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CONSISTENCY IN THE LOCATION MODEL: THE UNDOMINATED CASE[1]

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Consistency in the undominated location model is investigated from a Bayesian point of view and a proof on the consistency of the Bayes procedures with respect to the invariant prior is provided. The consistency of Bayes procedures with respect to other prior measures is established as a corollary.

1. Introduction. Let $X_1, \cdots, X_n, \cdots$ be a sequence of random variables with distribution P_θ where θ is a location parameter in the following sense: for each n, the joint cumulative distribution function of $X_1, \cdots, X_n$ satisfies $F_\theta(x_1, \cdots, x_n) = F_0(x_1 - \theta, \cdots, x_n - \theta)$ where F_0 is known and θ is real. Let R be the real line and $\mathscr{B}$ its Borel σ-field. Denote

$$\mathbf{x} = (x_1, \cdots, x_n), \; \mathbf{x} + t = (x_1 + t, \cdots, x_n + t)$$

and define $Q(A) = \int P_s(A)\, ds$ for all $A \in \mathscr{B}^n$. Let $\mathscr{F}_n$ be the σ-field generated by the ancillary statistics $X_2 - X_1, \cdots, X_n - X_1$ and $\mathscr{F}_\infty$ be the P_0-completion of $\bigvee_{n=2}^\infty \mathscr{F}_n$. Note that $\mathscr{F}_\infty$ is P_θ-complete for all θ and is also Q-complete. For brevity, denote the regular conditional distribution of X_1 given $\mathscr{F}_n$ computed with respect to P_0 by $P_0(\cdot \mid y_2, \cdots y_n)$.

Let $P^n(ds \mid \mathbf{x})$ be a function defined on $\mathscr{B} \times R^n$ by

$$P^n(A \mid x_1, \cdots, x_n) = P_0(-A + x_1 \mid x_2 - x_1, \cdots, x_n - x_1).$$

Then $P^n(ds \mid \mathbf{x})$ is in fact the posterior distribution of the location parameter s given $\mathbf{x}$ with respect to the Lebesgue prior and satisfies the following conditions:

(1.1) (i) $P^n(\cdot \mid \mathbf{x})$ is a probability on $\mathscr{B}$ for each $\mathbf{x} \in R^n$,

(1.2) (ii) $P^n(B \mid \cdot)$ *is* $\mathscr{B}^n$-measurable for each $B \in \mathscr{B}$,

(1.3) (iii) $\int_{R^{n+1}} g(s, \mathbf{x}) P^n(ds \mid \mathbf{x}) Q(d\mathbf{x}) = \int_{R^{n+1}} g(s, \mathbf{x}) P_s(d\mathbf{x})\, ds$ for all non-negative and $\mathscr{B}^{n+1}$-measureable functions g and

(1.4) (iv) $P^n(B + t \mid \mathbf{x} + t) = P^n(B \mid \mathbf{x})$ for all $B \in \mathscr{B}$, all t and all $\mathbf{x}$.

Condition (1.3) can be established by the usual monotone class arguments beginning with g of the form $g(s, \mathbf{x}) = I_A(s) I_B(x_1) g_1(x_2 - x_1, \cdots, x_n - x_1)$. We call $\{P^n(ds \mid \mathbf{x}),\ n \geq 1\}$ the sequence of Pitman distributions. Note that $\hat{\theta}_n = \int s\, P^n(ds \mid \mathbf{x})$, if it exists, is the classical Pitman estimate and the invariance and unbiasedness of θ_n is well-known (Pitman, 1939, and Blackwell and Girschick, 1954).

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In Section 2, we prove that $P^n(B \mid \mathbf{x})$ converges to 1 or 0 according as $\theta \in B$ for B open or $\theta \notin B$ for B closed. The proof is based on a forward martingale argument in the spirit of Doob (1949). That a posterior distribution with respect to smooth prior measure behaves similarly and the corresponding Bayes estimate is consistent are obtained as a corollary. The consistency of the Pitman estimate is also established under no absolute moment conditions.

Note that previous work pioneered by Le Cam (1953) and refined by subsequent workers, notably Schwartz (1965), Berk (1966) as well as the work of Farrell (1964) and Strasser (1981) does not apply here since they assume the existence of a σ-finite measure which dominates P_θ for all θ.

2. Consistency in the location model. Denote that a function g is measurable with respect to a σ-field $\mathscr{F}$ by $g \in \mathscr{F}$. We also denote a property which holds almost everywhere with respect to a measure μ by a.s. $[\mu]$. The key condition in establishing consistency of P^n is that $X_1 \in \mathscr{F}_\infty$. Our first proposition shows that this is a consequence of one of the following laws of large numbers:

(A) For any bounded and measurable h, $(1/n) \sum_1^n h(X_i) \to C$ in P_0-probability where C is a finite constant (possibly depending on h).

This assumption is satisfied in the usual iid case or if the X's are stationary and ergodic (Doob, 1953).

(B) $(1/n) \sum_1^n X_i \to C$ in P_0-probability where C is a finite constant.

PROPOSITION 1. *Assume* (A) *or* (B). *Then* $X_1 \in \mathscr{F}_\infty$.

PROOF. Let $g_n = (1/(n-1)) \sum_{j=2}^n e^{it(X_j - X_1)}$. Note that $g_n \in \mathscr{F}_n$. Now rewrite g_n as $e^{-itX_1}(1/(n-1)) \sum_{j=2}^n e^{itX_j}$. By (A), g_n converges to $e^{-itX_1}C_0$ in P_0-probability. Since $\mathscr{F}_\infty$ is P_0-complete we conclude $e^{-itX_1} \in \mathscr{F}_\infty$. Now differentiate with respect to t and put $t = 0$ to conclude $X_1 \in \mathscr{F}_\infty$. If (B) is assumed, let

$$g_n = (1/(n-1)) \sum_{j=2}^n (X_j - X_1)$$

and argue similarly. □

THEOREM 1. *Assume* $X_1 \in \mathscr{F}_\infty$. *For each* θ, $P^n(B \mid \mathbf{x})$ *converges to* 1 *or* 0 *a.s.* $[P_\theta]$ *according as* $\theta \in B$ *for* B *open or* $\theta \notin B$ *for* B *closed.*

PROOF. The proof is based on a characteristic function argument. First we let $\phi_n(t) = \int e^{its} P^n(ds \mid \mathbf{x})$. The definition of P^n implies $e^{-itX_1}\phi_n(t) = E_0(e^{-itX_1} \mid \mathscr{F}_n)$ a.s. $[P_0]$. It follows from the forward martingale convergence theorem of Doob (1953) that $e^{-itX_1}\phi_n(t)$ converges to $E_0(e^{-itX_1} \mid \mathscr{F}_\infty)$ a.s. $[P_0]$. Since $X_1 \in \mathscr{F}_\infty$, $e^{-itX_1}\phi_n(t) \to e^{-itX_1}$ a.s. $[P_0]$. Thus for each fixed t, $\phi_n(t) \to 1$ a.s. $[P_0]$. Finally, an application of Fubini's theorem gives $P_0\{\phi_n(t) \to 1$, almost all $t\} = 1$, implying $P_0\{P^n(ds \mid \mathbf{x}) \to_{\mathscr{L}} \delta_0\} = 1$. This implies $P_\theta\{P^n(ds \mid \mathbf{x}) \to_{\mathscr{L}} \delta_\theta\} = 1$ for all θ by (1.4), completing the proof. □

The following corollary depicts the consistency of Bayes procedures for smooth priors in the undominated location family situation. Let π be a prior σ-finite measure and $\pi^n(ds \mid \mathbf{x})$ be the regular conditional distribution that satisfies the following condition:

$$\int_{R^{n+1}} g(s, \mathbf{x})\pi^n(ds \mid \mathbf{x})Q_\pi(d\mathbf{x}) = \int_{R^{n+1}} g(s, \mathbf{x})P_s(d\mathbf{x})\pi(ds) \tag{2.1}$$

where g is any nonnegative function on $(R^{n+1}, \mathscr{B}^{n+1})$ and Q_π is defined by $Q_\pi(A) = \int P_s(A)\pi(ds)$, for all $A \in \mathscr{B}^n$. The $\pi^n(ds \mid \mathbf{x})$ is also called the posterior distribution of s given $\mathbf{x}$ with respect to the prior $\pi(ds)$.

Assume that π is absolutely continuous with respect to the Lebesque measure. Then there is a Q_π-version of $\pi^n(ds \mid \mathbf{x})$ which is given by

$$\pi^n(B \mid \mathbf{x}) = \frac{\int_B \pi'(s)P^n(ds \mid \mathbf{x})}{\int \pi'(s)P^n(ds \mid \mathbf{x})}, \quad \text{for all} \quad B \in \mathscr{B}. \tag{2.2}$$

COROLLARY. *Let $\pi^n(ds \mid \mathbf{x})$ be defined by* (2.2) *and assume $X_1 \in \mathscr{F}_\infty$.*

(i) *For any θ, $\pi'(s)$ is bounded, continuous and positive at θ implies $\pi^n(B \mid \mathbf{x})$ converges to* 1 *or* 0 *a.s. $[P] \to [P_\theta]$ according as $\theta \in B$ for B open or $\theta \notin B$ for B closed.*

(ii) *If in addition to the conditions in* (i) *$s\,\pi'(s)$ is bounded, then $\int s\,\pi^n(ds \mid \mathbf{x})$ converges to θ a.s. $[P_\theta]$ for all θ.*

PROOF. First note that

$$\int e^{its}\pi^n(ds \mid \mathbf{x}) = \frac{\int e^{its}\pi'(s)P^n(ds \mid \mathbf{x})}{\int \pi'(s)P^n(ds \mid \mathbf{x})} \text{ by (2.2).}$$

Next, note that by Theorem 1, $P^n(ds \mid \mathbf{x})$ converges weakly to a point mass at θ a.s. $[P_\theta]$. Therefore, by standard weak convergence arguments $\int e^{its}\pi^n(ds \mid \mathbf{x}) \to e^{it\theta}$ a.s. $[P_\theta]$. Finally an application of Fubini's theorem as in Theorem 1 proves (i). The proof of (ii) is similar and is omitted. □

REMARK. The conditions on π given in the corollary are imposed because they are convenient to apply. They are far from necessary. Take for example the genuine Bayes situation, i.e. $\pi(R) = 1$. One expects that if Bayes consistency holds for one parameter value (this is guaranteed by Doob, 1949), by invariance it must hold for all parameter values lying in the support of π. However, the author does not know a proof of this phenomenon.

A conclusion of the above corollary is that if a Bayesian statistician is willing to use smooth priors, he can be assured of consistent estimates. The situation for the Pitman estimate is not as simple since $\hat{\theta}_n = \int sP^n(ds \mid \mathbf{x})$ need not exist. Nevertheless, we will show that if $X_1 \in \mathscr{F}_\infty$ and $\hat{\theta}_n$ exists a.s. $[P_0]$ for some n then the Pitman estimate is consistent.

THEOREM 2. *Assume $X_1 \in \mathscr{F}_\infty$ and $E_0(|X_1| \| \mathscr{F}_{n_0}) < \infty$ a.s. $[P_0]$ for some n_0. Then $P_\theta\{\hat{\theta}_n \to \theta\} = 1$ for all θ.*

PROOF. According to Proposition II-2-7 and Corollary II-2-13 of Neveu (pages 23 and 31) we have:

$$E_0(X_1^+ \mid \mathscr{F}_n) \to E_0(X_1^+ \mid \mathscr{F}_\infty) \quad \text{a.s. } [P_0] \tag{2.3}$$

and

$$\sup_{n \ge n_0} E_0(X_1^+ \mid \mathscr{F}_n) < \infty \quad \text{a.s. } [P_0]$$

and similarly for X_1^-. Hence $E_0(X_1 \mid \mathscr{F}_n)$ exists for all $n \ge n_0$. Now use the definition of P^n to check

$$\hat{\theta}_n = \int sP^n(ds \mid \mathbf{x}) = X_1 - E_0(X_1 \mid \mathscr{F}_n)$$

for all $n \ge n_0$. Apply (2.3) and $X_1 \in \mathscr{F}_\infty$ to conclude $\hat{\theta}_n \to 0$ a.s. $[P_0]$. Now apply (1.4) to conclude the proof. □

EXAMPLE. Let $X_1, \cdots, X_n$ be a sample from P_θ, where P_0 is defined by $P_0\{X_1 = k\} = C(1 + k^2)^{-1}$ for $k = 0, \pm 1, \cdots$ and C is a normalizing constant. Then, $E_0 \mid X_1 \mid = \infty$ but $E_0(\mid X_1 \mid \mid \mathscr{F}_n) < \infty$ a.s. $[P_0]$ for $n = 2$.

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