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ON THE MAXIMUM NUMBER OF CONSTRAINTS IN ORTHOGONAL ARRAYS[1]

BY A. HEDAYAT AND J. STUFKEN

University of Illinois at Chicago and University of Georgia

We show that Bush's bound for the maximum number of constraints in an orthogonal array of index unity is uniformly better than Rao's bound.

1. Introduction. Orthogonal arrays were introduced by Rao (1947a) because of their statistical properties when used as fractional factorial designs. See also Rao (1946, 1947b). Rao's papers have been a source of inspiration for many other researchers, who continued to study these arrays and various generalizations. It is mainly due to the work by Taguchi and his colleagues that orthogonal arrays have gained a renewed interest in industrial experimentation for product improvement during recent years, also among researchers in this country.

For the sake of completeness we start with a formal definition of an orthogonal array.

DEFINITION. A $k \times N$ array with entries from $S = \{0, 1, \dots, s-1\}$ is called an orthogonal array of strength t and index λ if in any t rows of the array each (ordered) t-tuple based on S appears λ times.

We will denote such an array by $OA(N, k, s, t)$. Observe that $N = \lambda s^t$. Often N is called the number of runs, k the number of constraints and s the number of levels.

One of the more interesting problems, both from a mathematical and statistical point of view, is the following. For given values of N, s and t, what is the maximum number of constraints that can be accommodated in an orthogonal array? Various results on upper bounds for this number are known. Some are for very general settings, others for very specific ones. For some it is known that equality can be achieved, for others this is an open problem. In this paper we will compare two such upper bounds, one by Rao (1947a), the other by Bush (1952). Rao's result may be stated as follows.

THEOREM 1. *In an $OA(\lambda s^t, k, s, t)$ the following inequalities hold*:
If $t = 2u$,

$$\lambda s^t \geq 1 + \binom{k}{1}(s-1) + \binom{k}{2}(s-1)^2 + \cdots + \binom{k}{u}(s-1)^u.$$

If $t = 2u + 1$,

$$\lambda s^t \geq 1 + \binom{k}{1}(s-1) + \binom{k}{2}(s-1)^2 + \cdots + \binom{k}{u}(s-1)^u + \binom{k-1}{u}(s-1)^{u+1}.$$

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The upper bound for k, implicitly given by these inequalities, is the most general result on the maximum number of constraints in the literature. Improvements have been obtained for special cases of the parameters N, s and t. Bush (1952) considered the special, but still important case of $N = s^t$ arrays of index unity. He proved the following result.

THEOREM 2. *In an $OA(s^t, k, s, t)$ the following inequalities hold*:

$$\begin{cases} k \le t + 1, & \text{if } s \le t, \\ k \le s + t - 1, & \text{if } s > t \text{ and } s \text{ is even, or if } t = 2, \\ k \le s + t - 2, & \text{if } s > t \ge 3 \text{ and } s \text{ is odd.} \end{cases}$$

Improvements on this result for special cases have been obtained, for example, by Kounias and Petros (1975). Other improvements result from studying the case $t = 2$, a problem that leads its own life. We refer the reader to Dénes and Keedwell (1974) for more on this.

Numerical examples seem to indicate that for orthogonal arrays of index unity the bound for the maximum number of constraints from Theorem 2 is sharper than that from Theorem 1 [Bush (1952)]. Personal communications [Bush (1986) and Raghavarao (1986)] made the authors aware that there is apparently no analytical proof of this observation. The objective of this paper is to provide such a proof.

2. The result. The following result provides a comparison of the bounds in Theorems 1 and 2.

THEOREM 3. *For orthogonal arrays of index unity, Bush's bound on the maximum number of constraints is uniformly better than Rao's bound.*

PROOF. If $s \le t$ an $OA(s^t, t + 1, s, t)$ can be constructed as follows. Start with a $t \times s^t$ array $A = (a_{ij})$ in which the columns consist of all possible t-tuples based on $\{0, 1, \ldots, s - 1\}$. Augment a $(t + 1)$th row with the jth entry defined by

$$a_{(t+1)j} \equiv \sum_{i=1}^{t} a_{ij} (\operatorname{mod} s), \qquad j = 1, \ldots, s^t.$$

This gives the desired orthogonal array. Thus Bush's bound can not be improved for this case. So we can restrict ourselves to the case $s > t$. To prove the theorem it suffices to show that if we use $k = s + t - 1$ in the inequalities of Theorem 1, we obtain valid inequalities. We distinguish between $t = 2u$ and $t = 2u + 1$.

CASE 1. $t = 2u$. We would like to show that $s^{2u} \ge 1 + \binom{s + 2u - 1}{1}(s - 1) + \cdots + \binom{s + 2u - 1}{u}(s - 1)^u$. First notice that

$$\binom{u}{i}(s + 1)^i \ge \binom{s + 2u - 1}{i}, \qquad i = 0, 1, \ldots, u.$$

This is obvious for $i = 0$. For $1 \le i \le u - 1$ it follows if we can show that

$$(u - j)(s + 1) \ge s + 2u - j - 1, \qquad j = 0, \dots, u - 2$$

or

$$(u - j - 1)s \ge u - 1, \qquad j = 0, \dots, u - 2.$$

This is true since $j \le u - 2$ and $s > 2u$. Finally, for $i = u$ the above inequality follows since

$$u!(s+1)^u = u(s+1)^2 \prod_{i=2}^{u-1} \{i(s+1)\}$$
$$\ge u(s+1)^2 \prod_{i=2}^{u-1} (s + 2u - i) \ge \prod_{i=1}^{u} (s + 2u - i).$$

The proof for Case 1 is now completed by noting that

$$s^{2u} = (s^2 - 1 + 1)^u = \sum_{i=0}^{u} \binom{u}{i} (s^2 - 1)^i = \sum_{i=0}^{u} \binom{u}{i} (s+1)^i (s-1)^i$$
$$\ge \sum_{i=0}^{u} \binom{s + 2u - 1}{i} (s-1)^i.$$

CASE 2. $t = 2u + 1$. Here we would like to show that

$$s^{2u+1} \ge 1 + \binom{s + 2u}{1}(s - 1) + \cdots + \binom{s + 2u}{u}(s-1)^u$$
$$+ \binom{s + 2u - 1}{u}(s - 1)^{u+1}.$$

As in Case 1 we have that $(s + 1)^u \ge \binom{s + 2u - 1}{u}$. Observe further that

$$\binom{u}{i-1}(s+1)^{i-1} + \binom{u}{i}(s+1)^i \ge \binom{s + 2u - 1}{i - 1} + \binom{s + 2u - 1}{i}$$
$$= \binom{s + 2u}{i}.$$

Hence,

$$s^{2u+1} = s \sum_{i=0}^{u} \binom{u}{i} (s+1)^i (s-1)^i$$
$$= \sum_{i=0}^{u} \binom{u}{i} (s+1)^i (s-1)^{i+1} + \sum_{i=0}^{u} \binom{u}{i} (s+1)^i (s-1)^i$$
$$= 1 + \sum_{i=1}^{u} \left[\binom{u}{i-1} (s+1)^{i-1} + \binom{u}{i} (s+1)^i \right] (s-1)^i$$
$$+ (s+1)^u (s-1)^{u+1}$$
$$\ge 1 + \sum_{i=1}^{u} \binom{s + 2u}{i} (s-1)^i + \binom{s + 2u - 1}{u} (s-1)^{u+1}.$$

This concludes the proof of Case 2 and establishes the result. □

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DEPARTMENT OF MATHEMATICS, STATISTICS
AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS
BOX 4348
CHICAGO, ILLINOIS 60680

DEPARTMENT OF STATISTICS
IOWA STATE UNIVERSITY
AMES, IOWA 50011