a denumerable partition $\{A_1, A_2, \dots\}$ of S with $\pi(A_i) > 0$ for all i. We refer to such a partition as a *positive denumerable partition.* A positive denumerable partition can be constructed from the partitions of (B.1)* by a sequential procedure that at each step chooses a positive probability event disjoint from its chosen predecessors in such a way that the complement of the union of events chosen thus far admits a sequence of positive-probability partitions whose number of parts increases without bound. A new sequence of partitions of the unchosen complement is constructed at each step.

Our two states-dominance axioms are, for all f, g, f' and g' in F:

(B.4) *If* $f(s) \succsim g(s)$ *for all* $s \in S$, *then* $f \succsim g$.

(B.4)* *If* $f \sim g$ *and* $\phi(f'(s), g'(s)) \geq \phi(f(s), g(s))$ *for all* $s \in S$, *then* $f' \succsim g'$.

Axiom (B.4), which is implied by (B.4)* when we take $f = g$ in (B.4)*, has a strong intuitive appeal. It says that if f is preferred or indifferent to g regardless of which state obtains, then f as a whole is preferred or indifferent to g.

The stronger (B.4)* is less compelling but is still appealing. In the SSB context it is not unreasonable to interpret $\phi(p, q)$ as a utility differential between p and q *in the lottery context.* [If the von Neumann–Morgenstern expected utility axioms hold for $\succ$ on P, then ϕ decomposes as $\phi(p,q) = u(p) - u(q)$.] Under this interpretation, (B.4)* says that if f is indifferent to g and if, for every s, the utility differential of $f'(s)$ over $g'(s)$ is at least as great as that of $f(s)$ over $g(s)$, then f' as a whole is preferred or indifferent to g'. It is easily seen that (B.4)* is necessary for the S^3B model.

The obvious problem with (B.4)* is its direct reference to the functional ϕ. Although $\phi(p', q') \geq \phi(p, q)$ in its statement could be replaced by conditions that refer solely to $\succ$ on P, it is messy to do so [see the construction of ϕ in Fishburn (1982)] and would add nothing to its intuitive interpretation.

We had hoped to obtain the complete S^3B model from (B.1)* and (B.4) but have been unable to do so. We have also not found a counterexample to the assertion that the model follows from (B.1)* and (B.4), so the question remains open. The main facts we have been able to establish from these two axioms are summarized in the following theorem, which is proved in the next section.

THEOREM 3. *Suppose* (B.1)* *and* (B.4) *hold. Then, for all* $p \in P$ *and all* $f, g \in F$

(a) ϕ *is bounded on* $P \times P$;
(b) $\inf_S \phi(f(s), p) \leq \phi(f, p) \leq \sup_S \phi(f(s), p)$;
(c) *if at least one of* f *and* g *is simple*,

$$\phi(f, g) = \int_S \phi(f(s), g(s))\, d\pi(s); \tag{3}$$

(d) ϕ *is bounded on* $F \times F$.

On replacing (B.4) with (B.4)*, we obtain the complete S^3B model.

THEOREM 4. *Suppose* (B.1)* *and* (B.4)* *hold. Then* (3) *holds for all* f *and* g *in* F.

Bounded utilities on P, or on $P \times P$ as in Theorem 3(a), are not unusual in utility theory. Although not implied by the von Neumann–Morgenstern (1947) axioms, boundedness is a consequence of the Blackwell–Girschick (1954) axioms for expected utility and of some subsequent axiomatizations such as Fishburn (1967). Savage's (1954) subjective expected utility theory in the states context also entails bounded utilities [Fishburn (1970), Chapter 14].

In the present theory, bounded SSB utilities are a consequence of our partition axiom (B.1)* in conjunction with the weak states-dominance axiom (B.4). As stated in Theorem 3, these axioms also yield the S^3B model when at least one of f and g is simple. Our inability to obtain (3) for all f and g from (B.1)* and (B.4) stems from our inability to derive the ensuing conclusion of Lemma 2 from these axioms.

Lemma 2. (B.1)* *and* (B.4)* *imply that, for all* $f, g \in F$,

$$\inf_S \phi(f(s), g(s)) \le \phi(f, g) \le \sup_S \phi(f(s), g(s)).$$

As shown by Theorem 3(b), this is partly implied by (B.4) in place of (B.4)*. We conclude this section by proving Lemma 2 and then showing how it yields (3) for all pairs of acts in F, thus completing the proof of Theorem 4. The conclusions of Theorem 3 are used freely in these derivations.

Proof of Lemma 2. Let A be an event in 2^S for which $\pi(A) > 0$ and $\pi(A^c) > 0$, as guaranteed by Theorem 2 and (B.1)*. By Theorem 1, (B.1) and Theorems 3(a) and 3(d), we can assume with no loss in generality that

$$\sup_{P \times P} \phi(p, q) = 1, \qquad \sup_{F \times F} \phi(f, g) = K \ge 1, \quad K \text{ finite}.$$

Choose $y, z \in P$ such that $\phi(y, z) > \frac{1}{2}$, and define x as $\frac{1}{2}y + \frac{1}{2}z$ so that $\phi(x, z) = \phi(y, x) > \frac{1}{4}$. Also let λ be any positive real number that does not exceed $\min\{\pi(A), \pi(A^c)\}/(4K)$.

Given $f, g \in F$, let $g_0 = \lambda g + (1 - \lambda)f$, so $\phi(f, g_0) = \lambda\phi(f, g)$ and $\phi(f(s), g_0(s)) = \lambda\phi(f(s), g(s))$ for all s. We show that the conclusion of Lemma 2 holds for (f, g_0), so it must also hold for (f, g).

By Lemma 1,

$$\phi(f, g_0) = \phi(fAx, g_0Ax) + \phi(fA^cx, g_0A^cx),$$

with $\phi(fAx, g_0Ax) = \lambda\phi(fAx, gAx)$ and $\phi(fA^cx, g_0A^cx) = \lambda\phi(fA^cx, gA^cx)$. Consequently,

$$\max\{|\phi(fAx, g_0Ax)|, |\phi(fA^cx, g_0A^cx)|\} \le \min\{\pi(A), \pi(A^c)\}/4.$$

It follows from the construction of x that there are p and r in P such that $\pi(A^c)\phi(p, x) = \phi(fAx, g_0Ax)$ and $\pi(A)\phi(r, x) = \phi(fA^cx, g_0A^cx)$, and therefore

$$\phi(f, g_0) = \pi(A)\phi(r, x) + \pi(A^x)\phi(p, x).$$

Moreover, by Lemma 1 and Theorem 2 or (3),

$$\begin{aligned} \phi(fAx, g_0Ap) &= \phi(fAx, g_0Ax) + \phi(xA^cx, pA^cx) \\ &= \phi(fAx, g_0Ax) + \pi(A^c)\phi(x, p) = 0, \\ \phi(fA^cx, g_0A^cr) &= \phi(fA^cx, g_0A^cx) + \phi(xAx, rAx) \\ &= \phi(fA^cx, g_0A^cx) + \pi(A)\phi(x, r) = 0, \end{aligned}$$

so that $fAx \sim g_0Ap$ and $fA^cx \sim g_0A^cr$.

We now apply (B.4)* to each of these indifference statements to obtain the desired sup conclusion. (The inf conclusion is obtained similarly.) It follows from our constructions that there are y' and z' in P such that

$$\phi(y', x) = \sup_A \phi(f(s), g_0(s)),$$

$$\phi(z', x) = \sup_{A^c} \phi(f(s), g_0(s)).$$

The first of these used with $fAx \sim g_0Ap$ in (B.4)* yields $y'Ax \gtrsim xAp$. Hence, by Theorems 1 and 2,

$$\phi(y'Ax, xAp) = \pi(A)\phi(y', x) + \pi(A^c)\phi(x, p) \geq 0$$

or

$$\pi(A)\sup_A \phi(f(s), g_0(s)) \geq \pi(A^c)\phi(p, x).$$

Similarly, the defining equation for z' used with $fA^cx \sim g_0A^cr$ in (B.4)* yields

$$\pi(A^c)\sup_{A^c} \phi(f(s), g_0(s)) \geq \pi(A)\phi(r, x).$$

Therefore,

$$\begin{aligned}\sup_S \phi(f(s), g_0(s)) &= \pi(A)\sup_S \phi(f(s), g_0(s)) + \pi(A^c)\sup_S \phi(f(s), g_0(s))\\ &\geq \pi(A)\sup_A \phi(f(s), g_0(s)) + \pi(A^c)\sup_{A^c} \phi(f(s), g_0(s))\\ &\geq \pi(A^c)\phi(p, x) + \pi(A)\phi(r, x)\\ &= \phi(f, g_0).\end{aligned}$$

□

Proof of Theorem 4. Given $f, g \in F$, let

$$a = \inf_S \phi(f(s), g(s)), \qquad b = \sup_S \phi(f(s), g(s)).$$

If $a = b$, then $\phi(f, g) = a$ by Lemma 2, and

$$\int \phi(f(s), g(s))\, d\pi(s) = \int a d\pi(s) = a\pi(S) = a,$$

so (3) holds for this case.

Assume henceforth that $a < b$. For a given positive integer n let

$$A_1 = \left\{s\colon a \leq \phi(f(s), g(s)) \leq a + \frac{1}{n}(b - a)\right\},$$

$$A_i = \left\{s\colon a + \frac{i-1}{n}(b - a) < \phi(f(s), g(s)) \leq a + \frac{i}{n}(b - a)\right\},$$

$$2 \leq i \leq n.$$

According to Lemma 1,

$$\phi(f, g) = \sum_{i=1}^{n} \phi(fA_ix, gA_ix), \qquad x \in P.$$

Consider one term in this sum where $A_i \neq \varnothing$, and let $a_i = a + (i - 1)(b - a)/n$, $b_i = a + i(b - a)/n$. For every $p, r \in P$ for which $a_i \leq \phi(p, r) \leq b_i$, Lemma 2 implies that

$$a_i \leq \phi(fA_ip, gA_ir) \leq b_i.$$

By Lemma 1 and Theorem 2,

$$\phi(fA_ip, gA_ir) = \phi(fA_ix, gA_ix) + \phi(pA_i^cx, rA_i^cx)$$
$$= \phi(fA_ix, gA_ix) + \pi(A_i^c)\phi(p, r).$$

Take $\phi(p, r)$ close to a_i and then close to b_i to get

$$\pi(A_i)a_i - n^{-2} \leq \phi(fA_ix, gA_ix) \leq \pi(A_i)b_i + n^{-2}.$$

Since $\phi(f, g) = \Sigma\phi(fA_ix, gA_ix)$, it follows that

$$\sum_{i=1}^{n} \pi(A_i)\left[a + \frac{i-1}{n}(b-a)\right] - \frac{1}{n}$$
$$\leq \phi(f, g) \leq \sum_{i=1}^{n} \pi(A_i)\left[a + \frac{i}{n}(b-a)\right] + \frac{1}{n}.$$

Moreover, by the definition of expectation,

$$\sum_{i=1}^{n} \pi(A_i)\left[a + \frac{i-1}{n}(b-a)\right]$$
$$\leq \int_S \phi(f(s), g(s))\, d\pi(s) \leq \sum_{i=1}^{n} \pi(A_i)\left[a + \frac{i}{n}(b-a)\right].$$

Hence,

$$\left|\phi(f, g) - \int_S \phi(f(s), g(s))\, d\pi(s)\right| \leq (b - a + 1)/n.$$

We then obtain (3) by letting $n \to \infty$. □

4. Proof of Theorem 3. The conclusions of Theorem 3 will be established by a series of lemmas. We assume that axioms (A.1)–(A.3), (B.1)–(B.3) and (B.4) hold, with ϕ and π as in Theorems 1 and 2. Axiom (B.1)* is presumed from Lemma 4 onward. Throughout, $f, g \in F$ and $p, r \in P$.

LEMMA 3a. *Let $a = \inf_S \phi(f(s), p)$. Then $a \leq \phi(f, p)$ if either $a = 0$ or ($a > 0$, $p \succ r$ for some r) or ($a < 0$, $r \succ p$ for some r).*

LEMMA 3b. *Let $b = \sup_S \phi(f(s), p)$. Then $\phi(f, p) \leq b$ if either $b = 0$ or ($b > 0$, $p \succ r$ for some r) or ($b < 0$, $r \succ p$ for some r).*

PROOF OF LEMMA 3b. (The proof of Lemma 3a is similar.) If $b = \infty$, then there is nothing to prove, so assume henceforth that b is finite. If $b = 0$, then $p \succsim f(s)$ for all s, so $p \succsim f$ (i.e., $p^* \succsim f$) by (B.4), and therefore $\phi(f, p) \leq 0$.

Suppose next that $b > 0$ and $p \succ r$. Define λ by $\lambda b + (1 - \lambda)\phi(r, p) = 0$. By SSB and the definition of b,

$$\phi(\lambda f(s) + (1-\lambda)r, p) = \lambda\phi(f(s), p) + (1-\lambda)\phi(r, p)$$
$$= \lambda\phi(f(s), p) - \lambda b$$
$$\leq \lambda b - \lambda b = 0,$$

for all s. (B.4) implies $p \succsim \lambda f + (1-\lambda)r$, hence $\lambda b = (1-\lambda)\phi(p, r) \geq \lambda\phi(f, p)$, so $b \geq \phi(f, p)$.

Finally, suppose $b < 0$ and $r \succ p$. Again define λ by $\lambda b + (1-\lambda)\phi(r, p) = 0$. Then $\phi(\lambda f(s) + (1-\lambda)r, p) = \lambda\phi(f(s), p) - \lambda b \leq \lambda b - \lambda b = 0$, so (B.4) $p \succsim \lambda f + (1-\lambda)r$ and, as before, $\phi(f, p) \leq b$. □

REMARK. We now assume (B.1)* and will let $\{A_1, A_2, \dots\}$ be a positive denumerable partition of S ordered so that $\pi(A_1) \geq \pi(A_2) \geq \cdots$ with $\pi(A_i) > 0$ for all i.

LEMMA 4a. *If $r \succ p$ for some r, then ϕ is bounded below on $P \times \{p\}$.*

LEMMA 4b. *If $p \succ r$ for some r, then ϕ is bounded above on $P \times \{p\}$.*

PROOF OF LEMMA 4b. (The proof of Lemma 4a is similar.) Given $p \succ r$, suppose to the contrary of Lemma 4b that ϕ is not bounded above on $P \times \{p\}$. Then $[0, \infty) \subseteq \phi(P \times \{p\})$. Choose $p_i \in P$ for each i so that

$$\phi(p_i, p) = \pi(A_i)^{-1}$$

and define f by $f(s) = p_i$ for all $s \in A_i$, $i = 1, 2, \dots$. We shall obtain the contradiction to Theorem 1 that $\phi(f, p)$ is infinite.

Let g_n be a simple act in F that is constant on each A_i for $i \leq n$ with

$$\phi(g_n(s), p) = \pi(A_n)^{-1} - \pi(A_i)^{-1}, \quad \text{for all } s \in A_i,$$

and that has $g_n(s) = p$ for all $s \in S \setminus (A_1 \cup \cdots \cup A_n)$. Then, by Theorem 2,

$$\begin{aligned}\phi(g_n, p) &= \sum_{i=1}^{n} \pi(A_i)\left[\pi(A_n)^{-1} - \pi(A_i)^{-1}\right]\\ &= \pi(A_n)^{-1}\sum_{i=1}^{n}\pi(A_i) - n.\end{aligned}$$

Note also that for $s \in A_i$, $i \leq n$,

$$\begin{aligned}\phi\left(\left(\tfrac{1}{2}f + \tfrac{1}{2}g_n\right)(s), p\right) &= \tfrac{1}{2}\phi(p_i, p) + \tfrac{1}{2}\phi(g_n(s), p)\\ &= \tfrac{1}{2}\pi(A_n)^{-1},\end{aligned}$$

and for $s \in A_i$, $i > n$,

$$\phi\left(\left(\tfrac{1}{2}f + \tfrac{1}{2}g_n\right)(s), p\right) = \tfrac{1}{2}\pi(A_i)^{-1} \geq \tfrac{1}{2}\pi(A_n)^{-1}.$$

Hence,

$$\inf_S \phi\left(\left(\tfrac{1}{2}f + \tfrac{1}{2}g_n\right)(s), p\right) = \pi(A_n)^{-1}/2 > 0$$

and, since $p \succ r$ by hypothesis, it follows from Lemma 3a that $\phi(\frac{1}{2}f + \frac{1}{2}g_n, p) \geq \pi(A_n)^{-1}/2$. Therefore,

$$\begin{aligned}\phi(f, p) &\geq \pi(A_n)^{-1} - \phi(g_n, p)\\ &= \pi(A_n)^{-1} - \pi(A_n)^{-1}\sum_{i=1}^{n}\pi(A_i) + n\\ &\geq n.\end{aligned}$$

Consequently, $\phi(f, p) \geq n$ for all n, for the claimed contradiction. □

LEMMA 5 [Theorem 3(a)]. *ϕ is bounded on $P \times P$.*

PROOF. (See the preceding Remark.) Suppose to the contrary that ϕ on $P \times P$ is unbounded, so $\phi(P \times P) = (-\infty, \infty)$, and for definiteness let p_i and q_i for $i = 1, 2, \ldots$ satisfy

$$\phi(p_i, q_i) = \pi(A_i)^{-1}.$$

Also let $f = p_i$ on A_i and $g = q_i$ on A_i for each i. We shall obtain the contradiction that $\phi(f, g)$ is infinite.

Fix r with $q \succ r \succ p$ for some q and p in P. Define simple f_n and g_n in F by

$$f_n = p_i \text{ on } A_i,\ i \le n; \qquad f_n = r \text{ on } S \setminus (A_1 \cup \cdots \cup A_n),$$
$$g_n = q_i \text{ on } A_i,\ i \le n; \qquad g_n = r \text{ on } S \setminus (A_1 \cup \cdots \cup A_n).$$

Then, by Theorem 2,

$$\phi(f_n, g_n) = \sum_{i=1}^{n} \pi(A_i)\left[\pi(A_i)^{-1}\right] = n.$$

In addition, note that $(\frac{1}{2}f + \frac{1}{2}g_n)(s) = (\frac{1}{2}g + \frac{1}{2}f_n)(s)$ for all $s \in A_1 \cup \cdots \cup A_n$ and that, for all $s \in S \setminus (A_1 \cup \cdots \cup A_n)$,

$$\begin{aligned}
&\phi\left(\left(\tfrac{1}{2}f + \tfrac{1}{2}g_n\right)(s), \left(\tfrac{1}{2}g + \tfrac{1}{2}f_n\right)(s)\right)\\
&\quad = \phi\left(\tfrac{1}{2}p_i + \tfrac{1}{2}r, \tfrac{1}{2}q_i + \tfrac{1}{2}r\right)\\
&\quad = \left[\phi(p_i, q_i) + \phi(p_i, r) + \phi(r, q_i)\right]/4,
\end{aligned}$$

when $s \in A_i$. Since $\phi(p_i, q_i)$ approaches ∞ as i gets large, and since ϕ is bounded on $P \times \{r\}$ by Lemmas 4a and 4b (recall that $q \succ r \succ p$), it follows that there is an N such that

$$\phi(p_i, q_i) + \phi(p_i, r) + \phi(r, q_i) > 0, \quad \text{for all } i \ge N.$$

This N does not depend on the particular n under consideration. Hence, for all $n \ge N$,

$$\left(\tfrac{1}{2}f + \tfrac{1}{2}g_n\right)(s) \succsim \left(\tfrac{1}{2}g + \tfrac{1}{2}f_n\right)(s), \quad \text{for all } s \in S.$$

Thus, by (B.4), $\frac{1}{2}f + \frac{1}{2}g_n \succsim \frac{1}{2}g + \frac{1}{2}f_n$ whenever $n \ge N$, so

$$\begin{aligned}
\phi(f, g) &\ge \phi(f_n, g_n) + \phi(f_n, f) + \phi(g, g_n)\\
&= n + \phi(f_n, f) + \phi(g, g_n), \quad \text{for } n \ge N.
\end{aligned}$$

We claim that $\phi(f_n, f)$ and $\phi(g, g_n)$ are bounded. Consider (f, f_n), which equals (p_i, p_i) on A_i for $i \le n$ and (p_i, r) on A_i for $i > n$. Since $\phi(x, r)$ is bounded on $P \times \{r\}$ by Lemma 4, let $a = \inf\{\phi(x, r)\colon x \in P\}$ and $b = \sup\{\phi(x, r)\colon x \in P\}$ with a and b finite. If $b \le 0$, then $\phi(f, f_n) \le 0$ by (B.4). If $b > 0$, define λ by $\lambda b + (1 - \lambda)\phi(p, r) = 0$, let $f' = p_i$ on A_i, $i \le n$, and $f' = \lambda p_i + (1 - \lambda)p$ on A_i, $i > n$, observe that $\phi(f'(s), f_n(s)) \le 0$ for all s, and thus conclude from (B.4) that $\phi(f', f_n) \le 0$. Since

$$\begin{aligned}
\phi(f', f_n) &= \phi(\lambda f + (1 - \lambda)\{p_i, i \le n;\ p, i > n\}, f_n)\\
&= \lambda\phi(f, f_n) + (1 - \lambda)\left[1 - \sum_{i=1}^{n} \pi(A_i)\right]\phi(p, r),
\end{aligned}$$

it follows that

$$\phi(f, f_n) \le \left[1 - \sum_1^n \pi(A_i)\right]\phi(r, p)(1 - \lambda)/\lambda$$
$$= b\left[1 - \sum_1^n \pi(A_i)\right].$$

Hence, $\phi(f, f_n) \le \max\{0, b\}$. By a similar proof, $\min\{0, a\} \le \phi(f, f_n)$.

Thus $\phi(f_n, f)$ and (by analogy) $\phi(g, g_n)$ are bounded as n gets large. Since $\phi(f, g) \ge n + \phi(f_n, f) + \phi(g, g_n)$ for $n \ge N$, we obtain the contradiction that $\phi(f, g)$ is infinite. □

LEMMA 6 [Theorem 3(b)]. $\inf_S \phi(f(s), p) \le \phi(f, p) \le \sup_S \phi(f(s), p)$.

PROOF. We show that $\phi(f, p) \le \sup_S \phi(f(s), p) = b$, where b is finite by Lemma 5. The only cases not already covered by Lemma 3b are ($b > 0$, $r \succsim p$ for all $r \in P$) and ($b < 0$, $p \succsim r$ for all $r \in P$).

Suppose first that $b > 0$, $r \succsim p$ for all $r \in P$, and let $t \in P$ satisfy $t \succ p$. Such a t is guaranteed by $b > 0$. Let $c = \sup_S \phi(f(s), t)$. For all $0 < \lambda < 1$, $\sup_S \phi(f(s), \lambda t + (1 - \lambda)p) \le \lambda c + (1 - \lambda)b$. Since $t \succ \lambda t + (1 - \lambda)p \succ p$, it follows from Lemma 3b that

$$\phi(f, \lambda t + (1 - \lambda)p) = \lambda\phi(f, t) + (1 - \lambda)\phi(f, p) \le \lambda c + (1 - \lambda)b.$$

Let λ approach 0 to conclude that $\phi(f, p) \le b$.

Suppose next that $b < 0$, $p \succsim r$ for all $r \in P$, and let t satisfy $p \succ t$. Let $c = \sup_S \phi(f(s), t)$ so, for $0 < \lambda < 1$, $\sup_S \phi(f(s), \lambda t + (1 - \lambda)p) \le \lambda c + (1 - \lambda)b$. Since $b \succ \lambda t + (1 - \lambda)p \succ t$, Lemma 3b yields

$$\phi(f, \lambda t + (1 - \lambda)p) = \lambda\phi(f, t) + (1 - \lambda)\phi(f, p) \le \lambda c + (1 - \lambda)b,$$

and again we get $\phi(f, p) \le b$. □

LEMMA 7. $\phi(f, p) = \int_S \phi(f(s), p)\, d\pi(s)$.

PROOF. This proof mimics the proof of Theorem 4 with g replaced by p and with Lemma 2 replaced by Lemma 6. □

LEMMA 8 [Theorem 3(c)]. *Equation* (3) *holds if g is simple.*

PROOF. Suppose g is simple with $g = p_i$ on A_i and $\{A_1, \dots, A_n\}$ a partition of S. By Lemma 1, for any $x \in P$,

$$\phi(f, g) = \sum_{i=1}^n \phi(fA_i x, p_i A_i x).$$

Consider $\phi(fA_1x, p_1A_1x)$. Write $fA_1x = (f, x, \dots, x)$ and $p_1A_1x =$

$(p_1, x, \ldots, x)$, where the jth argument refers to A_j. By Lemma 1,

$$\begin{aligned}
&\phi((f, x, \ldots, x), (p_1, x, \ldots, x)) \\
&\quad = \phi((f, p_1, x, \ldots, x), (p_1, p_1, x, \ldots, x)) \\
&\qquad + \phi((p_1, x, p_1, \ldots, p_1), (p_1, x, p_1, \ldots, p_1)) \\
&\quad = \phi((f, p_1, x, \ldots, x), (p_1, p_1, x, \ldots, x)) \\
&\quad = \phi((f, p_1, p_1, x, \ldots, x), (p_1, p_1, p_1, x, \ldots, x)) \\
&\quad \vdots \\
&\quad = \phi((f, p_1, \ldots, p_1), (p_1, p_1, \ldots, p_1)).
\end{aligned}$$

Hence, by Lemma 7, $\phi(fA_1x, p_1A_1x) = \int_{A_1}\phi(f(s), p_1)\, d\pi(s)$. Since a similar expression holds for each A_i,

$$\begin{aligned}
\phi(f, g) &= \sum_{i=1}^{n} \int_{A_i} \phi(f(s), p_i)\, d\pi(s) \\
&= \int_S \phi(f(s), g(s))\, d\pi(s). \qquad \square
\end{aligned}$$

LEMMA 9 [Theorem 3(d)]. *ϕ is bounded on $F \times F$.*

PROOF. Assume for definiteness that $\sup_S \phi(p, r) = 1$, and let $p_1, p_2 \in P$ satisfy $\phi(p_1, p_2) > \frac{7}{9}$. For any $f, g \in F$,

$$\begin{aligned}
&\phi\left(\tfrac{1}{4}f(s) + \tfrac{3}{4}p_2, \tfrac{1}{4}g(s) + \tfrac{3}{4}p_1\right) \\
&\quad = \tfrac{1}{16}\left[\phi(f(s), g(s)) + 3\phi(p_2, g(s)) + 3\phi(f(s), p_1) + 9\phi(p_2, p_1)\right] \\
&\quad < 0,
\end{aligned}$$

since the first three terms in the brackets sum to 7 or less and the final term is less than -7. It follows from (B.4) that $\phi(f, g) \le 3\phi(g, p_2) + 3\phi(p_1, f) + 9\phi(p_1, p_2)$. By Lemma 7, the right-hand side cannot exceed $3 + 3 + 9$, so $\phi(f, g) \le 15$. Since f and g are arbitrary, $\phi(g, f) \le 15$, or $-15 \le \phi(f, g)$, so ϕ is bounded on $F \times F$. $\square$

REFERENCES

BELL, D. E. (1982). Regret indecision making under uncertainty. *Oper. Res.* **30** 961–981.

BLACKWELL, D. and GIRSHICK, M. A. (1954). *Theory of Games and Statistical Decisions.* Wiley, New York.

FISHBURN, P. C. (1967). Bounded expected utility. *Ann. Math. Statist.* **38** 1054–1060.

FISHBURN, P. C. (1970). *Utility Theory for Decision Making.* Wiley, New York. Reprinted (1979). Krieger, Huntington, N.Y.

FISHBURN, P. C. (1982). Nontransitive measurable utility. *J. Math. Psych.* **26** 31–67.

FISHBURN, P. C. (1984). SSB utility theory and decision-making under uncertainty. *Math. Social Sci.* **8** 253–285.

FISHBURN, P. C. (1985). Uncertainty aversion and separated effects in decision making under uncertainty. Technical Report, AT & T Bell Labs, Murray Hill, N.J.

GILBOA, I. (1985). Expected utility with purely subjective non-additive probabilities. Working paper 6-85, Foerder Institute for Economic Research, Tel-Aviv Univ., Ramat-Aviv, Israel.

HERSTEIN, I. N. and MILNOR, J. (1953). An axiomatic approach to measurable utility. *Econometrica* **21** 291–297.

LOOMES, G. and SUGDEN, R. (1982). Regret theory: an alternative theory of rational choice under uncertainty. *Econom. J.* **92** 805–824.

LUCE, R. D. and NARENS, L. (1985). Classification of concatenation measurement structures according to scale type. *J. Math. Psych.* **29** 1–72.

RAMSEY, F. P. (1931). Truth and probability. In *The Foundations of Mathematics and Other Logical Essays* 156–198. Harcourt Brace, New York. Reprinted (1964) in *Studies in Subjective Probability* (H. E. Kyburg, Jr. and H. E. Smokler, eds.) 61–92. Wiley, New York.

SAVAGE, L. J. (1954). *The Foundations of Statistics*. Wiley, New York. 2nd rev. ed. (1972). Dover, New York.

SCHMEIDLER, D. (1984). Subjective probability and expected utility without additivity. Preprint 84, Institute for Mathematics and Its Applications, Univ. of Minnesota, Minneapolis.

VON NEUMANN, J. and MORGENSTERN, O. (1947). *Theory of Games and Economic Behavior*, 2nd ed. Princeton Univ. Press.

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