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TREND-FREE BLOCK DESIGNS FOR VARIETAL AND FACTORIAL EXPERIMENTS

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Some general results on the existence of trend-free and partially trend-free designs are given for both varietal and factorial experiments. In particular, trend-free properties of cyclic and generalized cyclic designs are investigated. It is shown that, for factorial experiments, certain designs which are not completely trend-free are nevertheless trend-free for estimating a subset of the main effect and interaction contrasts.

1. Introduction. In many industrial and agricultural experiments, treatments (or factorial treatment combinations) are applied to experimental units sequentially in time or space. It is sometimes preferable in such situations to use a systematic, rather than a randomized, ordering of the treatments [see Daniel and Wilcoxon (1966) for discussion]. It is often possible to find an ordering which will allow estimation of treatment effects independently of any polynomial time trends or spatial trends that might be present in the experiment. Such an ordering of the treatments is known as a *trend-free design*.

The study of trend-free and nearly trend-free designs was begun by Cox (1951, 1952) and has been addressed by a number of different authors, for example, Box (1952), Hill (1960), Daniel and Wilcoxon (1966), Draper and Stoneman (1968), Dickinson (1974), Bradley and Yeh (1980), Yeh, Bradley and Notz (1985), Cheng (1985), Chatterjee and Mukerjee (1986), Cheng and Jacroux (1988) and Coster and Cheng (1988).

In this article we shall be concerned with multireplicate varietal and factorial experiments arranged in blocks. Bradley and Yeh (1980) gave a necessary and sufficient condition for a binary block design with equal-size blocks to be trend-free for a common within-block polynomial trend. Lin (1989) showed that this condition also holds for designs which are not necessarily binary and which do not necessarily have equal-size blocks. The optimality results of Bradley and Yeh (1980) and many of the existence results of Yeh and Bradley (1983) also hold in this general setting. In Section 2 we discuss further existence results for trend-free block designs of given sizes.

In Section 3 we investigate the trend-free properties of three well-known and efficient classes of designs, namely, cyclic designs, generalized cyclic (GC/s) designs and generalized cyclic incomplete block (GCIB) designs. Complete cyclic and GC/s designs are always trend-free before the treatment labels within blocks are randomly ordered. It is shown that fractional cyclic designs,

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fractional GC/s designs and GCIB designs possess an arrangement of treatment labels within blocks so that the design is trend-free for odd-degree (i.e., linear, cubic, etc.) components of a within-block polynomial trend. A limited randomization of labels within blocks can be performed.

Multireplicate factorial experiments arranged in blocks are considered in Section 4. It is shown that certain arrangements of treatment combinations within the blocks of a non-trend–free design allow the design to become trend-free for estimating a subset of the main effects and interactions. Specific results for factorial experiments in GC/s designs are given.

2. Conditions for trend-free designs. Let $\mathscr{D}(v, b, k_1, \ldots, k_b, r_1, \ldots, r_v)$ be the class of block designs with v treatments and b blocks where the ith treatment is observed r_i times and the jth block is of size k_j, $i = 1, \ldots, v$, $j = 1, \ldots, b$. We consider the problem of estimating treatment contrasts in the presence of a polynomial trend of order p_j $(\leq k_j - 1)$ over the jth block, $j = 1, \ldots, b$. We assume the trend can be expressed as, or approximated by, a linear function of the orthogonal polynomials $\phi_{\alpha_j}(t_j)$, $1 \leq \alpha_j \leq p_j$, $1 \leq t_j \leq k_j$, on k_j equally spaced points, satisfying

$$\sum_{t_j=1}^{k_j} \phi_{\alpha_j}(t_j) = 0, \tag{2.1a}$$

$$\sum_{t_j=1,}^{k_j} \phi_{\alpha_j}(t_j)\phi_{\delta_j}(t_j) = \begin{cases} 1, & \text{if } \alpha_j = \delta_j, \\ 0, & \text{if } \alpha_j \neq \delta_j, \end{cases} \tag{2.1b}$$

for all $\alpha_j, \delta_j = 1, \ldots, p_j$.

REMARK 2.1. Let $s_1, s_2, \ldots, s_d$ be the distinct values among $k_1, \ldots, k_b$. Partition the blocks of the design $\delta \in \mathscr{D}(v, b, k_1, \ldots, k_b, r_1, \ldots, r_v)$ into d sets of blocks $S_1, S_2, \ldots, S_d$, where the blocks in S_j are of size s_j. If the blocks in S_j have a common within-block trend of order $p_j \leq s_j - 1$, $j = 1, \ldots, d$, then it can be shown that δ is trend-free if and only if each of $S_1, \ldots, S_d$ forms a trend-free design. Consequently, we need only consider designs in $\mathscr{D}(v, b, k, r_1, \ldots, r_v)$, that is, designs with equal block size k and with a common within-block trend.

Let $X_\theta = [1_b \otimes \Phi_p]$ and $X_\beta = [I_b \otimes 1_k]$ where 1_n is a $n \times 1$ vector of unit elements, I_b is an $b \times b$ identity matrix, $\otimes$ denotes a Kronecker product and Φ_p is a $k \times p$ matrix with element $\phi_\alpha(t)$ in row t and column α, $t = 1, \ldots, k$, $\alpha = 1, \ldots, p$. Then listing the response variables in the vector Y in order of plot position within successive blocks, the standard model for a block design under a common within-block trend is

$$E[Y] = X_\mu \mu + X_\tau \tau + X_\beta \beta + X_\theta \theta, \tag{2.2}$$

where $\theta = [\theta_1, \theta_2, \ldots, \theta_p]$ is the vector of regression coefficients for $\phi_1(t), \ldots, \phi_p(t)$, β is the $b \times 1$ vector of block parameters, τ is the $v \times 1$ vector

of treatment parameters, μ is a constant, $X_\mu = 1_{bk}$ and $X_\tau = [\Delta_1', \ldots, \Delta_b']'$, where for $j = 1, \ldots, b$, Δ_j is a $k \times v$ matrix whose (t, i)th element is unity if treatment i is applied to plot position t of block j, and zero otherwise. The following definition for a block design to be trend-free was given by Bradley and Yeh (1980).

DEFINITION 2.1. Under model (2.2) a design $d \in \mathscr{D}(v, b, k, r_1, \ldots, r_v)$ is *trend-free* if the block sum of squares and adjusted treatment sum of squares may be calculated as though the trend effects were omitted from the model.

Bradley and Yeh (1980) proved that a necessary and sufficient condition for a binary design $d \in \mathscr{D}(v, b, k, r_1, \ldots, r_v)$ to be trend-free for a common within-block trend of order p under model (2.2) is that

$$X_\tau' X_\theta = 0; \tag{2.3}$$

that is,

$$\left(\sum_{j=1}^{b} \Delta_j \right)' \Phi_p = \Delta_+' \Phi_p = 0. \tag{2.4}$$

Note that (2.3) gives the "time-count," which was discussed by Draper and Stoneman (1968) for $p = 1$ and subsequently used for $p \geq 1$ in several of the references listed in Section 1 in the context of single replicate and fractional factorial experiments. Yeh and Bradley (1983) and Stufken (1988) have given a number of interesting results concerning the existence of trend-free binary incomplete block designs (some of which are applicable to m-dimensional trends). The following existence theorem and its corollaries are needed in Sections 3 and 4.

THEOREM 2.1. *Let P denote the set of vectors $\{\phi_\alpha, \alpha = 0, 1, \ldots, k-1\}$, where $\phi_0 = [1, 1, \ldots, 1]'$. For $p \leq k-1$, let $P_\Phi = \{\phi_1, \phi_2, \ldots, \phi_p\}$ so that P_Φ contains those vectors in P corresponding to the columns of Φ_p, and let $\overline{P}_\Phi = \{\phi_0, \phi_{p+1}, \ldots, \phi_{k-1}\}$. Then under model (2.2) there exists a trend-free block design $d \in \mathscr{D}(v, b, k, r_1, \ldots, r_v)$ if and only if there exists a matrix Δ_+ with nonnegative integer elements satisfying $\Delta_+ 1_v = b1_k$ and $1_k' \Delta_+ = [r_1, \ldots, r_v]$ and such that each column of Δ_+ is a linear combination of vectors in $\overline{P}_\Phi$.*

COROLLARY 2.1.1. *A block design $d \in \mathscr{D}(v, b, k, r_1, \ldots, r_v)$ which is trend-free for a $(k-1)$th order polynomial trend exists if and only if it is possible to arrange the treatments so that the ith treatment occurs $k^{-1} r_i$ times in each plot position, $i = 1, \ldots, v$.*

Corollary 2.1.1 is a generalization of Theorem 2.4 of Yeh and Bradley (1983), which was proved for binary designs. Bradley and Odeh (1988) provide a computer algorithm which reorders the treatments within the blocks of a

binary incomplete block design to give a linear trend-free design if it can be achieved, and otherwise to give the arrangement which minimizes the sum of squares of the elements in $X_\tau' X_\theta$. When $p \geq 1$, this minimization produces a nearly trend-free design of type A as defined by Yeh, Bradley and Notz (1985). These authors also discuss nearly trend-free designs of type B where they sequentially minimize the sum of squares of the elements of $X_\tau' X_\theta^{(1)}$ and $X_\tau' X_\theta^{(2)}$, where $X_\theta = [X_\theta^{(1)}, X_\theta^{(2)}] = [1_b \otimes \Phi_{p-1}, 1_b \otimes \phi_p]$. If the trend is believed to be of order $p - 1$ but one wishes to guard against a trend of order p, then type-B minimization is more useful than type A. A generalization of type-B minimization is to set $X_\theta^{(i)} = 1_b \otimes \Phi_p^{(i)}$, $i = 1, 2$, where $\Phi_p^{(1)}$ contains those orthogonal polynomial components of trend believed to be nonnegligible and $\Phi_p^{(2)}$ contains the remaining components (of order p or less). With this in mind we define the classes of odd- and even-degree trend-free designs as follows.

DEFINITION 2.2. Under model (2.2) a design $d \in \mathscr{D}(v, b, k, r_1, \ldots, r_v)$ is *odd-degree* (*even-degree*) *trend-free* if (2.3) holds for $X_\theta = 1_b \otimes \Phi_p$, where the columns of Φ_p represent the odd-degree (even-degree) polynomial components of trend (2.1).

The class of odd-degree trend-free designs includes the important class of linear trend-free designs. An even-degree trend-free design would be used when a symmetric within-block trend is suspected due to, say, the symmetrical placement of heating and lighting elements in a laboratory, in a commercial oven or in a greenhouse, or due to the use of a symmetrical piece of equipment, such as a fertilizer spreader or water sprayer with a row of nozzles fed from a central pipe.

The following corollary to Theorem 2.1 gives a necessary and sufficient condition for the existence of odd-degree trend-free designs. [The sufficiency was also noted by Mitra and Saha (1983, 1987).] Some even-degree trend-free designs are discussed in Section 4.

COROLLARY 2.1.2. *A block design $d \in \mathscr{D}(v, b, k, r_1, \ldots, r_v)$ which is odd-degree trend-free exists if and only if it is possible to arrange the treatments so that treatment i occurs s_{ti} times in plot positions t and $(k - t + 1)$ for nonnegative integers s_{ti}, $i = 1, \ldots, v$, $t = 1, 2, \ldots, [(k + 1)/2]$, where the square brackets denote integer part.*

3. Classes of trend-free and odd-degree trend-free designs. In this section we show that designs which are based on the cyclic method of construction are trend-free or odd-degree trend-free when the treatment labels are systematically ordered within the blocks. The construction of efficient block designs using the cyclic method dates back to the 1930's [e.g., Bose and Nair (1939)]. A catalogue of cyclic designs with high average efficiency factors was provided by John, Wolock and David (1972) and updated by John (1981).

There are two generalizations of cyclic designs in common use. The first generalization involves the representation of treatment labels by s-tuples

($s > 1$), whilst the second generalization essentially involves selecting every mth block ($m > 1$) of a cyclic design. Designs produced under the first generalization have been called *generalized cyclic designs* [John (1971), Chapter 15], *GC/s designs* [Dean and Lewis (1980)], *s-cyclic designs* [John (1987)] and *Abelian-group designs* [Bailey (1988)]. In this article we shall use the first two of these terms. A cyclic design is merely a GC/1 design, and therefore no results for cyclic designs will be given explicitly.

DEFINITION 3.1. Let T be the set of $v = m_1 m_2 \cdots m_s$ lexicographically ordered treatment labels $\{a: a = a_1 a_2 \ldots a_s;\ 0 \le a_i \le m_i - 1\}$. A generalized cyclic (GC/s) design consists of a selection of k labels from T (not necessarily distinct) to form the generating block. The jth block of the design ($j = 1, \ldots, v$) is obtained by adding the jth label in T to each label in the generating block, where addition of a and b in T is defined as

$$a_1 a_2 \ldots a_s + b_1 b_2 \ldots b_s = c_1 c_2 \ldots c_s,$$

where $a_i + b_i = c_i \bmod m_i$, $i = 1, 2, \ldots, s$. Duplicate blocks are ignored.

The set T forms an Abelian group. The generating block B can be expressed as $B = S[+]R$, where $[+]$ denotes the set of all elements $a + b$, $a \in S$, $b \in R$, including repetitions, S is a subgroup of T of order d and R is a subset of T of size k/d. Dean and Lewis (1980) show that if S is the largest subgroup that allows B to be expressed in the form $S[+]R$, then the GC/s design has v/d distinct blocks. If $d = 1$, we shall call such a design a *full* GC/s design, and if $d > 1$ we shall call the design a *fractional* GC/s design.

We allow a random ordering of treatment labels within the generating block B unless otherwise stated. It is understood that if $a \in T$ is in position q in B, $1 \le q \le k$, then in the jth block treatment label $a + b^{(j)}$ is in position q, where $b^{(j)}$ is the jth ordered label in T, $j = 1, \ldots, v$. Duplicate blocks are ignored and the distinct blocks are arranged in a random order. *The order of treatment labels within blocks remains fixed.* The following theorem follows from Corollary 2.1.1.

THEOREM 3.1. *All full GC/s designs are trend-free.*

Theorem 3.1 would also hold for fractional GC/s designs if duplicate blocks were retained. However, designs with duplicate blocks would not normally be used in practice for efficiency considerations. The following theorem shows that many fractional GC/s designs are odd-degree trend-free.

THEOREM 3.2. *If k/d is even, there exists an arrangement of treatment labels within the blocks of any GC/s design so that the design is at least odd-degree trend-free.*

PROOF. Let the generating block be $B = S[+]R$, where S is the largest subgroup of T that allows B to be expressed in this form, $|S| = d$, $|R| = k/d$

and k/d is even. Let the cosets of S in T be $S + h_1, S + h_2, \ldots, S + h_{v/d}$. The distinct blocks of the generalized cyclic design are $B + h_i$, $i = 1, \ldots, v/d$ [see Dean and Lewis (1980), Theorem 1]. Without loss of generality suppose that the elements of B are in the order $S + h_1^*, S + h_2^*, \ldots, S + h_{k/d}^*$, where $h_i^* \in H = \{h_1, h_2, \ldots, h_{v/d}\}$ and where $S + h_i^* = S + a_i$ for some $a_i \in R$, $i = 1, \ldots, k/d$. Now consider the first d positions in block j, $1 \leq j \leq v/d$. These contain the treatment labels in one of the cosets of S. Since the set of cosets of S is itself a group, every coset appears in the first d positions in some block of the design. Similarly, the treatment labels in each coset occur in the set of positions $P_m = \{md + j;\ j = 1, \ldots, d\}$ in some block of the design, for every $m = 0, 1, \ldots, (k/d) - 1$. Now, suppose that the treatment labels in the coset $S + h_i$ occur in the set of positions P_m in some block of the design (for fixed m). Find the occurrence of the same coset $S + h_i$ in P_{m^*}, where $m^* = (k/d) - m - 1$. Randomly order the elements of $S + h_i$ in P_m and reverse the order of these elements in P_{m^*}. Repeat this procedure for every coset and every $m = 0, 1, \ldots, (k/2d) - 1$. The result follows from Corollary 2.1.2. □

EXAMPLE 3.1. Consider a GC/2 design with $v = 3 \times 6 = 18$ treatment labels in six blocks of size $k = 6$. The design listed in Table 3 of Dean and Lewis (1980) has generating block $B = \{00\ 12\ 24\ 01\ 13\ 25\} = S[+]R$, where $S = \{00\ 12\ 24\}$ and $R = (00\ 01)$. The design constructed by the method described in Definition 3.1 is shown in Table 1(a), and an odd-degree trend-free arrangement of the design is shown in Table 1(b). In each case, rows denote blocks.

As mentioned earlier, a different generalization of a cyclic design is that of using every mth block of the design, where the cyclic design is generated as in Definition 3.1 with $s = 1$ and $m_1 = v$. Such a design exists if $v = nm$ for some integer n and is called a *generalized cyclic incomplete block (GCIB) design* by Jarrett and Hall (1978) and Hall and Jarrett (1981). The construction of a GCIB design is given precisely in Definition 3.2. We use the notation GCIB_m design. Note that a cyclic design is a GCIB_1 design.

TABLE 1
A GC/2 design with $v = 3 \times 6$ treatment labels in six blocks of size 6

(a) Generated design	(b) Odd-degree trend-free design
00 12 24 01 13 25	00 12 24 25 13 01
01 13 25 02 14 20	01 13 25 20 14 02
02 14 20 03 15 21	02 14 20 21 15 03
03 15 21 04 10 22	03 15 21 22 10 04
04 10 22 05 11 23	04 10 22 23 11 05
05 11 23 00 12 24	05 11 23 24 12 00

DEFINITION 3.2. A GCIB_m design consists of a selection of k treatment labels (not necessarily distinct) from $T = \{0, 1, \ldots, v - 1\}$ to form the generating block. The jth block of the design is obtained by adding (modulo v) jm to every label in the generating block, $j = 0, \ldots, n - 1$. The value of m is known as the *incrementing number*.

In general, a GCIB_m design has n blocks. We consider only the case where these n blocks are distinct, and we call such a design *a full* $GCIB_m$ design. The following theorem shows that certain full GCIB_m designs are odd-degree trend-free. Divide the $v = mn$ treatment labels into m groups of n labels using the residue classes modulo m; that is, $S_a = \{a, a + m, \ldots, a + (n - 1)m\}$, $a = 0, 1, \ldots, m - 1$. Then the following theorem follows from Corollary 2.1.2.

THEOREM 3.3. *A full* $GCIB_m$ *design is odd-degree trend-free if the treatment labels in positions* q *and* $k - q + 1$ *of the generating block belong to the same residue class, for all* $q = 1, 2, \ldots, k$.

GCIB_m designs tend to be small in terms of the number of blocks. Jarrett and Hall (1978) give examples of efficient "composite" designs formed from g ($g \geq 2$) GCIB_m designs. These GCIB_m designs need not have the same value of m nor the same block sizes. We call the resulting composite design a *composite GCIB* $(m_1, \ldots, m_g; k_1, \ldots, k_g)$ design and, if $m_1 = m_2 = \cdots = m_g = m$ and $k_1 = k_2 = \cdots = k_g = k$, we abbreviate this to a *composite GCIB* $(m; k; g)$ design. If each GCIB_{m_i} design is odd-degree trend-free, $i = 1, \ldots, g$, then the composite GCIB $(m_1, \ldots, m_g; k_1, \ldots, k_g)$ design is also odd-degree trend-free (see Remark 2.1). If $m_i = m$ and $k_i = k$ for all $i = 1, \ldots, g$, more flexibility in constructing odd-degree trend-free composite GCIB designs is obtained as detailed by the following corollary to Theorem 3.3.

COROLLARY 3.3.1. *Consider a composite GCIB* $(m; k; g)$ *design* δ *with generating blocks* $B_1, B_2, \ldots, B_g$. *If treatment labels* $a_1, a_2, \ldots, a_g$ *are in position* q *of the* g *generating blocks, then* δ *is odd-degree trend-free if the treatment labels in position* $k - q + 1$ *of the generating blocks can be matched in pairs with* $a_1, \ldots, a_g$ *so that the pairs are in the same residue class, and if this holds for all* $q = 1, \ldots, k$.

EXAMPLE 3.2. Consider the composite GCIB (3; 6; 2) design in Table 2.C1 of Hall and Jarrett (1981) with eight blocks of size 6, made up from the two GCIB_3 designs with generating blocks (0 1 2 3 4 7) and (0 3 5 8 10 11), respectively. The residue classes are $S_0 = \{0, 3, 6, 9\}$, $S_1 = \{1, 4, 7, 10\}$ and $S_2 = \{2, 5, 8, 11\}$. It can be verified that if the generating blocks are reordered as (0 1 2 4 7 3) and (5 3 10 8 0 11), Corollary 3.3.1 is satisfied and the design is odd-degree trend-free.

REMARK 3.1. In a full GC/s design, if the treatment labels in any two plot positions are interchanged in every block the design remains trend-free. In a fractional GC/s design, if treatment labels in plot positions j_1 and j_2 are interchanged in some block, the design remains odd-degree trend-free provided that the matching treatment labels in plot positions $k - j_1 + 1$ and $k - j_2 + 1$ are also interchanged in some block. Thus some limited randomization of treatment labels within blocks can be performed. Similarly, if treatment labels in positions j_1 and j_2 of a GCIB$_m$ or a composite GCIB $(m; k; g)$ design are interchanged in every block, the design remains odd-degree trend-free provided that treatment labels in positions $k - j_1 + 1$ and $k - j_2 + 1$ are also interchanged in every block.

4. Designs for factorial experiments. Consider a factorial experiment with s factors where the ith factor has m_i levels, $i = 1, \ldots, s$. Let the vector τ in model (2.2) represent the effects of the $v = m_1 m_2 \cdots m_s$ treatment combinations in lexicographical order. Suppose that the experiment is arranged in b blocks of size k subject to a common within-block polynomial trend (see Remark 2.1 for more general designs).

Let $x = (x_1 x_2 \ldots x_s)$ denote a binary vector where $x_i = 0$ or 1 and let α^x denote the effect of the interaction between those factors for which $x_i = 1$, $i = 1, \ldots, s$. Throughout this section it will be convenient, when considering an f-factor interaction α^x, to temporarily reorder the labels on the factors so that α^x represents the interaction between the first f factors $(1 \le f \le s)$. To avoid confusion, we label α^x as $\alpha_{(f)}$ and τ as $\tau_{(f)}$ after such a reordering. If a design is to be used for an s-factor factorial experiment, then its treatment labels can be written as s-tuples, so that $T = \{a;\ a = a_1 a_2 \cdots a_s,\ 0 \le a_i \le m_i - 1,\ i = 1, \ldots, s\}$. We denote by $T_{(f)}$ the group of treatment labels T after deleting the last $s - f$ digits from each treatment label. Similarly the design $d_{(f)}$ is obtained from the design d by deleting the last $s - f$ digits from each treatment label, and $X_{\tau(f)}$ is the corresponding X_τ matrix.

Let K^x be a matrix whose rows are orthonormal and span the vector space of contrasts associated with α^x; that is, $\alpha^x = K^x \tau$. It is well known [cf. Kurkjian and Zelen (1962)] that for any given order of the factors, K^x can be expressed as $K^x = K_1^{x_1} \otimes K_2^{x_2} \otimes \cdots \otimes K_s^{x_s}$ when the treatment combinations are listed in lexicographical order, where $K_i^{x_i}$ is a constant row vector of m_i elements equal to $(m_i)^{-1/2}$ if $x_i = 0$, and $K_i^{x_i}$ is an $(m_i - 1) \times m_i$ matrix K_i with orthonormal rows and zero row sums if $x_i = 1$, $i = 1, \ldots, s$. Let the vectors $y_1, y_2, \ldots, y_n$ denote the distinct possibilities for the binary vector x. If $K = [K^{y_1}, K^{y_2}, \ldots, K^{y_{n'}}]$, then K is a $v \times v$ orthogonal matrix. Let $\Psi = K\tau$ denote a vector of factorial contrasts (including the general mean); then $\tau = K\Psi$ and, as in (2.3), a design is trend-free for estimating Ψ if and only if $(X_\tau K')' X_\theta = 0$. Using (2.4) this occurs if and only if

$$K X_\tau' X_\theta = K \Delta_+' \Phi_p = 0. \tag{4.1}$$

It follows that a design is trend-free for estimating $\alpha^x = K^x \tau$ if and only if

$$K^x K_\tau' X_\theta = K^x \Delta_+' \Phi_p = 0. \tag{4.2}$$

REMARK 4.1. We have not discussed the issue of estimability of contrasts here. If $k'\tau$ is nonestimable, where k' is some row of K^x, then it may be irrelevant whether or not the design is trend-free for that contrast. In this case the row k' could be deleted from K^x in (4.2).

Since (2.3) implies (4.2), all trend-free designs are also trend-free for estimating factorial contrasts. Therefore, in this section, we will only consider the case of designs which do not satisfy (2.3). For an unblocked single replicate or fractional factorial design, $X_\tau K'$ represents a reordering of rows (or subset of rows) of K'. This is the starting point for the construction of trend-free designs for factorial experiments in the articles by Daniel and Wilcoxon (1966), and Cheng and Jacroux (1988). A different method of construction is described by Coster and Cheng (1988) and Coster (1988). This method, which they call the generalized foldover method, produces a special case of the class of GC/s designs when applied to single replicate and fractional factorial experiments arranged in blocks. In this section we consider general multireplicate factorial experiments in blocks where the ith treatment combination is observed r_i (≥ 1) times, $i = 1, \ldots, v$.

As in Section 3, if (4.2) holds when Φ_p contains only the odd-degree (linear, cubic, etc.) orthogonal polynomial components of trend, we will call the design *odd-degree trend-free for estimating* α^x. Similarly, if Φ_p contains only the even-degree (quadratic, quartic, etc.) orthogonal polynomial components of trend, we will call the design *even-degree trend-free for estimating* α^x.

REMARK 4.2. Using (4.2), it is easy to show that a design d is completely (odd-degree/even-degree) trend-free for estimating contrasts associated with the interaction between the first f factors if and only if $d_{(f)}$ is completely (odd-degree/even-degree) trend-free, where $d_{(f)}$ is defined above.

DEFINITION 4.1. The interaction α^x is called an *odd-factor* (*even-factor*) *interaction* if $x = (x_1 x_2 \ldots x_s)$ is such that $\sum_{i=1}^{s} x_i$ is odd (even).

Consider now the case of estimating $\alpha_{(f)}$ in a 2^s factorial experiment. Then K_i is the row vector $2^{-1/2}(-1\ 1)$ for $i = 1, \ldots, f$. It is easy to prove by induction that $K_{(f)} = K_1 \otimes \cdots \otimes K_f$ is a symmetric row vector if f is even and antisymmetric if f is odd. Also, it can be verified that if element q of $K_{(f)}$ is indexed by $a \in T_{(f)}$, then element $w - q + 1$ is indexed by $\bar{a} \in T_{(f)}$ for all $q = 1, \ldots, w$, where $\bar{a} = (\bar{a}_1 \bar{a}_2 \ldots \bar{a}_f)$ and $\bar{a}_i = 1 - a_i$ for $i = 1, \ldots, f$ and $w = 2^f$. Similarly, if column q of $X_{\tau(f)}$ is indexed by $a \in T_{(f)}$, then column $w - q + 1$ of $X_{\tau(f)}$ is indexed by $\bar{a} \in T_{(f)}$.

THEOREM 4.1. *Let* $d \in \mathcal{D}(2^s, b, k, r_1, \ldots, r_v)$ *be a block design for a* 2^s *factorial experiment, with the property that for every occurrence of treatment label* a *in position* t *of some block of* d, *treatment label* $\bar{a}$ *also occurs in position* t *of some block of* d, *for all* $a \in T$ *and* $t = 1, \ldots, k$. *Then* d *is trend-free for estimating every main-effect and odd-factor interaction.*

PROOF. Without loss of generality consider the f-factor interaction $\alpha_{(f)}$, $1 \le f \le s$, and the corresponding design $d_{(f)}$. From the description of the design d, the rows of $\Delta_{+(f)}$ are symmetric. If f is odd, the vector $K'_{(f)}$ is antisymmetric and hence $\Delta_{+(f)}K'_{(f)} = 0$. The result follows from Remark 4.2. □

EXAMPLE 4.1. Consider the following design in four blocks of size 5 for a 2^3 factorial experiment with each treatment combination occurring either two or three times. (Rows denote blocks.) The design is connected and satisfies the conditions of Theorem 4.1 and thus the design is trend-free for estimating each main effect and the three-factor interaction.

111	110	101	001	000
100	101	010	011	111
000	010	111	100	011
011	001	000	110	100

If the third digit of each treatment label is deleted, then Corollary 2.1.1 is satisfied. Hence the design is also trend-free for estimating the interaction between the first two factors.

THEOREM 4.2. *Let $d \in \mathcal{D}(2^s, b, k, r_1, \ldots, r_v)$ be a block design for a 2^s factorial experiment with the property that for every occurrence of treatment label a in position t of some block of d, treatment label $\bar{a}$ occurs in position t of some block of d and in position $k - t + 1$ of some block of d, for all $a \in T$ and $t = 1, \ldots, k$. Then*:

(i) *d is trend-free for estimating every main-effect and odd-factor interaction*;

(ii) *d is odd-degree trend-free for estimating every even-factor interaction*.

PROOF. Without loss of generality consider the f-factor interaction $\alpha_{(f)}$, $1 \le f \le s$, and the corresponding design $d_{(f)}$. Let $X_{\tau(f)} = [\Delta'_{1(f)}, \ldots, \Delta'_{b(f)}]'$ and $\Delta_{+(f)} = \sum_{i=1}^{b} \Delta_{i(f)}$. From the description of the design, the elements in row t of $\Delta_{+(f)}$ are repeated in reverse order in row $k - t + 1$ of $\Delta_{+(f)}$. Consequently, if f is even, the tth element of the vector $\Delta_{+(f)}K'_{(f)}$ is identical to the $(k - t + 1)$th element, for all $t = 1, \ldots, k$. Since the columns of Φ_p are antisymmetric for all odd-degree trends, result (ii) follows from Remark 4.2. Result (i) follows from Theorem 4.1. □

EXAMPLE 4.2. Modifying the treatment labels in the design of Example 4.1 we obtain the following design, which satisfies the conditions of Theorem 4.2. Consequently, the design is trend-free for estimating the three main-effects

and the three-factor interactions, and odd-degree trend-free for estimating the three two-factor interactions.

101	011	111	000	110
010	100	000	111	001
110	000	100	011	101
001	111	011	100	010

REMARK 4.3. Designs which satisfy Theorem 4.2 are odd-degree trend-free. If the polynomial trend is expected to be of degree $p \geq 2$, then an arrangement of treatment labels within blocks should be sought which not only satisfies Theorem 4.2 but also minimizes the dependence of the odd-factor interactions on the even-degree polynomials, and also produces an efficient design for the estimation of the even-factor interactions. It is possible that, by sacrificing the trend-free properties guaranteed by Theorem 4.2, a design could be found which performs better under some average optimality criterion such as the type-A criterion mentioned in Section 2. However, it is not obvious which optimality criteria are the most suitable in the present setting. The minimum time count alone is not sufficient since it ignores the efficiency of a design. The class of designs satisfying Theorem 4.2 is large and we conjecture that the "best" of these designs will be among the best possible under a criterion that is concerned with efficiency of estimation of factorial effects. Similar remarks apply to the designs in the remainder of this section. We leave a thorough discussion of optimality criteria for future work.

Now consider a general $m_1 \times m_2 \times \cdots \times m_s$ factorial experiment. For $x_i = 1$, the contrast matrix K_i has $(m_i - 1)$ orthonormal rows. If the levels of the ith factor F_i are quantitative, the contrasts of interest are often the orthogonal polynomial trends in the levels of F_i. In order to avoid confusion, we will call these orthogonal polynomial contrasts the first, second, ..., $(m_i - 1)$th order contrasts in F_i. Thus the odd-numbered rows of K_i corresponding to the odd-order contrasts are antisymmetric and the even-numbered rows of K_i corresponding to the even-order contrasts are symmetric. It can be verified that, as for the 2^n case, if column q of $K_{(f)}$ and $X_{\tau(f)}$ are indexed by $a \in T_{(f)}$, then column $w - q + 1$ of $K_{(f)}$ and $X_{\tau(f)}$ are indexed by $\bar{a} \in T_{(f)}$, for all $q = 1, \ldots, w$, where $\bar{a} = (\bar{a}_1 \bar{a}_2 \ldots \bar{a}_f)$ and $\bar{a}_i = m_i - 1 - a_i$ for $i = 1, \ldots, f$ and $w = m_1 m_2 \cdots m_f$, We call $\bar{a}$ the *complement* of a.

LEMMA 4.1. *Consider the design $d_{(f)}$ corresponding to factors $F_1, F_2, \ldots, F_f$. The row of $K_{(f)}$ corresponding to the (q_1th order contrast in $F_1 \times q_2$th order contrast in $F_2 \times \cdots \times q_f$th order contrast in F_f) is*:

(i) *symmetric if $\sum_{i=1}^{f} q_i$ is even*;
(ii) *antisymmetric if $\sum_{i=1}^{f} q_i$ is odd*.

The following lemmas follow from Remark 4.2, Lemma 4.1 and Theorem 2.1.

LEMMA 4.2. *Let $d \in \mathscr{D}(m_1 m_2 \cdots m_s, b, k, r_1, \ldots, r_v)$ be a block design for an $m_1 \times m_2 \times \cdots \times m_s$ factorial experiment with the property that for every occurrence of treatment label a in position t of a block of d, treatment label $\bar{a}$ occurs in position t of some block of d, for all $a \in T$ and $t = 1, \ldots, k$. Then the design is trend-free for estimating the (q_1th order contrast in $F_1 \times \cdots \times q_s$th order contrast in F_s) if $\sum_{i=1}^{s} q_i$ is odd, where q_i is set to zero if F_i does not occur in the interaction of interest.*

LEMMA 4.3. *Let $d \in \mathscr{D}(m_1 m_2 \cdots m_s, b, k, r_1, \ldots, r_v)$ be a block design for an $m_1 \times m_2 \times \cdots \times m_s$ factorial experiment with the property that for every occurrence of treatment label a in position t of a block of d, treatment label $\bar{a}$ occurs in position $k - t + 1$ in some block of d for all $a \in T$ and $t = 1, \ldots, k$. Then:*

(i) *d is even-degree trend-free for estimating (q_1th order contrast in $F_1 \times \cdots \times q_s$th order contrast in F_s) if $\sum_{i=1}^{s} q_i$ is odd;*

(ii) *d is odd-degree trend-free for estimating (q_1th order contrast in $F_1 \times \cdots \times q_s$th order contrast in F_s) if $\sum_{i=1}^{s} q_i$ is even;*

where q_i is set to zero if F_i does not occur in the interaction of interest.

The results of Corollary 2.1.2, Lemma 4.2 and Lemma 4.3 can be combined to provide the stronger result given in Theorem 4.3.

THEOREM 4.3. *Let $d \in \mathscr{D}(m_1 m_2 \cdots m_s, b, k, r_1, \ldots, r_v)$ be a block design for an $m_1 \times m_2 \times \cdots \times m_s$ factorial experiment with the property that for every occurrence of treatment label a in position t of a block of d, treatment label $\bar{a}$ occurs in position t in some block of d and also in position $k - t + 1$ in some block of d for all $a \in T$ and $t = 1, \ldots, k$. Then:*

(i) *d is odd-degree trend-free for estimating every factorial effect;*

(ii) *d is even-degree trend-free for estimating (q_1th order contrast in $F_1 \times \cdots \times q_s$th order contrast in F_s) if $\sum_{i=1}^{s} q_i$ is odd.*

Note that an alternative phrasing of (i) and (ii) in Theorem 4.3 is that the design is completely trend-free for estimating (q_1th order contrast in $F_1 \times \cdots \times q_s$th order contrast in F_s) if $\sum_{i=1}^{s} q_i$ is odd, and odd-degree trend free for $\sum_{i=1}^{s} q_i$ even. An example of a design which satisfies Theorem 4.3 is given in Table 2.

Any block design can be used for a factorial experiment by putting the treatment labels of the design into one–one correspondence with the treatment combinations of the experiment. Generalized cyclic designs, whose treatment labels are s-tuples, are natural choices for s-factor factorial experiments [see, e.g., John (1981, 1987)]. Although there is evidence to show that the obvious correspondence between treatment labels and factor levels may not lead to the most efficient designs [see Bailey (1985)], the trend-free properties of such designs are straightforward as shown in Theorem 4.4. First we need

TABLE 2
A GC/2 design for a 3 × 6 factorial experiment which satisfies Theorem 4.4

00	12	24	01	13	25
25	13	01	20	14	02
02	14	20	21	15	03
03	15	21	04	10	22
22	10	04	05	11	23
23	11	05	24	12	00

the notion of complementary cosets. As in the proof of Theorem 3.2, let the cosets of a subgroup S of order d in T be $S + h_1, S + h_2, \ldots, S + h_{v/d}$. Consider the coset $S + h_i = \{a_1 + h_i, a_2 + h_i, \ldots, a_d + h_i\}$ and let $u = (1, 1, \ldots, 1)$. Then the complement of $a_j + h_i \in S + h_i$ is $(-u - a_j - h_i) = (-a_j) - (h_i + u) = a_w + h_g \in S + h_g$ for some $1 \leq w \leq d$ and $1 \leq g \leq v/d$. Consequently the complements of all treatment labels in any given coset $S + h_i$ all belong to the same coset $S + h_g$ (where $S + h_g$ may or may not be different from $S + h_i$). The pair of cosets $(S + h_i, S + h_g)$ will be called *complementary cosets* and if $S + h_i = S + h_g$, then $S + h_i$ will be called a *self-complementary coset.*

THEOREM 4.4. *If v/d and k/d are both even, there exists an arrangement of treatment labels within the blocks of any GC/s design so that, if the treatment labels represent the treatment combinations of an $m_1 \times m_2 \times \cdots \times m_s$ factorial experiment, the design satisfies the conditions of Theorem 4.3, provided that the subgroup S in the generating block $S[+]R$ has no self-complementary cosets.*

PROOF. Arrange the treatment labels in the generating block as in the proof of Theorem 3.2. Then every coset of $S \subseteq T$ occurs in the set of positions $P_m = \{md + j;\ j = 1, \ldots, d\}$ in some block of the design, $0 \leq m \leq (k/d) - 1$. Order the cosets in P_m so that complementary treatment labels occur in the same plot positions, and correspondingly order the matching cosets in P_{m^*} in reverse order where $m^* = (k/d) - m - 1$. Repeat this procedure for all complementary pairs of cosets and every $m = 0, 1, \ldots, (k/2d) - 1$. Theorem 4.3 is then satisfied. □

EXAMPLE 4.3. Consider the GC/2 design of Example 3.1, and let the treatment labels represent the treatment combinations of a 3 × 6 factorial experiment. The cosets form complementary pairs as follows {(00, 12, 24); (25, 13, 01)}, {(02, 14, 20); (23, 11, 05)}, {(03, 15, 21); (22, 10, 04)}. If the treatments within the blocks of the design shown in Table 1(b) are reordered as in Table 2, then the design is odd-degree trend-free for estimating all factorial effects and, in addition, is even-degree trend-free for estimating linear F_1, linear F_2, cubic F_2, quintic F_2, linear F_1 × quadratic F_2, linear F_1 × quartic

F_2, quadratic $F_1 \times$ linear F_2, quadratic $F_1 \times$ cubic F_2 and quadratic $F_1 \times$ quintic F_2 (that is, the design is completely trend-free for estimating these nine factorial effects).

Note that if v/d is odd but k/d is even, a GC/s design can always be arranged so that either Corollary 2.1.2 is satisfied (as shown by Theorem 3.2) or Lemma 4.3 is satisfied (by modifying the proof of Theorem 3.2 to focus on $\bar{a}$ rather than a in position P_{m^*}). However, Theorem 4.3 can never be satisfied since the subgroup in the generating block must have at least one self-complementary coset.

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