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A MINIMAX APPROACH TO SAMPLE SURVEYS

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Suppose that there is a population of N identifiable units each with two associated values x_i and y_i. All N values of x are given but y_i is determined only after the ith unit is selected and observed. The objective is to estimate the population total $\sum_{i=1}^N y_i$. It is assumed that $y_i = \theta x_i + \delta_i g(x_i)$, $1 \le i \le N$, where $(\delta_1, \cdots, \delta_N)$ is in some bounded neighborhood of $(0, \cdots 0)$. The Rao-Hartley-Cochran and Hansen-Hurwitz strategies are shown to be approximately minimax under certain models with $g(x) = x^{1/2}$ and with $g(x) = x$, the latter relating to a problem considered by Scott and Smith (1975). These two strategies are then compared with some commonly-used strategies and are found to perform favorably when $g^2(x)/x$ is an increasing function of x. The problem of estimating θ is also considered. Finally, some exact minimax results are obtained for sample size one.

1. Introduction. Suppose that there is a population $U = \{1, 2, \cdots, N\}$ of N identifiable units and that two correlated values x_i and y_i, where $x_i \ge 0$, are associated with the ith unit. The values $x_1, x_2, \cdots, x_N$ are given but each y_i is determined only after the ith unit is selected and observed. The objective is to estimate the unknown population total $Y = \sum_{i=1}^N y_i$ based on a sample of size n. Without loss of generality, we shall assume that $x_1 \le x_2 \le \cdots \le x_N$. In practice, x_i is often the value of y_i at some previous time when a complete census was taken. Another application is in cluster sampling where y_i is the ith cluster total and x_i is the size of the ith cluster. Many procedures have been suggested for using the auxiliary information provided by x to increase the precision of the estimate either at the design stage (e.g., using a probability proportional to size design) or at the estimation stage (e.g., using a ratio estimator) or both. The superpopulation approach of Brewer (1963) and Royall (1970a) incorporates the auxiliary information into a random superpopulation model which often leads to the selection of a purposive sample. Whether one should randomize has been one of the major controversies between the fixed population approach and the superpopulation approach. In this paper, we shall formulate a regression type model and use a minimax criterion to justify randomization. We shall assume that

$$y_i = \theta x_i + \varepsilon_i, \qquad 1 \le i \le N, \tag{1.1}$$

where $\theta \in \Theta \subset R$ (usually $\Theta = R$ or $\Theta = (0, \infty)$), and ε_i is the "error" from the strict linear relationship between y and x. We shall further assume that

$$\varepsilon_i = \delta_i g(x_i), \qquad 1 \le i \le N, \tag{1.2}$$

where g is a known function of x and $(\delta_1, \cdots, \delta_N)'$ belongs to a fixed neighborhood L of $(0, \cdots, 0)'$. Usually $g(x)$ is an increasing function of x (e.g., $g(x) = x^\alpha$, $\alpha \ge 0$). There are many possible choices for the neighborhood L depending on the measure of distance used. We may assume L to be the L_2-ball $L_2(M) = \{(\delta_1, \cdots, \delta_N)' : \sum_{i=1}^N \delta_i^2 \le M\}$ or the L_∞-ball $L_\infty(M) = \{(\delta_1, \cdots, \delta_N)' : |\delta_i| \le M$, for all $i\}$ where $M > 0$ is fixed. This is *not* a superpopulation model though it is comparable to the commonly used superpopulation

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model in which $y_1, \cdots, y_N$ are assumed to be independent random variables with

$$E(y_i) = \theta x_i \quad \text{and} \quad \operatorname{Var}(y_i) = \sigma^2 v(x_i). \tag{1.3}$$

Note that $v(x_i)$ is the counterpart of $g^2(x_i)$.

Throughout this paper, we shall assume squared loss function and only consider sampling designs with fixed sample size n. Since a sampling design with replacement can always be improved by one without replacement, we *may consider sampling designs without replacement only*. Thus, without loss of generality, we define a *sampling design* to be a probability measure P on $\mathscr{S} = \{S : S \subset U \text{ and } \#(S) = n\}$, where $\#(S)$ is the cardinality of S. We shall also restrict to linear homogeneous estimators of the population total Y, i.e., estimators of the form $\sum_{i\in S} a_{i(S)} y_i$, where S is the selected sample. This kind of estimator was first put forward by Godambe (1955) and can also be written as $\sum_{i=1}^N a_{i(S)} y_i$ with $a_{i(S)} = 0$ for all $i \notin S$. Let $\mathbf{a}_S = (a_{1(S)}, \cdots, a_{N(S)})'$. Then a linear homogeneous estimator is specified by the vectors of coefficients $\{\mathbf{a}_S\}_{S\in\mathscr{S}}$. A pair $d = (P, \{\mathbf{a}_S\}_{S\in\mathscr{S}})$ of sampling design and estimator of Y is then called a *strategy*. The set of all such strategies will be denoted by $\mathscr{D}_n$.

For any $\theta \in \Theta$ and $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_N)' \in L$, the mean squared error (MSE) of a strategy $d = (P, \{\mathbf{a}_S\}_{S\in\mathscr{S}})$, denoted by $R_n(d; \theta, \boldsymbol{\delta})$ or $R_n(P, \{\mathbf{a}_S\}_{S\in\mathscr{S}}; \theta, \boldsymbol{\delta})$, is defined to be $\sum_{S\in\mathscr{S}} P(S)(\sum_{i\in S} a_{i(S)} y_i - \sum_{i=1}^N y_i)^2$. Our goal is to find a strategy to

$$\text{minimize} \quad \sup_{\theta\in\Theta, \boldsymbol{\delta}\in L} R_n(d; \theta, \boldsymbol{\delta}). \tag{1.4}$$

When exact minimax strategies are difficult to find, approximately minimax strategies will be desirable.

Note that all the strategies in $\mathscr{D}_n$ have nonrandomized estimators. One can also consider strategies with randomized linear homogeneous estimators. All such strategies will be denoted by $\mathscr{D}_n^R$. Clearly $\mathscr{D}_n \subset \mathscr{D}_n^R$. On the other hand, since the squared loss function is used, for each strategy d in $\mathscr{D}_n^R$, there is a Rao-Blackwellized strategy d' in $\mathscr{D}_n$ which is at least as good as d. However, sometimes Rao-Blackwellization can only provide a small amount of improvement at the cost of introducing a quite complicated estimator which makes the analysis difficult. In such cases the use of a randomized strategy is preferrable. In this spirit, the well-known Rao-Hartley-Cochran strategy (Rao, Hartley and Cochran 1962), a strategy with randomized estimator, will be shown in Section 3 to be approximately minimax over $\mathscr{D}_n^R$ under model (1.1)–(1.2) with $g(x) = x$, $L = L_\infty(M)$, and that with $g(x) = x^{1/2}$ and $L = L_2(M)$, when the n largest x values are not too extreme. Also if the sampling fraction is small, then the Hansen-Hurwitz strategy (Hansen and Hurwitz, 1943), which is only slightly less efficient than the Rao-Hartley-Cochran strategy, but much simpler to implement, is also shown to be approximately minimax.

Under model (1.1)–(1.2), θ is not identifiable, i.e., for given $\{(x_i, y_i)\}_{i=1}^N$, there exist more than one θ such that (1.1) and (1.2) are satisfied. To make θ identifiable, we must restrict $\boldsymbol{\delta}$ to a suitable subset $\tilde{L}$ of L. The choice of $\tilde{L}$ depends on what we think θ means. For instance, if θ is interpreted as the population ratio $\sum_{i=1}^N y_i / \sum_{i=1}^N x_i$ and we have a situation where the individual ratios $\theta_i = y_i/x_i$ are approximately equal, then ε_i represents the error of approximating $y_i (= \theta_i x_i)$ by θx_i. Under this interpretation, we have

$$\theta \sum_{i=1}^N x_i = \sum_{i=1}^N y_i = \sum_{i=1}^N \{\theta x_i + \delta_i g(x_i)\} = \theta \sum_{i=1}^N x_i + \sum_{i=1}^N \delta_i g(x_i),$$

which implies that $\sum_{i=1}^N \delta_i g(x_i) = 0$. Then we are led to $\tilde{L} = \{\boldsymbol{\delta} \in L : \sum_{i=1}^N \delta_i g(x_i) = 0\}$, i.e., it would be appropriate to

$$\text{minimize} \quad \sup_{\theta\in\Theta, \boldsymbol{\delta}\in L, \sum_{i=1}^N \delta_i g(x_i) = 0} R_n(d; \theta, \boldsymbol{\delta}). \tag{1.5}$$

Sometimes, instead of having a definite meaning for θ, we merely know that the line $y = \theta x$ would be a good approximation of the data $\{(x_i, y_i)\}_{i=1}^N$ should we have the chance to observe all the y_i's. If θ is such that

$$\sum_{i=1}^N (y_i - \theta x_i)^2 / \{g(x_i)\}^2 = \min_{\theta'} \sum_{i=1}^N (y_i - \theta' x_i)^2 / \{g(x_i)\}^2,$$

i.e., the value which gives the weighted least squares fit, then

$$\theta = \sum_{i=1}^{N} \{g(x_i)\}^{-2}x_i y_i / \sum_{i=1}^{N} x_i^2\{g(x_i)\}^{-2}, \tag{1.6}$$

and it is not hard to see that under model (1.1)–(1.2), we have

$$\sum_{i=1}^{N} \delta_i\{g(x_i)\}^{-1}x_i = 0. \tag{1.7}$$

Hence we should

$$\text{minimize} \quad \sup_{\theta\in\Theta,\delta\in L,\sum_{i=1}^{N}\delta_i\{g(x_i)\}^{-1}x_i=0} R_n(d;\,\theta,\,\boldsymbol{\delta}). \tag{1.8}$$

Clearly Problems (1.5) and (1.8) are the same when $g(x) = x^{1/2}$. It will be shown later that for unbounded Θ and $L = L_2(M)$, (1.4) and (1.8) are identical problems for any g. Thus the results on Rao-Hartley-Cochran strategy mentioned earlier also apply to (1.8).

In Section 2, the problem will be formulated in matrix notation. We shall introduce the important notion of *risk-generating matrix* and *adjusted risk-generating matrix*, which are very useful in assessing the performance of a sampling strategy when the estimator is linear. Section 3 is devoted to the approximate minimaxity of Rao-Hartley-Cochran strategy and Hansen-Hurwitz strategy. In Section 4, we shall compare the Rao-Hartley-Cochran strategy with two commonly-used procedures: simple random sampling (SRS) together with ratio estimator and sampling with probability proportional to aggregate size (PPAS) together with ratio estimator. We find that the Rao-Hartley-Cochran strategy performs favorably when $g^2(x)/x$ is an increasing function of x, e.g., $g(x) = x^\alpha$ with $\alpha \geq \frac{1}{2}$. In Section 5, we shall study the problem of estimating θ under the identifiability condition (1.7). For a quite arbitrary g, approximately minimax strategies are derived. It turns out that if instead of selecting units with probability proportional to size x_i, we select units with probability proportional to $x_i^2/\{g(x_i)\}^2$, then the modified versions of Rao-Hartley-Cochran and Hansen-Hurwitz strategies are approximately minimax for estimating θ.

Before closing this section, we shall relate our problem to earlier works in the literature. Blackwell and Girshick (1954) gave the first minimax result for justifying the use of simple random sampling. Under the assumption that the space of all possible $\mathbf{y} = (y_1, \cdots, y_N)'$ is permutation-invariant, they showed the minimaxity of simple random sampling for any permutation-invariant estimator and loss function. Works on minimax estimation of the population mean under simple random sampling include, e.g., Aggarawal (1959), Royall (1970b), Bickel and Lehmann (1981), Hodges and Lehmann (1981). Bickel and Lehmann (1981) essentially studied model (1.1)–(1.2) with $x_1 = x_2 = \cdots = x_N$, $\theta \in R$ and $L = L_2(M)$ while Hodges and Lehmann (1981) considered model (1.1)–(1.2) with $x_1 = \cdots = x_N$, θ known, and $L = L_\infty(M)$. They did not restrict to linear homogeneous estimators as we do. Stenger (1979) obtained the minimaxity of (SRS, sample mean) under the restriction that the estimators are linear homogeneous and satisfy a condition similar to (2.2) in Section 2 of the present paper.

Scott and Smith (1975) probably is the first and only paper on the minimaxity of unequal probability sampling schemes. Their interesting results to a certain extent stimulated our work. They considered the minimax estimation of a population total $Y = \sum_{i=1}^{N} y_i$, where $y_i = x_i z_i$, with the observable z_i taking values in a fixed interval, say $0 \leq z_i \leq B$. They fixed the estimator to be $X \sum_{i=1}^{n} z_i/n$, where $z_1, \cdots, z_n$ are the n observed values of z, and showed that under some condition, if a sampling scheme *with replacement* is to be used, then the probability proportional to size design is approximately minimax. Their restrictions on the competing strategies, however, rule out many commonly used procedures, e.g., (SRS, ratio estimator) and (PPAS, ratio estimator). In fact, in Scott and Smith's model, $0 \leq y_i \leq Bx_i$. Consequently y_i can be expressed as $y_i = (B/2)x_i + \varepsilon_i$ with $|\varepsilon_i| \leq (B/2)x_i$. Now it is clear that this is a special case of our model (1.1)–(1.2) with $\theta = B/2$, $g(x) = x$, and $L = L_\infty(B/2)$. Thus our results also apply to Scott and Smith's problem; the restrictions are then removed.

2. The risk-generating matrix and adjusted risk-generating matrix. In this section we shall formulate our problem in matrix notation and derive an important matrix for assessing the performance of a sampling strategy. For convenience, the following notations will be adopted: $\mathbf{1}$ = the $N \times 1$ vector of 1's; $\mathbf{G}$ = the $N \times N$ diagonal matrix with $g(x_i)$ as the ith diagonal element; $\mathbf{x}$ = the $N \times 1$ vector $(x_1, \cdots, x_N)'$; $\mathbf{y}$ = the $N \times 1$ vector $(y_1, \cdots, y_N)'$; and $X = \sum_{i=1}^N x_i$. Then for any $d = (P, \{\mathbf{a}_S\}_{S\in\mathscr{S}}) \in \mathscr{D}_n$,

$$
\begin{aligned}
R_n(P, \{\mathbf{a}_S\}_{S\in\mathscr{S}}; \theta, \boldsymbol{\delta}) &= \textstyle\sum_{S\in\mathscr{S}} P(S)(\sum_{i=1}^N a_{i(S)}y_i - \sum_{i=1}^N y_i)^2 \\
&= \textstyle\sum_{S\in\mathscr{S}} P(S)\mathbf{y}'(\mathbf{a}_S - \mathbf{1})(\mathbf{a}_S - \mathbf{1})'\mathbf{y} \qquad (2.1)\\
&= \textstyle\sum_{S\in\mathscr{S}} P(S)(\theta\mathbf{x} + \mathbf{G}\boldsymbol{\delta})'(\mathbf{a}_S - \mathbf{1})(\mathbf{a}_S - \mathbf{1})'(\theta\mathbf{x} + \mathbf{G}\boldsymbol{\delta}).
\end{aligned}
$$

Clearly (2.1) is a quadratic function of θ if $(\mathbf{a}_S - \mathbf{1})'\mathbf{x} \neq 0$ for some S with $P(S) > 0$. If Θ is unbounded, then the maximum MSE will be infinite unless we have $(\mathbf{a}_S - \mathbf{1})'\mathbf{x} \neq 0$ for all S with $P(S) > 0$, i.e.,

$$\textstyle\sum_{i=1}^N a_{i(S)}x_i = X \quad \text{for all} \quad P(S) > 0. \qquad (2.2)$$

Therefore for unbounded Θ, e.g., $\Theta = R$ or $(0, \infty)$, we may restrict to estimators satisfying (2.2). We shall also make the same restriction when Θ is bounded. Condition (2.2) in fact is equivalent to the unbiasedness of an estimator in the usual superpopulation model (see, e.g., Cochran, 1977, page 159). Such a restriction is reasonable and is indispensable when Θ is unbounded. Clearly (2.2) holds for the ratio estimator and the estimator used by Scott and Smith (1975). A strategy satisfying (2.2) was called a *representative strategy* by Hájek (1959). We shall denote the collection of all the representative strategies in $\mathscr{D}_n$ by $\bar{\mathscr{D}}_n$.

For a strategy in $\bar{\mathscr{D}}_n$, we obviously have

$$
\begin{aligned}
R_n(P, \{\mathbf{a}_S\}_{S\in\mathscr{S}}; \theta, \boldsymbol{\delta}) &= \textstyle\sum_{S\in\mathscr{S}} P(S)\boldsymbol{\delta}'\mathbf{G}'(\mathbf{a}_S - \mathbf{1})(\mathbf{a}_S - \mathbf{1})'\mathbf{G}\boldsymbol{\delta} \qquad (2.3)\\
&= \boldsymbol{\delta}'\mathbf{G}\{\textstyle\sum_{S\in\mathscr{S}} P(S)(\mathbf{a}_S - \mathbf{1})(\mathbf{a}_S - \mathbf{1})'\}\mathbf{G}\boldsymbol{\delta}
\end{aligned}
$$

which is independent of θ and is a quadratic form in the matrix $\mathbf{G}\{\sum_{S\in\mathscr{S}} P(S)(\mathbf{a}_S - \mathbf{1})(\mathbf{a}_S - \mathbf{1})'\}\mathbf{G}$. This matrix plays an important role in assessing the performance of the strategy $d = (P, \{\mathbf{a}_S\}_{S\in\mathscr{S}})$. We shall call $\sum_{S\in\mathscr{S}} P(S)(\mathbf{a}_S - \mathbf{1})(\mathbf{a}_S - \mathbf{1})'$ the *risk-generating matrix*, and $\mathbf{G}\{\sum_{S\in\mathscr{S}} P(S)(\mathbf{a}_S - \mathbf{1})(\mathbf{a}_S - \mathbf{1})'\}\mathbf{G}$ the *adjusted risk-generating matrix* of the strategy $(P, \{\mathbf{a}_S\}_{S\in\mathscr{S}})$; and *we shall denote the adjusted risk-generating matrix of a strategy d by* $\mathbf{R}_n(d)$, *or simply by* $\mathbf{R}(d)$, *when n is clear.*

We have the following basic property of $\mathbf{R}(d)$.

PROPOSITION 2.1. *If d is representative, then* $\mathbf{R}(d)$ *is singular and satisfies the condition* $\mathbf{R}(d)\mathbf{G}^{-1}\mathbf{x} = \mathbf{0}$.

This is a straightforward consequence of (2.2); the proof is thus omitted. The identifiability condition (1.7) can be written in vector form $\boldsymbol{\delta}'\mathbf{G}^{-1}\mathbf{x} = 0$. Now in view of (2.3) and Proposition 2.1, we have the following.

PROPOSITION 2.2. *If* (i) $L = L_2(M)$ *and* (ii) Θ *is unbounded or* Θ *is bounded and the estimators satisfy* (2.2), *then* (1.4) *and* (1.8) *are identical problems. Moreover, a minimax strategy minimizes the maximum eigenvalue of* $\mathbf{R}(d)$.

This brings out an interesting connection to the theory of optimum designs. A minimax strategy in Proposition 2.2 is like an E-optimum design; see Kiefer (1974) for some terminology of optimum design theory. In fact, when $x_1 = x_2 = \cdots = x_N$ (i.e., there is no auxiliary information), one has $\mathbf{R}(d)\,\mathbf{1} = \mathbf{0}$ under (2.2), i.e., $\mathbf{R}(d)$ has zero row sums. In this case, if d^* is the strategy (simple random sampling, $N\cdot$sample mean), then $\mathbf{R}(d^*)$ minimizes tr $\mathbf{R}(d)$ and is completely symmetric in the sense that all the diagonal elements are the same and all the off-diagonals are the same, i.e., $\mathbf{R}(d^*)$ behaves like a multiple of identity matrix in the nondegenerated directions. An argument similar to Proposition 1 of Kiefer 1975) then shows the minimaxity of d^* when the space of all $\mathbf{y} = (y_1, \cdots, y_N)'$ is permutation-invariant. The so-called C-matrix of a balanced incomplete block design is also completely symmetric. Thus d^* plays the same role as a balanced incomplete block design in block design setting.

For a strategy with randomized estimator, a representativeness condition similar to (2.2) is also necessary for the maximum MSE to be finite when Θ is unbounded. The mean squared error of a strategy $d \in \mathscr{D}_n^R$ satisfying the representativeness condition is also a quadratic form $\boldsymbol{\delta}'\mathbf{R}(d)\boldsymbol{\delta}$ with the adjusted risk-generating matrix $\mathbf{R}(d)$ satisfying $\mathbf{R}(d)\mathbf{G}^{-1}\mathbf{x} = \mathbf{0}$, too. Later on, we shall show that when $L = L_2(M)$ and $g(x) = x^{1/2}$, all the nonzero eigenvalues of the adjusted risk-generating matrix of the Rao-Hartley-Cochran strategy are the same. This may explain why the Rao-Hartley-Cochran strategy performs very well under the minimax criterion; it is like a balanced design in experimental design settings.

For convenience, hereafter the maximum eigenvalue of a matrix $\mathbf{A}$ will be denoted by $\lambda_{\max}(\mathbf{A})$.

3. Approximate minimaxity of Rao-Hartley-Cochran strategy. For a sample size n, the Rao-Hartley-Cochran strategy first forms n random groups of units, one unit to be drawn from each group. The number of units $N_1, N_2, \cdots, N_n$ in the respective groups are made as equal as possible, i.e., $|N_i - N_j| \le 1$ for all $i, j = 1, 2, \cdots, n$. Let $X_j = \sum_{i \in \text{group}_j} x_i$. Then the probability of selecting the ith unit in the jth group is x_i/X_j, and the estimate of the population total is

$$\hat{Y}_{\text{RHC}} = \textstyle\sum_{j=1}^n X_j y_j / x_j,$$

where y_j, x_j refer to the unit drawn from group j. This strategy will be denoted by d_{RHC}. Clearly d_{RHC} is not in $\mathscr{D}_n$ since the estimator $\hat{Y}_{\text{RHC}}$ is a randomized estimator. Therefore d_{RHC} can be improved by some strategy in $\mathscr{D}_n$. However, since d_{RHC} is well-known and is easy to implement, we shall ignore its improved version in $\mathscr{D}_n$ and state our main results in terms of d_{RHC}.

We shall calculate the adjusted risk-generating matrix of d_{RHC}, $\mathbf{R}(d_{\text{RHC}})$, through the Hansen-Hurwitz strategy (denoted by d_{HH}) since d_{RHC} is closely related to d_{HH} and that the mean squared error of d_{HH} is easy to calculate. Recall that strategy d_{HH} is a with replacement scheme in which the probability of selecting the ith unit at each stage is x_i/X and the estimator of the population total is $\hat{Y}_{\text{HH}} = n^{-1}X \sum_{i=1}^n y_i/x_i$, where $(x_1, y_1), \cdots, (x_n, y_n)$ are the observed values of (x, y) with possible repetitions. When $n > 1$, this strategy is in neither $\mathscr{D}_n$ nor $\mathscr{D}_n^R$ because of the "with replacement" feature.

For $n = 1$, the mean squared error of d_{HH} is $R_1(d_{\text{HH}}; \theta, \boldsymbol{\delta}) = \boldsymbol{\delta}'\mathbf{G}\{\sum_{i=1}^N P(i)(\mathbf{a}_i - \mathbf{1})\cdot(\mathbf{a}_i - \mathbf{1})'\}\mathbf{G}\boldsymbol{\delta}$ where $\mathbf{a}_i$ is the $N \times 1$ vector with X/x_i as the ith coordinate and all the other coordinates are zero. Through some simple calculation, we obtain

$$R_1(d_{\text{HH}}; \theta, \boldsymbol{\delta}) = \boldsymbol{\delta}'\mathbf{G}\{\text{diag}(x_1^{-1}X, \cdots, x_N^{-1}X) - \mathbf{J}_N\}\mathbf{G}\boldsymbol{\delta},$$

where $\mathbf{J}_N$ is the $N \times N$ matrix of ones. For $n > 1$, $\hat{Y}_{\text{HH}}$ is an unbiased estimator of Y and is essentially the average of a random sample. Therefore

$$R_n(d_{\text{HH}}; \theta, \boldsymbol{\delta}) = n^{-1}R_1(d_{\text{HH}}; \theta, \boldsymbol{\delta}) = n^{-1}\boldsymbol{\delta}'\mathbf{G}\{\text{diag}(x_1^{-1}X, \cdots, x_N^{-1}X) - \mathbf{J}_N\}\mathbf{G}\boldsymbol{\delta}. \tag{3.1}$$

Now, write $N = nR + k$, where R and k are integers with $0 \le k < n$, and let

$$\mu = (N-1)^{-1}(N-n) + N^{-1}(N-1)^{-1}k(n-k).$$

By (9A.66) and (9A.67) of Cochran (1977), we have

$$R_n(d_{\text{RHC}}; \theta, \boldsymbol{\delta}) = \mu \cdot R_n(d_{\text{HH}}; \theta, \boldsymbol{\delta}),$$

which leads to

$$\mathbf{R}_n(d_{\text{RHC}}) = \mu \cdot n^{-1}\mathbf{G}\{\text{diag}(x_1^{-1}X, \cdots, x_N^{-1}X) - \mathbf{J}_N\}\mathbf{G}. \tag{3.2}$$

Furthermore, when $g(x) = x^{1/2}$, we obtain

$$\mathbf{R}_n(d_{\mathrm{RHC}}) = \mu n^{-1}(X\mathbf{I} - \mathbf{G}\mathbf{J}_N\mathbf{G}),$$

which has all the nonzero eigenvalues equal to $\mu n^{-1}X$. Thus we obtain the following proposition.

PROPOSITION 3.1. *Let $g(x) = x^{1/2}$. Then for any $M > 0$, we have*

$$\sup_{\theta\in\Theta,\boldsymbol{\delta}\in L_2(M)} R_n(d_{\mathrm{HH}}; \theta, \boldsymbol{\delta}) = n^{-1}XM, \quad \text{and} \quad \sup_{\theta\in\Theta,\boldsymbol{\delta}\in L_2(M)} R_n(d_{\mathrm{RHC}}; \theta, \boldsymbol{\delta}) = \mu n^{-1}XM.$$

To establish the approximate minimaxity of d_{RHC}, we need to give a lower bound for the maximum MSE of an arbitrary strategy. The following proposition provides such a useful lower bound.

PROPOSITION 3.2. *Let $g(x) = x^{1/2}$ and Θ be unbounded. Then for any $M > 0$,*

$$\min_{d\in\mathscr{D}_n}\sup_{\theta\in\Theta,\boldsymbol{\delta}\in L_2(M)} R_n(d; \theta, \boldsymbol{\delta}) \geq n^{-1}XM\{(\textstyle\sum_{i=1}^{N-n} x_i)^2 + n\sum_{i=1}^{N-n} x_i^2\}/(X^2 - \sum_{i=1}^{N} x_i^2).$$

If Θ is bounded, then the above inequality holds for $d \in \bar{\mathscr{D}}_n$.

PROOF. For any $t > 0$, let

$$\mathbf{Z} = \mathbf{R}_n(d) + t(\sqrt{x_1}, \cdots, \sqrt{x_N})'(\sqrt{x_1}, \cdots, \sqrt{x_N}). \tag{3.3}$$

By Proposition 2.1, we have $\mathbf{R}_n(d)(\sqrt{x_1}, \cdots, \sqrt{x_N})' = 0$; i.e., each row of $\mathbf{R}_n(d)$ is orthogonal to $(\sqrt{x_1}, \cdots, \sqrt{x_N})'$. Therefore if $\lambda_1, \lambda_2, \cdots, \lambda_{N-1}$ and 0 are the eigenvalues of $\mathbf{R}_n(d)$, then the eigenvalues of $\mathbf{Z}$ are $\lambda_1, \lambda_2, \cdots, \lambda_{N-1}$, and $t\sum_{i=1}^{N} x_i = tX$. We shall first give a lower bound for the maximum eigenvalue of $\mathbf{Z}$ and then by suitably choosing t we may derive a good lower bound for the maximum eigenvalue of $\mathbf{R}_n(d)$.

Now, the maximum eigenvalue of $\mathbf{Z}$, denoted by $\lambda_{\max}(\mathbf{Z})$, satisfies

$$\begin{aligned} \lambda_{\max}(\mathbf{Z}) \geq &\ \text{the maximum diagonal element of } \mathbf{Z} \\ &= \max_{1\leq i\leq N}\{\textstyle\sum_{S\in\mathscr{S}} x_i P(S)(a_{i(S)} - 1)^2 + tx_i\} \\ &= \max_{1\leq i\leq N}[\{\textstyle\sum_{S\in\mathscr{S}} x_i^2 P(S)(a_{i(S)} - 1)^2 + tx_i^2\}/x_i] \\ &\geq [\textstyle\sum_{i=1}^{N} \{\sum_{S\in\mathscr{S}} x_i^2 P(S)(a_{i(S)} - 1)^2 + tx_i^2\}]/\sum_{i=1}^{N} x_i \\ &= [\{\textstyle\sum_{S\in\mathscr{S}} P(S)(\sum_{i\in S} (a_{i(S)}x_i - x_i)^2 + \sum_{i\notin S} x_i^2)\} + t\sum_{i=1}^{N} x_i^2]/X. \end{aligned} \tag{3.4}$$

Next, as was demonstrated in Section 2, (2.2) must hold if we want the maximum MSE finite. An important consequence of (2.2) is the following inequality:

$$\begin{aligned} \textstyle\sum_{i\in S} (a_{i(S)}x_i - x_i)^2 &\geq n(n^{-1}\textstyle\sum_{i\in S} a_{i(S)}x_i - n^{-1}\sum_{i\in S} x_i)^2 \\ &= n(n^{-1}X - n^{-1}\textstyle\sum_{i\in S} x_i)^2 = n^{-1}(\sum_{i\notin S} x_i)^2. \end{aligned} \tag{3.5}$$

Now, due to the assumption that $x_1 \leq x_2 \leq \cdots \leq x_N$, we have $\sum_{i\notin S} x_i \geq \sum_{i=1}^{N-n} x_i$ and $\sum_{i\notin S} x_i^2 \geq \sum_{i=1}^{N-n} x_i^2$. Therefore, back to (3.4), we obtain

$$\lambda_{\max}(\mathbf{Z}) \geq [\{n^{-1}(\textstyle\sum_{i=1}^{N-n} x_i)^2 + \sum_{i=1}^{N-n} x_i^2\} + t\sum_{i=1}^{N} x_i^2]/X. \tag{3.6}$$

Consider any t such that $t < \{n^{-1}(\sum_{i=1}^{N-n} x_i)^2 + \sum_{i=1}^{N-n} x_i^2\}/(X^2 - \sum_{i=1}^{N} x_i^2)$. A simple computation leads to

$$tX < [\{n^{-1}(\textstyle\sum_{i=1}^{N-n} x_i)^2 + \sum_{i=1}^{N-n} x_i^2\} + t\sum_{i=1}^{N} x_i^2]/X.$$

Therefore, tX can not be the maximum eigenvalue of $\mathbf{Z}$. Thus from the discussions following (3.3) we obtain $\lambda_{\max}(\mathbf{R}_n(d)) = \lambda_{\max}(\mathbf{Z})$. Finally, in (3.6), by letting t tend to $\{n^{-1}(\sum_{i=1}^{N-n} x_i)^2 + \sum_{i=1}^{N-n} x_i^2\}/(X^2 - \sum_{i=1}^{N} x_i^2)$, we get the desired bound. □

By Propositions 3.1 and 3.2, we have the following.

THEOREM 3.1. *Let $g(x) = x^{1/2}$ and Θ be unbounded. If*

$$\mu(X^2 - \textstyle\sum_{i=1}^{N} x_i^2)/\{(\sum_{i=1}^{N-n} x_i)^2 + n\sum_{i=1}^{N-n} x_i^2\} \leq 1 + \varepsilon,$$

then the Rao-Hartley-Cochran strategy is $(1 + \varepsilon)$-approximately minimax in the sense that for any sampling strategy $d \in \mathscr{D}_n$,

$$\sup_{\theta\in\Theta,\delta\in L_2(M)} R_n(d_{\mathrm{RHC}}; \theta, \boldsymbol{\delta}) \leq (1 + \varepsilon) \sup_{\theta\in\Theta,\delta\in L_2(M)} R_n(d; \theta, \boldsymbol{\delta}).$$

If Θ is bounded, then the above inequality still holds if $d \in \bar{\mathscr{D}}_n$. If $n \mid N$ and $x_1 = x_2 = \cdots = x_N$, then $\mu(X^2 - \sum_{i=1}^{N} x_i^2)/\{(\sum_{i=1}^{N-n} x_i)^2 + n\sum_{i=1}^{N-n} x_i^2\} = 1$ and d_{RHC} is the same as (SRS, $N\cdot$sample mean); the minimaxity of the latter follows. In general, a rough calculation shows that $\mu(X^2 - \sum_{i=1}^{N} x_i^2)/\{(\sum_{i=1}^{N-n} x_i)^2 + n\sum_{i=1}^{N-n} x_i^2\} \leq \mu(1 + \sum_{i=N-n+1}^{N} x_i/\sum_{i=1}^{N-n} x_i)^2$. Thus d_{RHC} is approximately minimax if the n largest x values are not too extreme.

Note that because of Proposition 2.2, the Rao-Hartley-Cochran strategy is also $(1 + \varepsilon)$-approximately minimax when the identifiability condition (1.7) is imposed. Furthermore, the Hansen-Hurwitz strategy will also be approximately minimax if μ is close to 1, i.e., if the sampling fraction is small.

Now let us turn to the case $g(x) = x$ and $L = L_\infty(M)$. We need the following propositions.

PROPOSITION 3.3. *If* $\mathbf{M}$ *is a* $N \times N$ *matrix such that* $\mathbf{1}'\mathbf{M}\mathbf{1} = 0$, *then there is an* $N \times 1$ *vector* $\boldsymbol{\delta}$ *with* $+1, -1$ *entries such that* $\boldsymbol{\delta}'\mathbf{M}\boldsymbol{\delta} \geq N(N-1)^{-1}\cdot$ trace $\mathbf{M}$ *when* N *is even, and* $\boldsymbol{\delta}'\mathbf{M}\boldsymbol{\delta} \geq (N+1)N^{-1}$ trace $\mathbf{M}$ *when* N *is odd.*

PROOF. Let $\mathbb{P}$ be the set of all the $N \times N$ permutation matrices $\mathbf{P}$. Clearly, the matrix $(N!)^{-1}\sum_{\mathbf{P}\in\mathbb{P}} \mathbf{P}'\mathbf{M}\mathbf{P}$ is of the form $a\mathbf{I}_N + b\mathbf{J}_N$ for some real numbers a and b. Moreover, a and b can be determined by the fact that trace $(a\mathbf{I} + b\mathbf{J}_N)$ = trace $\mathbf{M}$ and that $\mathbf{1}'(a\mathbf{I} + b\mathbf{J}_N)\mathbf{1} = (N!)^{-1}\sum_{\mathbf{P}\in\mathbb{P}} \mathbf{1}'\mathbf{P}'\mathbf{M}\mathbf{P}\mathbf{1} = (N!)^{-1}\sum_{\mathbf{P}\in\mathbb{P}} \mathbf{1}'\mathbf{M}\mathbf{1} = 0$. After some computation, we get

$$a = (N-1)^{-1}\,\mathrm{trace}\,\mathbf{M}, \quad \text{and} \quad b = -N^{-1}(N-1)^{-1}\,\mathrm{trace}\,\mathbf{M}.$$

Consider first the case where N is even. Let $\boldsymbol{\delta}^0$ be the vector with the first $N/2$ coordinates equal to 1 and the last $N/2$ coordinates equal to -1. Then we have

$$\begin{aligned}\mathrm{Max}\{\boldsymbol{\delta}'\mathbf{M}\boldsymbol{\delta} \mid \boldsymbol{\delta} = \mathbf{P}\boldsymbol{\delta}^0 \text{ for some } \mathbf{P} \text{ in } \mathbb{P}\} &\geq (N!) \textstyle\sum_{\mathbf{P}\in\mathbb{P}} \boldsymbol{\delta}^{0\prime}\mathbf{P}'\mathbf{M}\mathbf{P}\boldsymbol{\delta}^0 \\ &= \boldsymbol{\delta}^{0\prime}(a\mathbf{I} + b\mathbf{J})\boldsymbol{\delta}^0 = Na = N(N-1)^{-1}\,\mathrm{trace}\,\mathbf{M}.\end{aligned}$$

This proves the case where N is even. Next, consider the case where N is odd. Let $\boldsymbol{\delta}^0$ be the vector with the first $(N+1)/2$ coordinates equal to 1 and the last $(N-1)/2$ coordinates equal to -1. A similar computation leads to $\mathrm{Max}\{\boldsymbol{\delta}'\mathbf{M}\boldsymbol{\delta} \mid \boldsymbol{\delta} = \mathbf{P}\boldsymbol{\delta}^0$ for some $\mathbf{P}$ in $\mathbb{P}\} \geq Na + b = (N+1)N^{-1}$ trace $\mathbf{M}$. The proof is now complete. □

PROPOSITION 3.4. *Let $g(x) = x$ and Θ be unbounded. Then we have*

$$\min_{d\in\mathscr{D}_n}\sup_{\theta\in\Theta,\delta\in L_\infty(M)} R_n(d; \theta, \boldsymbol{\delta}) \geq \{n^{-1}(\textstyle\sum_{i=1}^{N-n} x_i)^2 + \sum_{i=1}^{N-n} x_i^2\}N(N-1)^{-1}M^2,$$

if N is even, and

$$\min_{d\in\mathscr{D}_n}\sup_{\theta\in\Theta,\delta\in L_\infty(M)} R_n(d; \theta, \boldsymbol{\delta}) \geq \{n^{-1}(\textstyle\sum_{i=1}^{N-n} x_i)^2 + \sum_{i=1}^{N-n} x_i^2\}(N+1)N^{-1}M^2,$$

if N is odd. If Θ is bounded, then the above inequalities hold if $d \in \bar{\mathscr{D}}_n$.

PROOF. Without loss of generality, we may assume $M = 1$. By Proposition 2.1, we have $\mathbf{R}_n(d)\mathbf{1} = \mathbf{0}$; i.e., all row sums of $\mathbf{R}_n(d)$ are zero. Therefore by Proposition 3.3, there exists a $\boldsymbol{\delta}^* \in L_\infty(1)$ such that $\boldsymbol{\delta}^{*\prime}\mathbf{R}_n(d)\boldsymbol{\delta}^* \geq N(N-1)^{-1}\operatorname{tr}\mathbf{R}_n(d)$ if N is even and $\boldsymbol{\delta}^{*\prime}\mathbf{R}_n(d)\boldsymbol{\delta}^* \geq (N+1)N^{-1}\operatorname{tr}\mathbf{R}_n(d)$ if N is odd. It remains to be established a lower bound for $\operatorname{tr}\mathbf{R}_n(d)$.

By a straightforward computation, we have

$$\operatorname{tr}\mathbf{R}_n(d) = \sum_{i=1}^N \sum_{S\in\mathcal{S}} P(S)(a_{i(S)} - 1)^2 x_i^2 = \sum_{S\in\mathcal{S}} P(S) \sum_{i=1}^N (a_{i(S)} - 1)^2 x_i^2$$

$$= \sum_{S\in\mathcal{S}} P(S)\{\sum_{i\in S} (a_{i(S)}x_i - x_i)^2 + \sum_{i\notin S} x_i^2\} \qquad \text{(by (3.5))}$$

$$\geq \sum_{S\in\mathcal{S}} P(S)\cdot\{n^{-1}(\sum_{i\notin S} x_i)^2 + \sum_{i\notin S} x_i^2\} \geq n^{-1}(\sum_{i=1}^{N-n} x_i)^2 + \sum_{i=1}^{N-n} x_i^2.$$

This completes the proof.

PROPOSITION 3.5. *Assume $g(x) = x$. Then for any $M > 0$, we have*

$$\sup_{\theta\in\Theta,\boldsymbol{\delta}\in L_\infty(M)} R_n(d_{\mathrm{HH}};\, \theta, \boldsymbol{\delta}) \leq n^{-1}X^2M^2$$

and

$$\sup_{\theta\in\Theta,\boldsymbol{\delta}\in L_\infty(M)} R_n(d_{\mathrm{RHC}};\, \theta, \boldsymbol{\delta}) \leq \mu n^{-1}X^2M^2.$$

PROOF. Without loss of generality, we assume $M = 1$. From (3.1), we obtain $\mathbf{R}_n(d_{\mathrm{HH}}) = n^{-1}\{\operatorname{diag}(x_1X, \cdots, x_NX) - \mathbf{G}\mathbf{J}_N\mathbf{G}\}$. Since $\mathbf{G}\mathbf{J}_N\mathbf{G}$ is nonnegative definite, we conclude that for any $\boldsymbol{\delta} \in L_{\infty(1)}$, $\boldsymbol{\delta}'\mathbf{R}_n(d_{\mathrm{HH}})\boldsymbol{\delta} \leq n^{-1}\sum_{i=1}^N x_iX = n^{-1}X^2$ as desired. Similarly, we obtain the bound for d_{RHC}. □

THEOREM 3.2. *Let $g(x) = x$ and Θ be unbounded. If $\mu N^{-1}(N-1)X^2/\{(\sum_{i=1}^{N-n} x_i)^2 + n\sum_{i=1}^{N-n} x_i^2\} \leq 1 + \varepsilon$ when N is even, and $\mu(N+1)^{-1}NX^2/\{(\sum_{i=1}^{N-n} x_i)^2 + n\sum_{i=1}^{N-n} x_i^2\} \leq 1 + \varepsilon$ when N is odd, then the Rao-Hartley-Cochran strategy is $(1+\varepsilon)$-approximately minimax in the sense that for any sampling strategy $d \in \mathcal{D}_n$, we have*

$$\sup_{\theta\in\Theta,\boldsymbol{\delta}\in L_\infty(M)} R_n(d_{\mathrm{RHC}};\, \theta, \boldsymbol{\delta}) \leq (1+\varepsilon)\sup_{\theta\in\Theta,\boldsymbol{\delta}\in L_\infty(M)} R_n(d;\, \theta, \boldsymbol{\delta}).$$

If Θ is bounded, then the above inequality holds if $d \in \bar{\mathcal{D}}_n$.

Again the Rao-Hartley-Cochran and Hansen-Hurwitz strategies are approximately minimax if the n largest x values are not too extreme and the sampling fraction is small.

Now take $\Theta = \{B/2\}$ and $M = B/2$. Then our result applies to the problem of Scott and Smith (1975), for which the Rao-Hartley-Cochran strategy is approximately minimax over all the strategies in $\bar{\mathcal{D}}_n$ (note that $\Theta = \{B/2\}$ is bounded) if $\mu X^2/\{(\sum_{i=1}^{N-n} x_i)^2 + n\sum_{i=1}^{N-n} x_i^2\}$ is close to 1. Note that we do not put any restriction on the sampling designs and the only restriction on the estimators is the representativeness condition (2.2) which is reasonable, while Scott and Smith (1975) fixed the estimator and restricted to sampling schemes with replacement. Furthermore, our result applies to more general models.

So far the approximate minimaxity results were only established for $g(x) = x^{1/2}$, $L = L_2(M)$ and $g(x) = x$, $L = L_\infty(M)$. Results for other forms of $g(x)$ are rather difficult to derive. We do not have satisfactory results in this direction. Therefore in the next Section, we shall compare the Rao-Hartley-Cochran strategy with two commonly-used strategies (SRS, ratio estimator) and (PPAS, ratio estimator) under a variety of functions g. It turns out that the Rao-Hartley-Cochran strategy performs favorably when $g(x) = x^\alpha$ with $\alpha \geq 1/2$.

4. Comparisons. Comparisons of various sampling strategies had been done in the literature, see, e.g., Chapter 7 of Cassel, Särndal, and Wretman (1977) and the references cited there. They were mostly empirical studies or based on some superpopulations. The criteria used were often expected (with respect to superpopulation) mean squared error and hence were "average" type criteria. The comparison we shall make here is different and is based on a minimax criterion. Borrowing optimum design theory terminology, one can say that our criterion is like an E-criterion and the earlier comparisons were more or less based on something like the A-criterion.

We have seen that the Rao-Hartley-Cochran strategy is approximately minimax under model (1.1)–(1.2) with $L = L_2(M)$ and $g(x) = x^{1/2}$. A comparable superpopulation model is (1.3) with $\nu(x) = x$. Under this model Brewer (1963) and Royall (1970a) showed that the best strategy is to select a purposive sample S^* which consists of the n units with the largest x values and then use the ratio estimator. Under our assumption that $x_1 \le \cdots \le x_N$, we have $S^* = \{x_{N-n+1}, \cdots, x_N\}$. Let this strategy be denoted by d_{BR}. Now let us first compare d_{BR} with d_{RHC} and d_{HH}, when $g(x) = x^{1/2}$.

Since the rank of $\mathbf{R}_n(d_{\mathrm{BR}})$ is 1 ($\mathbf{R}_n(d_{\mathrm{BR}}) = \mathbf{G}(\mathbf{a}_{S^*} - \mathbf{1})(\mathbf{a}_{S^*} - \mathbf{1})'\mathbf{G}$),

$$\lambda_{\max}(\mathbf{R}_n(d_{\mathrm{BR}})) = \mathrm{tr}(\mathbf{R}_n(d_{\mathrm{BR}})) = (X \textstyle\sum_{i=1}^{N-n} x_i)/(\sum_{i=N-n+1}^{N} x_i).$$

Comparing this value with the result of Proposition 3.1, we get the following proposition.

PROPOSITION 4.1. *Assume* $g(x) = x^{1/2}$. If $\sum_{i=1}^{N-n} x_i \ge n^{-1} \sum_{i=N-n+1}^{N} x_i$, *then for any* $M > 0$,

$$\sup_{\theta\in\Theta,\delta\in L_2(M)} R_n(d_1; \theta, \delta) \ge \sup_{\theta\in\Theta,\delta\in L_2(M)} R_n(d_{\mathrm{HH}}; \theta, \delta) \ge \sup_{\theta\in\Theta,\delta\in L_2(M)} R_n(d_{\mathrm{RHC}}; \theta, \delta).$$

Therefore, if the n largest x values are not too extreme, then d_{BR} is worse than d_{HH} and d_{RHC}.

Two sampling designs which are commonly used together with the ratio estimator are SRS and PPAS. Let (SRS, ratio estimator) and (PPAS, ratio estimator) be denoted by d_{SR} and d_{PR}, respectively. Recall that in a PPAS sampling scheme, each sample S is selected with probability proportional to $X_S \equiv \sum_{i\in S} x_i$. Such a sampling design was proposed to make the ratio estimator design-unbiased, see, e.g., Cochran (1977, page 175). We shall show that both d_{BR} and d_{PR} are inferior to d_{RHC} under model (1.1)–(1.2) with $g(x) = x^{1/2}$ and $L = L_2(M)$. Later we shall extend the result to an arbitrary function g such that $g^2(x)/x$ is increasing in x. Some conditions on the configuration of $x_1, \cdots, x_N$ are needed there.

Now assume $g(x) = x^{1/2}$ and write $X_S = \sum_{i\in S} x_i$. Then we have

$$\begin{aligned}\mathrm{tr}\,\mathbf{R}_n(d_{\mathrm{SR}}) &= \textstyle\sum_{i=1}^{N} [\{\sum_{S:i\in S} P(S) x_i (X^2/X_S^2 - 2X/X_S)\} + x_i] \\ &= \binom{N}{n}^{-1} \textstyle\sum_{S\in\mathcal{S}} \sum_{i\in S} x_i (X^2/X_S^2 - 2X/X_S) + X \\ &= \binom{N}{n}^{-1} \textstyle\sum_{S\in\mathcal{S}} X^2/X_S - X \ge \binom{N}{n} X^2/(\sum_{S\in\mathcal{S}} X_S) - X = n^{-1}(N-n)X.\end{aligned}$$

Since $\mathbf{R}_n(d_{\mathrm{SR}})$ is singular, we conclude that $\lambda_{\max}(\mathbf{R}_n(d_{\mathrm{SR}})) \ge (N-1)^{-1}\,\mathrm{tr}\,\mathbf{R}_n(d_{\mathrm{SR}}) \ge n^{-1}(N-1)^{-1}(N-n)X$, and the equality holds only if $x_1 = x_2 = \cdots = x_N$. Comparing with the result of Proposition 3.1, we establish the following proposition.

PROPOSITION 4.2. *Assume* $g(x) = x^{1/2}$ *and* N/n *is integral. Then we have*

$$\sup_{\theta\in\Theta,\delta\in L_2(M)} R_n(d_{\mathrm{SR}}; \theta, \delta) \ge \sup_{\theta\in\Theta,\delta\in L_2(M)} R_n(d_{\mathrm{RHC}}; \theta, \delta),$$

and the equality holds only if $x_1 = x_2 = \cdots = x_N$.

After some computation we can show that $\mathrm{tr}\,\mathbf{R}_n(d_{\mathrm{PR}}) = \mathrm{tr}\,\mathbf{R}_n(d_{\mathrm{RHC}})$. Therefore a similar argument leads to the following proposition.

PROPOSITION 4.3 *Assume* $g(x) = x^{1/2}$ *and* N/n *is integral. Then we have*

$$\sup_{\theta\in\Theta,\delta\in L_2(M)} R_n(d_{\mathrm{PR}}; \theta, \delta) \ge \sup_{\theta\in\Theta,\delta\in L_2(M)} R_n(d_{\mathrm{RHC}}; \theta, \delta),$$

and the equality holds only if $x_1 = \cdots = x_N$.

Now consider a general function g such that $g^2(x)/x$ is increasing in x. Substituting $\mathbf{G} = \mathrm{diag}(g(x_1), \cdots, g(x_N))$ into (3.2), we have

$$\mathbf{R}_n(d_{\mathrm{RHC}}) = \mu n^{-1}\{\mathrm{diag}(X g^2(x_1)/x_1, \cdots, Xg^2(x_N)/x_N) - \mathbf{G}\,\mathbf{J}_N\mathbf{G})\}.$$

Since $\mathbf{G}\,\mathbf{J}_N\mathbf{G}$ is non-negative definite and $g^2(x)/x$ is an increasing function of x, it follows that

$$\lambda_{\max}(\mathbf{R}_n(d_{\mathrm{RHC}})) \le \mu n^{-1}X\, g^2(x_N)/x_N.$$

On the other hand, we have

$$\begin{aligned}\lambda_{\max}(\mathbf{R}_n(d_{\mathrm{SR}})) &\ge \text{the } N\text{th diagonal element of } \mathbf{R}_n(d_{\mathrm{SR}}) \\ &\ge g^2(x_N)\binom{N}{n}^{-1}\cdot\textstyle\sum_{S:N\in S}(X_S^{-1}X - 1)^2 \\ &\ge g^2(x_N)\binom{N}{n}^{-1}\binom{N-1}{n-1}\left\{\binom{N-1}{n-1}X/(\textstyle\sum_{S:N\in S}X_S) - 1\right\}^2 \\ &= g^2(x_N)\frac{n}{N}[(N-1)X/\{(N-n)x_N + (n-1)X\} - 1]^2.\end{aligned}$$

Therefore, to show that $\lambda_{\max}(\mathbf{R}_n(d_{\mathrm{RHC}})) \le \lambda_{\max}(\mathbf{R}_n(d_{\mathrm{SR}}))$, it suffices to demonstrate that

(4.1) $$n^2N^{-1}x_NX^{-1}[(N-1)X/\{(N-n)x_N + (n-1)X\} - 1]^2 \ge 1.$$

Let $a = Nx_N/X$ and $f = \dfrac{n}{N}$. Then we may rewrite (4.1) as

(4.2) $$N^{-1}(1-f)^2fa^3 - (1-f)^2(2nf+1)a^2 + (1-f)\{n^2(1-f) - 2(n-1)\}a - (n-1)^2 \ge 0.$$

Discarding the first term of (4.2) and changing $(n-1)$ and $(n-1)^2$ to n and n^2 respectively, we obtain the following sufficient condition for (4.2) to hold:

(4.3) $$-(1-f)^2(2nf+1)a^2 + (1-f)n\{n(1-f) - 2\}a - n^2 \ge 0.$$

Now, letting

(4.4) $$l = ½ + ½\,\{1 - 4(1+f)/n(1-f)^2\}^{1/2} - n^{-1}(1-f)^{-1},$$

and solving (4.3), we get

(4.5) $$(1-f)^{-2}l^{-1} \le a \le ln^2N/(2n^2+N).$$

This leads to the following result.

PROPOSITION 4.4 *Assume that* $g^2(x)/x$ *is non-decreasing in x. Suppose* (4.5) *holds, where a is the ratio of* x_N *and the average of the* x_i*'s, i.e.,* $a = x_N/(N^{-1}X)$ *and l is defined by* (4.4). *Then we have*

$$\sup_{\theta\in\Theta,\boldsymbol{\delta}\in L_2(M)} R_n(d_{\mathrm{SR}};\theta,\boldsymbol{\delta}) \ge \sup_{\theta\in\Theta,\boldsymbol{\delta}\in L_2(M)} R_n(d_{\mathrm{RHC}};\theta,\boldsymbol{\delta}).$$

Note that the possible values of a are between 1 and N; and l is very close to 1 if the sampling fraction f is small. Therefore, (4.5) almost amounts to saying that $X_N \le (N^{-1}X)\{n^2N/(2n^2+N)\}$, which is a reasonable condition if n or N is large enough, because it simply means that the largest x value is not too far away from the average x value. We further remark that because of the asymmetry of $\mathbf{R}_n(d_{\mathrm{SR}})$, we would expect that even if (4.5) does not hold, d_{RHC} may still be better than d_{SR}.

A similar argument also applies to the comparison between d_{RHC} and d_{PR}. So d_{RHC} performs favorably when $g(x) = x^{\alpha}$ with $\alpha \geq 1/2$. Some comparisons of d_{RHC} with strategies using the ratio estimator can be found in Chapter 7 of Cassel, Särndal, and Wretman. The comparison there was based on superpopulation model (1.3) with $\nu(x) = x^{\beta}$ (comparable to our model with $g(x) = x^{\beta/2}$) and the assumption that N is very large and that the frequency distribution of the auxiliary variable values $x_1, \cdots, x_N$ is approximately a gamma distribution. On page 171, the authors wrote that d_{RHC} was a good choice for $1 \leq \beta \leq 2$. This seems to be consistent with our finding that d_{RHC} is good for α (comparable to $\beta/2$) $\geq 1/2$.

5. Estimation of θ. Sometimes one may be more interested in estimating θ than the population total Y especially when x_i is the value of y_i at some previous time. In this section, we shall study the minimax estimation of θ under model (1.1)–(1.2) with identifiability condition (1.7). We shall focus the discussion on the case $L = L_2(M)$.

When $g(x) = x^{1/2}$, by (1.6), we have $\theta = Y/X$; the estimation of Y is then the same as that of θ. Therefore Theorem 3.1 is applicable. In Section 3, for the estimation of Y, we were only able to derive satisfactory results for $g(x) = x^{1/2}$. In this section, however, we shall show that for estimating θ, parallel minimax results can be established for an *arbitrary* g. Thus Theorem 3.1 could be viewed as a special case of the results in this section.

Let us again restrict ourselves to linear homogeneous estimators and use the same notation as before. By (1.6),

$$\theta = \sum_{i=1}^{N} \{g(x_i)\}^{-2} x_i y_i / \sum_{i=1}^{N} \{g(x_i)\}^{-2} x_i^2,$$

and hence θ can be viewed as a population total $\sum_{i=1}^{N} \theta_i$ with

$$\theta_i = \{g(x_i)\}^{-2} x_i y_i / \sum_{i=1}^{N} \{g(x_i)\}^{-2} x_i^2.$$

Let $z_i = \{g(x_i)\}^{-2} x_i^2$ and $Z = \sum_{i=1}^{N} z_i$. Then we have

$$\theta_i = Z^{-1}[\{g(x_i)\}^{-2} x_i (\theta x_i + \delta_i g(x_i))] = Z^{-1}(\theta z_i + \delta_i z_i^{1/2}).$$

Since Z is a known constant, the problem of estimating θ is now reduced to that of estimating a population total in model (1.1)–(1.2) with x and $g(x)$ replaced by z and $g(z) = z^{1/2}$, respectively. Therefore by Theorem 3.1 an approximately minimax strategy is to divide the N units into n random groups of sizes as equal as possible, choose one unit from each group with probability proportional to $z = x^2/\{g(x)\}^2$, and then estimate θ by $\sum_{j=1}^{n} \theta_j z_j^{-1} Z_j$ or, equivalently, $Z^{-1}(\sum_{j=1}^{n} x_j^{-1} y_j Z_j)$, where x_j, y_j refer to the unit drawn from group j, and $Z_j = \sum_{i \in \text{group}_j} z_i$.

We may also generalize the Hurwitz-Hansen strategy for the estimation of θ for arbitrary g as follows. The sampling scheme is to select n units with replacement such that at each stage, the probability of selecting the ith unit is proportional to $z_i = x_i^2/\{g(x_i)\}^2$, and the estimator is $n^{-1} \sum_{j=1}^{n} y_j/x_j$ where y_j, x_j refer to the unit selected at the jth stage. If the sampling fraction is small then this strategy will also be approximately minimax.

6. Some exact minimax results for $n = 1$. Scott and Smith (1975) derived some exact minimax results for $n = 1$. Similar results also hold for our problem.

When the sample size $n = 1$, there is only one estimator satisfying the representativeness condition (2.2), i.e., $\hat{Y} = x_i^{-1} y_i X$. Therefore the problem is purely the choice of designs. Let P^* be the design such that the ith unit is selected with probability proportional to x_i, then we have the following

THEOREM 6.1. *Let $n = 1$, $g(x) = x^{1/2}$, and $\hat{Y}$ be the estimator $x_i^{-1} y_i X$. If $\sum_{i \in S_0} x_i = \frac{1}{2} \sum_{i=1}^{N} x_i$ for some $S_0 \subset \{1, 2, \cdots, N\}$, then P^* minimizes $\sup_{\theta \in \Theta, \delta \in L_2(M)} R_1(P, \hat{Y}; \theta, \delta)$ over all sampling designs P for any non-empty Θ and $M > 0$.*

PROOF. By Proposition 2.2, without loss of generality, we may impose the identifiability condition (1.7) on θ, i.e., $\theta = Y/X$ and $\sum_{i=1}^{N} \delta_i x_i^{1/2} = 0$. Then the mean squared error of $(P^*, \hat{Y})$ is $X^{-1} \sum_{i=1}^{N} x_i(x_i^{-1}y_iX - Y)^2 = X \sum_{i=1}^{N} \delta_i^2$. Let $\tilde{L}_2(M) = \{\boldsymbol{\delta} \in L_2(M): \sum \delta_i x_i^{1/2} = 0\}$. Then $\sup_{\theta\in\Theta,\boldsymbol{\delta}\in\tilde{L}_2(M)} R_1(P^*, \hat{Y}; \theta, \boldsymbol{\delta}) \le MX$. On the other hand, let

$$\delta_i^* = \begin{cases} (Mx_i/X)^{1/2}, & \text{if } i \in S_0 \\ -(Mx_i/X)^{1/2}, & \text{if } i \notin S_0. \end{cases}$$

Then $\boldsymbol{\delta}^* = (\delta_1^*, \cdots, \delta_N^*)' \in \tilde{L}_2(M)$ and for any P,

$$R_1(P, \hat{Y}; \theta, \boldsymbol{\delta}^*) = MX.$$

Therefore $\sup_{\theta\in\Theta,\boldsymbol{\delta}\in\tilde{L}_2(M)} R_1(P, \hat{Y}; \theta, \boldsymbol{\delta}) \ge MX \ge \sup_{\theta\in\Theta,\boldsymbol{\delta}\in\tilde{L}_2(M)} R_1(P^*, \hat{Y}; \theta, \boldsymbol{\delta})$. □

One can also write down a similar result for the problem of estimating θ. The condition needed is $\sum_{i\in S_0} z_i = \frac{1}{2} \sum_{i=1}^{N} z_i$ for some S_0, where $z_i = x_i^2/\{g(x_i)\}^2$.

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