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SIMULTANEOUS ESTIMATION OF LOCATION PARAMETERS UNDER QUADRATIC LOSS

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Simultaneous estimation of p $(p \geq 3)$ location parameters are considered under quadratic loss. Explicit estimators which dominate the best invariant one are given mainly when coordinates of the best invariant one are independently, identically and symmetrically distributed. Effectiveness of integration by parts in evaluating the risk function of the dominating estimator is shown for three typical continuous distributions (uniform, double exponential and t). Further explicit dominating estimators are given in terms of second and fourth moments of the best invariant one.

1. Introduction. Since Stein (1955) first showed that the best invariant estimator of a p-dimensional $(p \geq 3)$ normal mean was inadmissible under squared error loss, there has been considerable interest in improving upon the best invariant estimator of a location vector. Brown (1966) proved that the best invariant estimator of a location vector is inadmissible for a wide class of distributions and loss functions if the dimension is at least three.

James and Stein (1961) presented an explicit estimator $\{1 - (p-2)/\|X\|^2\}X$ which is better than the best invariant estimator X under squared error loss if X has a normal distribution with covariance matrix I. They also showed that the assumption of normality is unnecessary and suggested an estimator of the form

$$\delta = \left\{1 - \frac{b}{a + \|X\|^2}\right\}X, \tag{1.1}$$

which is better than X if a and b are suitably chosen. However they did not determine their values explicitly.

Outside of the normal case explicit estimators of a location vector which dominate the best invariant one seem to be given only for the case where the distribution is spherically symmetric. (See Brandwein and Strawderman (1980) and the papers in their references.) Since the normal distribution is the only spherically symmetric distribution with independent coordinates (see Kac, 1945), no explicit dominating estimator seems to be available when coordinates are independently distributed except for the normal case.

Here we mainly deal with the case where coordinates of the best invariant estimator X are independently, identically and symmetrically distributed. We give some sufficient conditions on a and b for the estimator δ to dominate X under the squared error loss function.

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In Section 2 we show that the use of integration by parts is effective in evaluating the risk function even for nonnormal continuous distributions. As a matter of fact we apply integration by parts to three typical distributions (uniform, double exponential and t). It should be mentioned that since Stein (1973) used integration by parts for the normal case, it was shown to apply to the general continuous exponential family in simultaneous estimation problems by many authors. (See, for example, Hudson, 1978, and Berger, 1980.)

In Section 3 we use only moments to determine the values of a and b for which the estimator δ dominates X. We note that the result obtained applies not only to continuous distribution, but also to discrete ones. (The admissibility of the best invariant estimator in the discrete case was discussed in Blackwell, 1951.)

2. Effectiveness of integration by parts. Let $X_i (i = 1, \cdots, \mathrm{p})$ be an observation from a density of the form $f_i(|x_i - \theta_i|)$. We assume that $X_1, \cdots, X_p$ are independent. Setting $Z_i = X_i - \theta_i$, we assume $E(Z_i) = 0$ without loss of generality. Suppose that the loss in estimating $\theta = (\theta_1, \cdots, \theta_p)'$ by $\hat{\theta} = (\hat{\theta}_1, \cdots, \hat{\theta}_p)'$ is

$$\| \hat{\theta} - \theta \|^2 = \textstyle\sum_{i=1}^{p} (\hat{\theta}_i - \theta_i)^2.$$

The problem is to determine a and b in (1.1) so that the risk $R(\delta, \theta)$ is uniformly smaller than the risk $R(X, \theta)$.

Here we choose three typical densities (uniform, double exponential and t) as f_i, and give an evaluation of $E(\delta_i - \theta_i)^2$ which leads to a determination of a and b. We evaluate $E(\delta_i - \theta_i)^2$ by applying integration by parts repeatedly. Although we deal with only three densities, we believe that the method of evaluation can be applied to a wide class of densities.

If we express δ_i as

$$\delta_i = \left\{ 1 - \frac{b}{S_i + X_i^2} \right\} X_i,$$

where $S_i = a + \sum_{k \neq i} X_k^2$, we have

$$\text{(2.1)} \qquad \frac{1}{2b} \{(X_i - \theta_i)^2 - (\delta_i - \theta_i)^2\} = \frac{X_i(X_i - \theta_i)}{S_i + X_i^2} - \frac{b}{2} \frac{X_i^2}{(S_i + X_i^2)^2}.$$

We use integration by parts to evaluate $E\{X_i(X_i - \theta_i)/(S_i + X_i^2)\}$. In the following E_Y will stand for the expectation with respect to Y.

2.1. *Uniform distribution.*

THEOREM 2.1. *Suppose that $Z_1, \cdots, Z_p$ are independently distributed and have the common uniform distribution on $(-r, r)$. Then the risk of δ given by* (1.1) *is uniformly smaller than that of X if $a \geq 9(2 - 2^{1/2})V(X_i)/p$ and $0 < b \leq 2(p - 2)V(X_i)$. (Note that $V(X_i) = r^2/3$.)*

PROOF. Without loss of generality we set $r = 1$. Then

$$E_{X_i}\left\{\frac{X_i(X_i-\theta_i)}{S_i+X_i^2}\right\} = \frac{1}{2}\int_{-1}^{1}\frac{(z_i+\theta_i)}{S_i+(z_i+\theta_i)^2}\, z_i\, dz_i \equiv I,$$

say. Applying integration by parts, we have

$$I = 4^{-1}\int_{-1}^{1} g(z_i)(1-z_i^2)\, dz_i,$$

where

$$g(z_i) = \frac{1}{S_i+(z_i+\theta_i)^2} - \frac{2(z_i+\theta_i)^2}{\{S_i+(z_i+\theta_i)^2\}^2}. \tag{2.2}$$

We express I as

$$I = 3^{-1}E_{Z_i}\{g(Z_i)\} + I_1, \tag{2.3}$$

where

$$I_1 = \int_{-1}^{1} g(z_i)(1/12 - z_i^2/4)\, dz_i.$$

If we apply integration by parts to I_1 twice, we have

$$I_1 = \int_{-1}^{1} h(z_i)(1-z_i^2)^2/8\, dz_i,$$

where

$$h(z_i) = \frac{1}{\{S_i+(z_i+\theta_i)^2\}^2} - \frac{8(z_i+\theta_i)^2}{\{S_i+(z_i+\theta_i)^2\}^3} + \frac{8(z_i+\theta_i)^4}{\{S_i+(z_i+\theta_i)^2\}^4}. \tag{2.4}$$

Using the inequalities $h(z_i) \geq -4(2-2^{1/2})(z_i+\theta_i)^2/\{S_i+(z_i+\theta_i)^2\}^3$ and $(1-z_i^2)^2 \leq 1$, we have

$$I_1 \geq -(2-2^{1/2})E_{X_i}\{X_i^2/(S_i+X_i^2)^3\}. \tag{2.5}$$

By (2.3) and (2.5) we see that

$$I \geq \frac{1}{3}E_{x_i}\left[\frac{1}{S_i+X_i^2} - \frac{2X_i^2}{(S_i+X_i^2)^2} - \frac{3(2-2^{1/2})X_i^2}{(S_i+X_i^2)^3}\right].$$

We note that $1/3$ is $V(X_i)$. Therefore we have

$$E\left\{\sum_{i=1}^{p}\frac{X_i(X_i-\theta_i)}{a+\|X\|^2}\right\} \geq \frac{1}{3}E\left[\frac{p-2}{a+\|X\|^2} + \frac{2a-3(2-2^{1/2})}{\{a+\|X\|^2\}^2}\right].$$

Hence from (2.1),

$$R(X,\theta) - R(\delta,\theta) \geq \frac{2}{3}\, b\, E\left[\frac{p-2}{a+\|X\|^2} + \frac{2a-3(2-2^{1/2})}{\{a+\|X\|^2\}^2} - \frac{3b\,\|X\|^2}{2\{a+\|X\|^2\}^2}\right],$$

which is nonnegative if $a \geq 9(2-2^{1/2})V(X_i)/p$ and $0 < b \leq 2(p-2)V(X_i)$.

2.2 *Double exponential distribution.*

THEOREM 2.2. *Suppose that $Z_1, \cdots, Z_p$ are independently distributed and have the common density*

$$(2\beta)^{-1}\exp(-|z_i|/\beta), \quad -\infty < z_i < \infty,$$

where $\beta > 0$. Then the risk of δ given by (1.1) *is uniformly smaller than that of X if $a \geq 3V(X_i)$ and $0 < b \leq 2(p-2)V(X_i)$. (Note that $V(X_i) = 2\beta^2$.)*

PROOF. Without loss of generality we set $\beta = 1$. Then

$$E_{X_i}\left\{\frac{X_i(X_i-\theta_i)}{S_i+X_i^2}\right\} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{z_i+\theta_i}{S_i+(z_i+\theta_i)^2}\, z_i\exp(-|z_i|)\,dz_i \equiv J,$$

say. By applying integration by parts we see that

$$J = E_{Z_i}\{g(Z_i)\} + J_1, \tag{2.6}$$

where $g(\cdot)$ is given by (2.2) and

$$J_1 = 2^{-1}\int_{-\infty}^{\infty}\{g(z_i)\,|z_i|\}\exp(-|z_i|)\,dz_i.$$

By applying integration by parts to J_1 twice and using (2.4), we have

$$\begin{aligned} J_1 &= E_{Z_i}\{g(Z_i)\} - 6\,E_{Z_i}\{h(Z_i)\} - 3\int_{-\infty}^{\infty} h(z_i)\,|z_i|\,\exp(-|z_i|)\,dz_i \\ &= E_{Z_i}[g(Z_i) - 6\{S_i+(Z_i+\theta_i)^2\}^{-2}] + 3J_2, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} J_2 = 8\int_{-\infty}^{\infty}&\frac{S_i(z_i+\theta_i)^2}{\{S_i+(z_i+\theta_i)^2\}^4}(1+|z_i|)\exp(-|z_i|)\,dz_i \\ &-\int_{\infty}^{\infty}\frac{|z_i|}{\{S_i+(z_i+\theta_i)^2\}^2}\exp(-|z_i|)\,dz_i. \end{aligned}$$

By applying integration by parts to the second term of J_2 twice, we see that

$$\begin{aligned} J_2 &= -2E_{Z_i}\left[\frac{1}{\{S_i+(Z_i+\theta_i)^2\}^2}\right] \\ &\quad + 4\int_{-\infty}^{\infty}\frac{S_i-5(z_i+\theta_i)^2+2S_i(z_i+\theta_i)^2}{\{S_i+(z_i+\theta_i)^2\}^4} \\ &\quad \times (1+|z_i|)\exp(-|z_i|)\,dz_i \\ &\geq -2E_{Z_i}[\{S_i+(Z_i+\theta_i)^2\}^{-2}], \end{aligned} \tag{2.8}$$

if S_i(or a) ≥ 2.5. From (2.6) (2.7) and (2.8) we see that

$$J \geq 2E_{X_i}\left\{\frac{1}{S_i+X_i^2} - \frac{2X_i^2+6}{(S_i+X_i^2)^2}\right\},$$

if $a \geq 2.5$. We note that 2 is $V(X_i)$. Hence we have

$$R(X, \theta) - R(\delta, \theta) \geq 4bE\left[\frac{p-2}{a + \|X\|^2} + \frac{2a - 6p}{(a + \|X\|^2)^2} - \frac{b\|X\|^2}{4(a + \|X\|^2)^2}\right],$$

which is nonnegative if $a \geq 3V(X_i)$ and $0 < b \leq 2(p-2)V(X_i)$.

2.3. *t distribution.*

THEOREM 2.3. *Suppose that $Z_1, \cdots, Z_p$ are independently distributed and have the common density*

$$c\{1 + z_i^2/(\beta\sigma^2)\}^{-(1+\beta)/2}, \quad -\infty < z_i < \infty,$$

where $c = \Gamma\{(\beta+1)/2\}/\{\sigma(\beta\pi)^{1/2}\Gamma(\beta/2)\}$. Then the risk of δ given by (1.1) is uniformly smaller than that of X if $\beta \geq 10$, $a \geq 4V(X_i)$ and $0 < b \leq 2(17p/21 - 2)V(X_i)$. (Note that $V(X_i) = \sigma^2\beta/(\beta-2)$.)

NOTE. From the proof below, it will be clear that even if $6 < \beta < 10$, we can choose a and b so that δ dominates X.

PROOF. Without loss of generality we set $\sigma = 1$. Then

$$E_{X_i}\left\{\frac{X_i(X_i - \theta_i)}{S_i + X_i^2}\right\} = c\int_{-\infty}^{\infty} \frac{z_i + \theta_i}{S_i + (z_i + \theta_i)^2} z_i\left(1 + \frac{z_i^2}{\beta}\right)^{-(1+\beta)/2} dz_i \equiv K,$$

say. Again by integration by parts, we see that

$$K = \beta(\beta-1)^{-1}E_{Z_i}\{g(Z_i)\} + (\beta-1)^{-1}K_1, \tag{2.9}$$

where $g(\cdot)$ is given by (2.2) and

$$K_1 = c\int_{-\infty}^{\infty} \{g(z_i)z_i\}\{z_i(1 + z_i^2/\beta)^{-(1+\beta)/2}\}\, dz_i.$$

If we apply integration by parts to K_1, we have

$$K_1 = (\beta-1)^{-1}K_1 + \beta(\beta-1)^{-1}E_{Z_i}\{g(Z_i)\} + \beta(\beta-1)^{-1}K_2,$$

where

$$\begin{aligned} K_2 &= c\int_{-\infty}^{\infty}\left[\frac{-6(z_i+\theta_i)}{\{S_i + (z_i+\theta_i)^2\}^2} + \frac{8(z_i+\theta_i)^3}{\{S_i + (z_i+\theta_i)^2\}^3}\right]\left\{z_i\left(1 + \frac{z_i^2}{\beta}\right)^{-(3+\beta)/2}\right\} dz_i \\ &= -6c\beta(\beta-3)^{-1}\int_{-\infty}^{\infty} h(z_i)(1 + z_i^2/\beta)^{-(5+\beta)/2}\, dz_i \end{aligned}$$

(again by integration by parts), where $h(\cdot)$ is given by (2.4). Thus we have

$$K_1 = \beta(\beta-2)^{-1}E_{Z_i}\{g(Z_i)\} + \beta(\beta-2)^{-1}K_2. \tag{2.10}$$

From (2.9) and (2.10) we see that

$$K = \frac{\beta}{\beta - 2} E_{X_i}\left\{\frac{1}{S_i + X_i^2} - \frac{2X_i^2}{(S_i + X_i^2)^2}\right\} - \frac{6\beta^2}{(\beta - 1)(\beta - 2)(\beta - 3)} E_{Z_i}\left\{h(Z_i)\left(1 + \frac{Z_i^2}{\beta}\right)^2\right\}. \tag{2.11}$$

We note that $\beta(\beta - 2)^{-1}$ is $V(X_i)$. In this case it is not so easy to get a lower bound for $-E_{Z_i}\{h(Z_i^2)(1 + Z_i^2/\beta)^2\}$, but such a bound is given by (A.3) in the Appendix. Thus from (2.11) and (A.3) we see that

$$\begin{aligned}
&\{V(X_i)\}^{-1}E_{X_i}\left\{\frac{X_i(X_i - \theta_i)}{S_i + X_i^2}\right\} \\
&\geq E\left(\frac{1}{S_i + X_i^2}\right) \\
&\quad\cdot\left[1 - \frac{6}{\beta(\beta - 1)(\beta - 3)}\left\{E(Z_i^2) + \frac{2E(Z_i^4)\{a + E(Z_i^2)\}}{a^2} + \frac{4E(Z_i^6)\{a + E(Z_i^2)\}}{a^3}\right\}\right] \\
&\quad - E\left[\frac{2X_i^2 + 6\beta^2/\{(\beta - 1)(\beta - 2)(\beta - 3)\}}{(S_i + X_i^2)^2}\right]
\end{aligned}$$

if $a \geq 2V(X_i)$. It can be easily checked that $R(X, \theta) - R(\delta, \theta) > 0$ if $\beta \geq 10$, $a \geq 4V(X_i)$ and $0 < b \leq 2(17p/21 - 2)V(X_i)$.

3. Determination of the range of a and b by using only moments. We cannot apply integration by parts unless the density function is specified. Here we only use information about moments of Z_i and determine a and b so that $R(\delta, \theta)$ is uniformly smaller than $R(X, \theta)$.

Here we assume that $Z_1, \cdots, Z_p$ are independent random variables with

$$E(Z_i) = 0, \quad E(Z_i^2) = 1, \quad E(Z_i^3) = 0, \quad \text{and } E(Z_i^4) = \kappa, \quad i = 1, \cdots, p.$$

Setting $b = 2(p - 2 - c)$, we may summarize the obtained result in the following.

THEOREM 3.1. *The risk of δ given by* (1.1) *is uniformly smaller than that of X if a and c satisfy either conditions* (I-1)—(I-4) *or* (II-1) *and* (II-2):

(I-1) $a \geq a_0$, *where*
$$\begin{aligned}
a_0 &= p - 2, && \text{if } p \geq 15, \\
&= p - 1, && \text{if } 12 \leq p \leq 14, \\
&= p, && \text{if } 9 \leq p \leq 11, \\
&= p + 1, && \text{if } 6 \leq p \leq 8, \\
&= p + 2, && \text{if } 3 \leq p \leq 5.
\end{aligned}$$

(I-2) $c_0(a) \leq c < p - 2$, *where* $c_0(a) = \dfrac{1}{a + 1}\left\{\dfrac{\max(3, \kappa)}{4} + (p - 2)\right\}$.

(I-3) $a + c \geq 5(\kappa - 3)/4 + 1/2$.

(I-4) $a - a_0 + (m + 1)(c - c_0(a)) \geq 2(\kappa - 3)$,

$$\begin{aligned} \textit{where } m &= 3(p-2)/p, && \textit{if } p \geq 7,\\ &= (3p-5)/p, && \textit{if } p = 5, 6,\\ &= 2, && \textit{if } p = 4,\\ &= 2.1, && \textit{if } p = 3. \end{aligned}$$

(II-1) $c \geq (p - 2)/2$,

$$\text{(II-2) } a \geq a_1 + 4(\kappa - 3)^+/3, \textit{ where } \begin{aligned} a_1 &= 4, && \textit{if } p \geq 5,\\ &= 4.2 && \textit{if } p = 4,\\ &= 4.5, && \textit{if } p = 3. \end{aligned}$$

PROOF. We only give a proof for the case when a and c satisfy (I-1)—(I-4) since the one for the other case is similar. In the proof we repeatedly use the following identity:

$$\frac{1}{a + \|z + \theta\|^2} = \frac{1}{a + p + \|\theta\|^2} - \frac{\|z\|^2 + 2z'\theta - p}{(a + p + \|\theta\|^2)(a + \|z + \theta\|^2)}. \tag{3.1}$$

Using (3.1) twice and taking expectation, we see that

$$E\left\{\frac{\|Z\|^2 + Z'\theta}{a + \|Z + \theta\|^2}\right\} = \frac{p - 2}{a + p + \|\theta\|^2} + \frac{2a - p(\kappa - 3)}{(a + p + \|\theta\|^2)^2} + R,$$

where

$$R = E\left\{\frac{(\|Z\|^2 + Z'\theta)(\|Z\|^2 + 2Z'\theta - p)^2}{(a + p + \|\theta\|^2)^2(a + \|Z + \theta\|^2)}\right\}.$$

Therefore we have

$$\begin{aligned} &\{R(X, \theta) - R(\delta, \theta)\}/(2b)\\ &\quad = E\left\{\frac{\|Z\|^2 + Z'\theta}{a + \|Z + \theta\|^2}\right\} - (p - 2 - c)E\left\{\frac{\|Z + \theta\|^2}{(a + \|Z + \theta\|^2)^2}\right\}\\ &\quad = \frac{p - 2 - c}{a + p + \|\theta\|^2} - E\left\{\frac{(p - 2 - c)\ \|Z + \theta\|^2}{(a + \|Z + \theta\|^2)^2}\right\}\\ &\quad\quad + \frac{c}{a + p + \|\theta\|^2} + \frac{2a - p(\kappa - 3)}{(a + p + \|\theta\|^2)^2} + R\\ &\quad \equiv D, \end{aligned}$$

say. Using (3.1) repeatedly, we can easily see that

$$\frac{1}{a+p+\|\theta\|^2} - E\left\{\frac{\|Z+\theta\|^2}{(a+\|Z+\theta\|^2)^2}\right\}$$

$$= \frac{a}{(a+p+\|\theta\|^2)^2} - E\left\{\frac{(\|Z\|^2+2Z'\theta-p)^2}{(a+p+\|\theta\|^2)^2(a+\|Z+\theta\|^2)}\right\}$$

$$+ E\left\{\frac{2a(\|Z\|^2+2Z'\theta-p)^2}{(a+p+\|\theta\|^2)^3(a+\|Z+\theta\|^2)}\right\}$$

$$+ E\left\{\frac{a(\|Z\|^2+2Z'\theta-p)^2}{(a+p+\|\theta\|^2)^2(a+\|Z+\theta\|^2)^2}\right\}.$$

Therefore we have

$$\text{(3.2)}\quad D = \frac{c}{a+p+\|\theta\|^2} + \frac{(p-c)\,a - p(\kappa-3)}{(a+p+\|\theta\|^2)^2} + R + \frac{p-2-c}{(a+p\ \|\theta\|^2)^2}S,$$

where

$$S = \left\{\frac{(\|Z\|^2+2Z'\theta-p)^2}{a+\|Z+\theta\|^2}\right\}\left\{\frac{2a}{a+p+\|\theta\|^2}-1\right\}$$

$$+ E\left\{\frac{a(\|Z\|^2+2Z'\theta-p)^2}{(a+\|Z+\theta\|^2)^2}\right\}.$$

We first give a lower bound of S. Again using (3.1) repeatedly, we see that

$$(a+\|\theta\|^2+p)^2S$$

$$= -(p+\|\theta\|^2-2a)\{p(\kappa-1)+4\|\theta\|^2\}$$

$$+ (p+\|\theta\|^2-3a)E\{(\|Z\|^2+2Z'\theta-p)^3(a+\|Z+\theta\|^2)^{-1}\}$$

$$+ aE\{(\|Z\|^2+2Z'\theta-p)^4(a+\|Z+\theta\|^2)^{-2}\}.$$

We notice that $(\|Z\|^2+2Z'\theta-p)^4(a+\|Z+\theta\|^2)^{-2} \geq (\|Z\|^2+2Z'\theta-p)^2\{-\ell^2 - 2\ell(\|Z\|^2+2Z'\theta-p)(a+\|Z+\theta\|^2)^{-1}\}$ for any ℓ. Choosing $\ell = (p+\|\theta\|^2-3a)/(2a)$, we have

$$\text{(3.3)}\quad \begin{aligned} S &\geq -\{p(\kappa-1)+4\|\theta\|^2\}\{1-2a/(a+p+\|\theta\|^2)\}^2/(4a) \\ &\geq -\{p(\kappa-1)+4\|\theta\|^2\}/(4a). \end{aligned}$$

To evaluate R, we consider the following two cases: $\|\theta\|^2 \leq mp$ and $\|\theta\|^2 > \mathrm{mp}$.

CASE 1. $\|\theta\|^2 \leq mp$. Noting that $R \geq -\|\theta\|^2\{p(\kappa-1)+4\|\theta\|^2\}/\{4a(a+p$

$+ \|\theta\|^2)^2\}$, from (3.2) and (3.3) we have

$$(3.4)\quad \begin{aligned}(a+p+\|\theta\|^2)^2 D \ge p(c+a-\kappa+3) + c\,\|\theta\|^2 \\ - (\|\theta\|^2 + p - 2 - c)\{p(\kappa-1) + 4\,\|\theta\|^2\}/(4a).\end{aligned}$$

Since the right hand side of (3.4) is a quadratic function of $\|\theta\|^2$, we can verify that it is nonnegative under the conditions (I-1)—(I-4) by examining the cases $\|\theta\|^2 = 0$ and $\|\theta\|^2 = mp$.

CASE 2. $\|\theta\|^2 > mp$. We first decompose R as $R = R_1 + R_2$, where

$$R_1 = E\{(\|Z\|^2 + Z'\theta)(\|Z\|^2 + 2Z'\theta)^2(a+p+\|\theta\|^2)^{-2}(a+\|Z+\theta\|^2)^{-1}\},$$

and

$$R_2 = -E[\{2p(\|Z\|^2 + Z'\theta)(\|Z\|^2 + 2Z'\theta - p) + p^2(\|Z\|^2 + Z'\theta)\}$$
$$(a+p+\|\theta\|^2)^{-2}\,(a+\|Z+\theta\|^2)^{-1}].$$

Since $(\|Z\|^2 + Z'\theta)(\|Z\|^2 + 2Z'\theta)^2(a+\|Z+\theta\|^2)^{-1} = \|Z\|^2\{\|Z\|^4 + 5Z'\theta\,\|Z\|^2 + 8(Z'\theta)^2\}\,(a+\|Z+\theta\|^2)^{-1} + 4(Z'\theta)^3(a+\|\theta\|^2)^{-1} - 4(Z'\theta)^3(\|Z\|^2 + 2Z'\theta)(a+\|\theta\|^2)^{-1}(a+\|Z+\theta\|^2)^{-1}$, we have

$$\begin{aligned}(a+p+\|\theta\|^2)^2 R_1 &= E\left[\frac{\|Z\|^2\{\|Z\|^4 + 5Z'\theta\,\|Z\|^2 + 8(Z'\theta)^2\}}{a+\|Z+\theta\|^2}\right] \\ &\quad - E\left\{\frac{4(Z'\theta)^3(\|Z\|^2 + 2Z'\theta)}{(a+\|\theta\|^2)(a+\|Z+\theta\|^2)}\right\} \\ &\ge E\,\frac{(Z'\theta)^2(\|Z\|^4 + \|Z\|^2 Z'\theta)}{(a+\|\theta\|^2)(a+\|Z+\theta\|^2)} \\ &\ge -(4a)^{-1}E\left\{\frac{(Z'\theta)^4}{(a+\|\theta\|^2)}\right\}.\end{aligned}$$

Since $E(Z'\theta)^4 \le \max(\kappa, 3)\,\|\theta\|^4$, we can easily see that

$$(3.5)\quad \begin{aligned}R_1 \ge \max(\kappa,3)[&-\{4a(a+p+\|\theta\|^2)\}^{-1} \\ &+ p\{4a(a+p+\|\theta\|^2)^2\}^{-1}\{2(a+p+\|\theta\|^2)^2\}^{-1} \\ &- a\{4(a+\|\theta\|^2)(a+p+\|\theta\|^2)^2\}^{-1}].\end{aligned}$$

On the other hand by using (3.1) we have

$$R_2 = -E\,\frac{(\|Z\|^2 + Z'\theta)(\|Z\|^2 + 2Z'\theta - p)}{(a+p+\|\theta\|^2)^2}\,r - E\,\frac{p^2(\|Z\|^2 + Z'\theta)}{(a+p+\|\theta\|^2)^3} + Rr,$$

where $r = p(a+p+\|\theta\|^2)^{-1}\{2 - p/(a+p+\|\theta\|^2)\}$. Since $R = R_1 + R_2$, we have

$$(3.6)\quad \begin{aligned}R = (1-r)^{-1}R_1 - r(1-r)^{-1}\{p(\kappa-1) + 2\,\|\theta\|^2\}(a+p+\|\theta\|^2)^{-2} \\ - p^3(1-r)^{-1}(a+p+\|\theta\|^2)^{-3}.\end{aligned}$$

From (3.5), we can easily check that if $2a \geq p$,

$$(1 - r)^{-1}R_1 \geq -\max(\kappa, 3)(4a)^{-1}(a + p + \|\theta\|^2)^{-1}. \tag{3.7}$$

Thus from (3.6) and (3.7), we have

$$\begin{aligned} R \geq & -\frac{\max(\kappa, 3)}{4a(a + p + \|\theta\|^2)} \\ & - \left\{\frac{2p}{a + \|\theta\|^2} + \frac{p^2}{(a + \|\theta\|^2)^2}\right\} \frac{p(\kappa - 1) + 2\|\theta\|^2}{(a + p + \|\theta\|^2)^2} \\ & - \frac{p^3}{(a + \|\theta\|^2)^2(a + p + \|\theta\|^2)}. \end{aligned} \tag{3.8}$$

By (3.2), (3.3) and (3.8) we have

$$\begin{aligned} D \geq\ & (a + p + \|\theta\|^2)^{-1}\{c - \max(\kappa, 3)/(4a) - (p - 2 - c_0)/a\} \\ & + (a + p + \|\theta\|^2)^{-2} \\ & \cdot [(p - c)a - p(\kappa - 3) \\ & \qquad - 4p - (p - 2 - c_0)\{p \max(0, \kappa - 3) - 4a - 2p\}/(4a)] \\ & - (a + \|\theta\|^2)^{-1}(a + p + \|\theta\|^2)^{-2}p[2\{p(\kappa - 1) - 2a\} + 2p + p^2] \\ & - (a + \|\theta\|^2)^{-2}(a + p + \|\theta\|^2)^{-2}p^2\{p(\kappa - 1) - 2a + p^2\}, \end{aligned}$$

which can be verified to be nonnegative under the conditions (I-1)—(I-4) if $\|\theta\|^2 \geq mp$.

NOTE. The values of a_0 and m given in the theorem were found by trial and error.

REMARK 1. The lower bounds of a and c given in the theorem are in no sense the minimum possible, and we should consider that they just give a guide post. Actually, when the values of κ (or its upper bound) and p are specified, we may show that δ dominates X under weaker conditions on a and c even if we adopt the same method of proof.

REMARK 2. It is well known that when $Z_1, \cdots, Z_p$ are normally distributed δ dominates X if $a \geq 0$ and $0 < b < 2(p - 2)$. By setting $\kappa = 3$ in conditions (I-1)—(I-4) in our theorem, we see that δ dominates X if $a \geq a_0$ and $c \geq c_0(a)$. We notice that $c_0(a_0) < \min\{1, (p - 2)/3\}$. If we put $b = p - 2$, we see from conditions (II-1) and (II-2) that δ dominates X if $a \geq a_1$ although the choice $a = 0$ corresponds to the Stein estimator.

REMARK 3. We can also apply the theorem to discrete distributions. For example, suppose that Y_i is a continuous random variable with density function $f(\cdot)$ and that Z_i is distributed as $[Y_i/d + 0.5]\, d$, where $[x]$ denotes the largest integer which is not larger than x. Even in this case we can apply the theorem if we can get upper and lower bounds of $E(Z_i^2)$ and also an upper bound of $E(Z_i^4)/E^2(Z_i^2)$. Without proof we state the following fact which may be useful for this purpose: If $f(\cdot)$ is a symmetric and unimodal density function, then

$$E(Y_i^2) - d^2/12 \le E(Z_i^2) \le E(Y_i^2) + d^2/4,$$

and

$$E(Z_i^4) \le E(Y_i^4) + 3d^2E(Z_i^2)/2.$$

REMARK 4. It should be mentioned that the theorem can be applied to the multiple observation case by the simple expedient of letting X_i denote an invariant estimator (ideally the best invariant estimator). We only need the values of $E(Z_i^2)$ and $E(Z_i^4)/E^2(Z_i^2)$. If X_i is the sample mean based on n observations, we can easily see that $E(Z_i^4)/E^2(Z_i^2) = 3 + (\kappa - 3)/n$, whose value is close to 3 for large n.

REMARK 5. Suppose that $E(Z_i^4) = \kappa_i$, $i = 1, \cdots, p$, are not necessarily the same. Even in this case we can show that the theorem is correct if we replace $\max(3, \kappa)$ by $\max(3, \kappa_{\max})$ in (I-2), $5(\kappa - 3)/4$ by $5(\bar{\kappa} - 3)/4$ in (I-3), $2(\kappa - 3)$ by $2(\bar{\kappa} - 3) + (\kappa_{\max} - \bar{\kappa})/(8p)$ in (I-4) and $4(\kappa - 3)^+/3$ by $(\bar{\kappa} - 3)^+ + (\kappa_{\max} - 3)^+/3$ in (II-2) respectively, where $\kappa_{\max} = \max(\kappa_1, \cdots, \kappa_p)$ and $\bar{\kappa} = \sum \kappa_i/p$.

REMARK 6. One implication of the theorem is that the property that δ dominates X is very robust as noted in James and Stein (1961, page 369). As a matter of fact, if we may assume that the density is symmetric and if we have some information about the upper bound for κ_i, $i = 1, \cdots, p$, we do not have to worry about the form of the density function. However, if we may assume a specific form of density, we will have a wide range of a and b (as in Section 2), which may lead to a larger improvement upon X.

APPENDIX

A lower bound of $-E_{Z_i}\{h(Z_i)(1 + Z_i^2/\beta)^2\}$ when Z_i has t distribution. We first note that

$$\begin{aligned}
-E_{Z_i}&\left\{h(Z_i)\left(1 + \frac{Z_i^2}{\beta}\right)^2\right\} \\
&= -E_{Z_i}\left[\frac{(1 + Z_i^2/\beta)^2}{\{S_i + (Z_i + \theta_i)^2\}^2}\right] + 8E_{Z_i}\left[\frac{S_i(Z_i + \theta_i)^2(1 + Z_i^2/\beta)^2}{\{S_i + (Z_i + \theta_i)^2\}^4}\right] \\
&\equiv L_1 + L_2,
\end{aligned}$$

say. For L_1 we have

$$L_1 = -E_{Z_i}\left[\frac{1}{\{S_i + (Z_i + \theta_i)^2\}^2}\right] - E_{Z_i}\left[\frac{2Z_i^2}{\beta\{S_i + (Z_i + \theta_i)^2\}^2}\right]$$

$$- E_{Z_i}\left[\frac{Z_i^4}{\beta^2\{S_i + (Z_i + \theta_i)^2\}^2}\right],$$

$$\equiv L_{11} + L_{12} + L_{13},$$

say. Applying integration by parts to L_{12}, we have

$$L_{12} = -\frac{2}{\beta - 2} E_{Z_i}\left[\frac{1}{\{S_i + (Z_i + \theta_i)^2\}^2}\right]$$

$$+ \frac{8\beta}{(\beta - 2)(\beta - 3)} E_{Z_i}\left[\frac{(1 + Z_i^2/\beta)^2}{\{S_i + (Z_i + \theta_i)^2\}^3} - \frac{6(Z_i + \theta_i)^2(1 + Z_i^2/\beta)^2}{\{S_i + (Z_i + \theta_i)^2\}^4}\right].$$

Thus it is easily seen that

$$L_2 + L_{12} \geq -2(\beta - 2)^{-1}E_{Z_i}[\{S_i + (Z_i + \theta_i)^2\}^{-2}],$$

if $a \geq 5\beta(\beta - 2)^{-1}(\beta - 3)^{-1}$. Therefore under the same condition on a we have

(A.1) $$L_1 + L_2 \geq -\beta(\beta - 2)^{-1}E_{Z_i}[\{S_i + (Z_i + \theta_i)^2\}^{-2}] + L_{13}.$$

To evaluate L_{13}, we first note that

$$\frac{1}{S_i + (Z_i + \theta_i)^2} = \frac{1}{S_i + \theta_i^2 + Z_i^2} - \frac{2\theta_i Z_i}{(S_i + \theta_i^2 + Z_i^2)\{S_i + (Z_i + \theta_i)^2\}}.$$

Using the above identity, we have

$$\beta^2 L_{13} = -E_{Z_i}\left\{\frac{Z_i^4}{(S_i + \theta_i^2 + Z_i^2)^2}\right\} + 4E_{Z_i}\left[\frac{\theta_i Z_i^5}{(S_i + \theta_i^2 + Z_i^2)^2\{S_i + (Z_i + \theta_i)^2\}}\right]$$

$$-4E_{Z_i}\left[\frac{\theta_i^2 Z_i^6}{(S_i + \theta_i^2 + Z_i^2)^2\{S_i + (Z_i + \theta_i)^2\}^2}\right]$$

$$\equiv L_{131} + L_{132} + L_{133},$$

say. We can easily see that

$$L_{131} \geq -E\left[\frac{Z_i^2}{\{S_i + \theta_i^2 + E(Z_i^2)\}}\right]$$

and

$$L_{132} \geq -2E_{Z_i}\left[\frac{Z_i^4(Z_i + \theta_i)^2}{(S_i + \theta_i^2 + Z_i^2)^2\{S_i + (Z_i + \theta_i)^2\}}\right]$$

$$\geq -2a^{-1}E(Z_i^4)E_{Z_i}\{(S_i + \theta_i^2 + Z_i^2)^{-1}\}.$$

Since

$$-E_{Z_i}\left(\frac{1}{S_i+\theta_i^2+Z_i^2}\right)=\frac{-1}{S_i+\theta_i^2+E(Z_i^2)}+\frac{1}{S_i+\theta_i^2+E(Z_i^2)}E_{Z_i}\left\{\frac{Z_i^2-E(Z_i^2)}{S_i+\theta_i^2+Z_i^2}\right\}$$

$$\geq -\{1+E(Z_i^2)/a\}/\{S_i+\theta_i^2+E(Z_i^2)\},$$

we have

$$L_{132}\geq -2E(Z_i^4)\{a+E(Z_i^2)\}/[a^2\{S_i+\theta_i^2+E(Z_i^2)\}].$$

In the same way we see that

$$L_{133}\geq -4E(Z_i^6)\{a+E(Z_i^2)\}/[a^3\{S_i+\theta_i^2+E(Z_i^2)\}].$$

Since $E\{S_i+(Z_i+\theta_i)^2\}^{-1}\geq\{S_i+\theta_i^2+E(Z_i^2)\}^{-1}$, we have

$$\begin{aligned}\beta^2L_{13}\geq -[&E(Z_i^2)+2a^{-2}E(Z_i^4)\{a+E(Z_i^2)\}\\ &+4a^{-3}E(Z_i^6)\{a+E(Z_i^2)\}]E\{S_i+(Z_i+\theta_i)^2\}^{-1}.\end{aligned}\tag{A.2}$$

From (A.1) and (A.2), we see that

$$\begin{aligned}&-E_{Z_i}\{h(Z_i)(1+Z_i^2/\beta)^2\}\\ &\geq -\frac{\beta}{\beta-2}E_{Z_i}\left[\frac{1}{\{S_i+(Z_i+\theta_i)^2\}^2}\right]\\ &\quad-\frac{1}{\beta^2}\left[E(Z_i^2)+\frac{2E(Z_i^4)\{a+E(Z_i^2)\}}{a^2}+\frac{4E(Z_i^6)\{a+E(Z_i^2)\}}{a^3}\right]\\ &\quad\cdot E\left\{\frac{1}{S_i+(Z_i+\theta_i)^2}\right\}\end{aligned}\tag{A.3}$$

if $a\geq 5\beta(\beta-2)^{-1}(\beta-3)^{-1}$.

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