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CURTAILED AND UNIFORMLY MOST POWERFUL SEQUENTIAL TESTS[1]

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The final decision of a fixed-sample likelihood ratio test is often determined before the entire sample is taken. Such a test can be curtailed as soon as the final decision becomes obvious. The construction and properties of these curtailed tests are described. A particular class of such tests contains sequential tests which are uniformly most powerful. The asymptotic efficiency of this class is investigated.

1. Introduction. The Neyman-Pearson lemma shows that a fixed-sample likelihood ratio test has the property that no other test based on the same information can improve upon its error probabilities. The Wald-Wolfowitz theorem shows that, under certain conditions, a sequential probability ratio test may be better than a fixed-sample likelihood ratio test in the sense of having no worse error probabilities but smaller average sample size. However, this improvement is achieved at the cost of using more observations some of the time. Besides, a sequential probability ratio test may not always be practical.

The purpose of this article is to discuss another type of improvement of a fixed-sample likelihood ratio test, whereby one can achieve the same error probabilities without using all the observations. Some definitions are needed.

Let $(\Omega, \mathcal{E})$ be an arbitrary measurable space, and P and Q be two mutually absolutely continuous probability measures on $\mathcal{E}$. The available observations X_1, $X_2 \cdots$ are assumed to be $\mathcal{E}$-measurable functions on Ω, and denote by $\mathcal{E}_n$ the smallest σ-field generated by $(X_1, \cdots, X_n)$ for $n \geqslant 1$. A *stopping time* N is any $\mathcal{E}$-measurable function, taking positive integer values, such that the event $N = n$ is in $\mathcal{E}_n$ for each n. Denote by $\mathcal{E}_N$ the σ-field of all events $E \in \mathcal{E}$ such that $E \cap [N = n]$ is in $\mathcal{E}_n$ for each n. A *decision* D under a given N is any $\mathcal{E}_N$-measurable function, taking values 0 or 1, where 0(1) represents the decision that $P(Q)$ is the true probability measure on $\mathcal{E}$. Thus, D is determined by events up to time N, and the events $[D = 0] \cap [N = n]$ and $[D = 1] \cap [N = n]$ are in $\mathcal{E}_n$ for each n. A (*sequential*) *test* is a pair (N, D), and $\alpha = P(D = 1)$ and $\beta = Q(D = 0)$ are the *error probabilities* of the test. A test (N, D) is called *weakly admissible* if there exists no test (N', D') having $\alpha' \leqslant \alpha$, $\beta' \leqslant \beta$ and $N' \leqslant N$ a.s. P, with at least one inequality strict. The inequality $N' \leqslant N$ a.s. P is considered strict if $P(N' < N) > 0$. If N equals some constant $m \geqslant 1$, we call (N, D) a *fixed-sample* test. Finally, the

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likelihood ratio (LR) for $\mathcal{E}_N$ is the unique $\mathcal{E}_N$-measurable function λ_N which satisfies $Q(E) = \int_E \lambda_N \, dP$ for all $E \in \mathcal{E}_N$. (See [2] and [3] for more details on these concepts).

Eisenberg and Simons ([2]) show that, under mild assumptions, any test (N, D) has a weakly admissible improvement, called a curtailed version of (N, D). In particular, a fixed-sample *level* k LR test has a unique weakly admissible improvement, provided only that $P(\lambda_N = k) = 0$. (A level k LR test is defined by $D = 0$ when $\lambda_N < k$, and $D = 1$ when $\lambda_N > k$). This improvement, denoted by (N^*, D^*), will be henceforth called a *curtailed test*. The decision of the curtailed test is identical to that of the fixed-sample LR test, but the procedure uses only as many observations as are needed to determine the decision. In this sense curtailed t-tests are considered in [1] and [5]. It should be emphasized that a fixed-sample LR test is optimal in terms of error probabilities alone, but it need not be weakly admissible ([2]).

In Section 2, some preliminary results about curtailed versions of LR test are proved. It is then shown in Section 3 that curtailed tests possess two useful optimal properties. In Section 4, the form of the curtailed test is completely described when the observations are independent under P and Q. It is shown that, if the observations are also identically distributed according to the one-parameter exponential family, then the curtailed test is also uniformly most powerful against one-sided alternatives. Finally, in Section 5, the average sample size of the uniformly most powerful curtailed test is compared with the sample size of the uniformly most powerful fixed-sample test.

2. Preliminaries. Let N be an arbitrary stopping time, and (N, D) be a level k LR test with $P(\lambda_N = k) = 0, 0 < k < \infty$ (then $Q(\lambda_N = k) = 0$ by absolute continuity of P and Q). We shall call (N, D) a *k-regular* (*sequential*) *test*. A k-regular test with $N \equiv m$ is the classical Neyman-Pearson test based on a predetermined sample of size m.

Theorems 2.1 and 2.2 below are closely related to results in Eisenberg and Simons ([2]), but they are presented here in a more useful form that takes advantage of the additional assumptions made in this paper.

THEOREM 2.1. *Let* (N, D) *be a k-regular test. Let* $\mathcal{E}'$ *be a subfield of* $\mathcal{E}_N$ *and* λ' *be the* LR *for* $\mathcal{E}'$. *If* D *is* $\mathcal{E}'$-*measurable, then*

(a) $P(\lambda' \leqslant k < \lambda_N) = 0 = P(\lambda' \geqslant k > \lambda_N)$,

(b) $P(\lambda' = k) = 0$,

(c) $[D = 0] = [\lambda' < k], [D = 1] = [\lambda' > k]$, a.s. P.

PROOF. If D is $\mathcal{E}'$-measurable, then $[\lambda_N > k] \in \mathcal{E}'$. Hence

$$\int_{\lambda' \leqslant k < \lambda_N} \lambda' \, dP = Q(\lambda' \leqslant k < \lambda_N) = \int_{\lambda' \leqslant k < \lambda_N} \lambda_N \, dP,$$

which implies that $kP(\lambda' \leqslant k < \lambda_N) \geqslant Q(\lambda' \leqslant k < \lambda_N) > kP(\lambda' \leqslant k < \lambda_N)$. The contradiction $P(\lambda' \leqslant k < \lambda_N) > P(\lambda' \leqslant k < \lambda_N)$ can be resolved only if $P(\lambda' \leqslant k < \lambda_N) = 0$. Similarly, we conclude that $P(\lambda' \geqslant k > \lambda_N) = 0$, which proves (a).

Assertion (b) follows from (a) and the assumption $P(\lambda_N = k) = 0$. Assertion (c) follows from (a), (b) and the same assumption. $\square$

Theorem 2.1 describes properties of an LR decision D which is measurable over a smaller σ-field than that over which D is defined. The following theorem describes properties of stopping times over whose σ-fields D is measurable.

THEOREM 2.2. *Let (N, D) be a k-regular test. Let N^* be the first n such that $A_n = [P(\lambda_N > k | \mathcal{S}_n) = 0]$ or $B_n = [P(\lambda_N < k | \mathcal{S}_n) = 0]$ occurs for $n \geqslant 1$, and define $D^* = 0$ in the first case and $D^* = 1$ in the second case. Then*

(a) *D^* is well defined, $\mathcal{S}_{N^*}$-measurable, and equals D* a.s. *P*,

(b) *$N^* \leqslant N'$* a.s. *P if $N' \leqslant N$* a.s. *P and D is $\mathcal{S}_{N'}$-measurable,*

(c) *the test (N^*, D^*) is weakly admissible and the curtailed version of (N, D).*

PROOF. Let $C_n = A_n \cap B_n$ for $n \geqslant 1$. Then

$$P(C_n) = \int_{C_n}[P(\lambda_N > k|\mathcal{S}_n) + P(\lambda_N < k|\mathcal{S}_n)]\, dP = 0, \qquad n \geqslant 1.$$

Hence D^* is well defined. $[D^* = 1] \cap [N^* = n] = B_n \cap [N^* = n] \in \mathcal{S}_n$ for each n so D^* is $\mathcal{S}_{N^*}$-measurable. Moreover,

$$P(D^* = 1, D = 0) \leqslant P(\lambda_N < k, \cup B_n) \leqslant \Sigma \int_{B_n} P(\lambda_n < k|\mathcal{S}_n)\, dP = 0,$$

and similarly $P(D^* = 0, D = 1) = 0$, which proves the last part of (a). Next, if $N' \leqslant N$ and D is $\mathcal{S}_{N'}$-measurable, then Theorem 2.1 shows that $P(\lambda_{N'} \leqslant k < \lambda_N) = 0 = P(\lambda_{N'} \geqslant k > \lambda_N)$. In particular, $P(\lambda_n \leqslant k < \lambda_N, N' = n) = 0$ for each n, which implies

$$\int_{[\lambda_n \leqslant k] \cap [N' = n]} P(\lambda_n > k|\mathcal{S}_n)\, dP = 0.$$

Consequently, $P(\lambda_N > k|\mathcal{S}_n) = 0$ on the set $[\lambda_n \leqslant k] \cap [N' = n]$, and similarly $P(\lambda_N < k|\mathcal{S}_n) = 0$ on the set $[\lambda_n \geqslant k] \cap [N' = n]$. Combining the two we conclude that, if $N' = n$, A_n or B_n occurs. But N^* is defined as the first n such that A_n or B_n occurs. Hence we must have $N^* \leqslant N'$ a.s. P, which proves (b). Finally, suppose that a test (N', D') satisfies $\alpha' \leqslant \alpha^*$, $\beta' \leqslant \beta^*$ and $N' \leqslant N^*$ a.s. P. Then $\alpha' \leqslant \alpha$, $\beta' \leqslant \beta$ and $N' \leqslant N$ a.s. P. But D is $\mathcal{S}_{N'}$-measurable by the Neyman-Pearson lemma (see [2]), and $N^* \leqslant N'$ a.s. P from (b). Hence we must have $N^* = N'$ and $D^* = D'$ a.s. P, which proves the assertions in (c). $\square$

3. Optimum properties of curtailed tests. A weakly admissible improvement (N^*, D^*) of an arbitrary test (N, D) has the property that there is no other test (N', D') with $\alpha' \leqslant \alpha^*$ and $\beta' \leqslant \beta^*$ such that $N' \leqslant N^*$ a.s. P. This does not generally imply that $N^* \leqslant N'$ a.s. P for all such tests (N', D') (see [2]). An interesting result of this nature does hold if (N, D) is a fixed-sample LR test with $N \equiv m$ and if the class of possible tests is restricted to $N' \leqslant m$. The result is essentially the weak admissibility property from a different viewpoint.

THEOREM 3.1. (*Optimum Property I*). *Let (N, D) be a k-regular test with $N \equiv m$, and (N^*, D^*) be the curtailed test* (i.e., *the curtailed version of (N, D)*). *If (N', D') has $\alpha' \leqslant \alpha$, $\beta' \leqslant \beta$ and $N' \leqslant m$, then $N^* \leqslant N'$* a.s. *P*.

PROOF. This is a special case of Theorem 2.2, part (b). ▯

The theorem above shows one way that curtailed test are preferable to truncated (at m) sequential probability ratio tests (see [4], page 221). It does not follow, however, that the only test to be used in every situation, where at most m observations are available, is the curtailed test. The reason is that the possible error probabilities of curtailed tests are limited to those generated by the fixed-sample LR test. If α is given and m is large, β would ordinarily be very small. There could very well be other tests based on at most m observations with preferable sample size at little additional cost in terms of β. Indeed, a fixed-sample LR test based on fewer than m observations could be such a test.

We next consider a second optimum property of curtailed tests. Let P_θ be a family of probability measures on $\mathfrak{S}$, indexed by a real-valued parameter θ. A test (N, D) is called *uniformly most powerful* (UMP) for $\theta = \theta_0$ against alternatives $\theta > \theta_0$ if, for all tests (N, D') with the same stopping time N, $\alpha' \leqslant \alpha$ implies $P_\theta(D = 0) \leqslant P_\theta(D' = 0)$ for all $\theta > \theta_0$.

THEOREM 3.2. (*Optimum Property II*). *If* (N, D) *is a* UMP *test for* $\theta = \theta_0$ *against* $\theta > \theta_0$ *with* P_θ *all equivalent, then its curtailed version* (N^*, D^*) *is also* UMP.

PROOF. If (N^*, D^*) is the curtailed version of (N, D), then any $\mathfrak{S}_{N^*}$-measurable decision is also $\mathfrak{S}_N$-measurable. But $D^* = D$ a.s. P, and D is UMP against all $\mathfrak{S}_N$-measurable decisions. Hence D^* must be UMP against all $\mathfrak{S}_{N^*}$-measurable decisions. ▯

It is well known that, if P_θ yields a monotone likelihood ratio family (see [7]}, the fixed-sample LR test based on m observations is UMP for $\theta = \theta_0$ against $\theta > \theta_0$. Theorem 3.2 shows that the curtailed test will then be UMP, too. The curtailed test in such situations is often sequential with N^* nonconstant (see the examples in the following section). This conclusion is remarkable in that, as a general rule, most of the common sequential procedures are not UMP for $\theta = \theta_0$ against $\theta > \theta_0$. For instance, a sequential probability ratio test or its truncated version is not UMP. Not only does the construction of the stopping time of such tests depend on the actual choice of $\theta > \theta_0$, so does the construction of the decision.

4. Curtailed tests in the independent model. The stopping time N^* and the decision D^* of a curtailed test can be described in a simple form when the observations $X_1, \cdots, X_m$ are independent under both P and Q.

Assume that the X_i are independent with distribution F_i under P and G_i under Q (and dG_i and dF_i are equivalent for each i). Let

$$(4.1) \qquad I_{n,m} = \operatorname{ess}_P \inf \Pi_{i=n+1}^m h_i(X_i), \qquad S_{n,m} = \operatorname{ess}_P \sup \Pi_{i=n+1}^m h_i(X_i),$$

where

$$h_i(x) = dG_i(x)/\,dF_i(x).$$

Then $\lambda_n = \Pi_{i=1}^n h_i$ for $1 \leqslant n \leqslant m$, $\lambda_m = \lambda_n \Pi_{i=n+1}^m h_i$ for $1 \leqslant n < m$, and $\lambda_n I_{n,m} \leqslant \lambda_m \leqslant \lambda_n S_{n,m}$ a.s P.

THEOREM 4.1. *If (N, D) is a k-regular test with $N \equiv m$ in the independent model, the curtailed test is a generalized sequential probability ratio test with upper boundary $B_n = k/I_{n,m}$ and lower boundary $A_n = k/S_{n,m}$ for $1 \leqslant n \leqslant m$. That is, N^* is the smallest n such that $\lambda_n \geqslant B_n$ or $\lambda_n \leqslant A_n$, and $D^* = 1$ in the former case and $D^* = 0$ in the latter case. Moreover, B_n is nonincreasing, A_n is nondecreasing, and $A_m = k = B_m$.*

The proof of the theorem is rather technical and is left to the Appendix.

COROLLARY. *If $\text{ess}_P \inf h_m(X_m) = 0$ and $\text{ess}_P \sup h_m(X_m) = \infty$, then the k-regular test with $N \equiv m$ is itself the curtailed test.*

PROOF. In this case, $A_n = 0$ and $B_n = \infty$ for $1 \leqslant n < m$. Since $P(0 < \lambda_n < \infty) = 1 = Q(0 < \lambda_n < \infty)$ for $n < m$, it follows that the generalized sequential probability ratio test does not stop before time m. ▯

We shall now apply Theorem 4.1 to derive curtailed tests which are UMP for a wide family of distributions. Assume that the X_i are independent with a common density

$$f_\theta(x) = A(\theta)B(x)e^{t(\theta)C(x)} \tag{4.2}$$

with respect to a σ-finite measure for some θ, where θ lies in some open interval and $t(\theta)$ is an increasing function of θ. It is well known that, under (4.2), P_θ and $P_{\theta'}$ are equivalent for all $\theta \neq \theta'$.

The k-regular test (N, D) with $N \equiv m$ of $\theta = \theta_0$ against $\theta = \theta_1$, $\theta_1 > \theta_0$, is also UMP for $\theta = \theta_0$ against alternatives $\theta > \theta_0$ (see [7]). The decision of this test is defined by $D = 0$ if $Y_m \leqslant c$ and $D = 1$ if $Y_m > c$ provided $P_\theta(Y_m = c) = 0$, where

$$Y_m = \Sigma_{i=1}^m C(X_i). \tag{4.3}$$

It is easily verified that the condition $P_\theta(Y_m = c) = 0$ is equivalent to the k-regularity condition $P_{\theta_0}(\lambda_m = k_c) = 0$ if one identifies $k_c = [A(\theta_1)/A(\theta_0)]^m \exp[c\{t(\theta_1) - t(\theta_0)\}]$. We may call this test the *fixed-sample* UMP *test* of $\theta = \theta_0$ against $\theta > \theta_0$. Let

$$a = \text{ess}_{P_\theta} \inf C(X_i), \qquad b = \text{ess}_{P_\theta} \sup C(X_i). \tag{4.4}$$

THEOREM 4.2. *If (N, D) is the fixed-sample* UMP *test with $N \equiv m$, then the curtailed test (N^*, D^*) is* UMP, *where N^* is the smallest n such that $Y_n \leqslant c - (m - n)b$ or $Y_n \geqslant c - (m - n)a$, and $D^* = 0$ in the first case and $D^* = 1$ in the second case. If a or b is infinite, then $0 \cdot \infty$ is interpreted as 0 so that $N^* \leqslant m$.*

PROOF. Applying Theorem 4.1 we note that the curtailed test stops the first time $\lambda_n \geqslant k_c/I_{n,m}$ or $\lambda_n \leqslant k_c/S_{n,m}$, where

$$I_{n,m} = \left[A(\theta_1)/A(\theta_0)\right]^{m-n} \exp\left[\{t(\theta_1) - t(\theta_0)\}(m - n)a\right],$$

$$S_{n,m} = \left[A(\theta_1)/A(\theta_0)\right]^{m-n} \exp\left[\{t(\theta_1) - t(\theta_0)\}(m - n)b\right].$$

Substituting for k_c this yields the rule given in the statement of the theorem. The curtailed test is UMP by Theorem 3.2. ▯

The description of the stopping rule in terms of Y_n, instead of λ_n, makes it clear that the form of the UMP curtailed test does not depend on the actual choice of $\theta > \theta_0$. We give below some applications of Theorem 4.2.

EXAMPLE 1. Suppose that the X_i are independent normal variables with zero mean and variance θ^2. To test $\theta = \theta_0 (\theta_0 > 0)$ against $\theta > \theta_0$ using m observations, the fixed-sample UMP test rejects $\theta = \theta_0$ if $Y_m > c$ and accepts $\theta = \theta_0$ if $Y_m \leqslant c$ for some $c \geqslant 0$, where $Y_m = \Sigma_{i=1}^m X_i^2$. Here, $P_\theta(Y_m = c) = 0$ for all c, and $a = 0$, $b = \infty$. Hence the UMP curtailed test rejects if $Y_n > c$ for any $n \leqslant m$ and accepts if $Y_m \leqslant c$.

EXAMPLE 2. Suppose that the X_i are independent normal variables with mean θ and variance 1. To test $\theta = \theta_0$ against $\theta > \theta_0$, the fixed-sample UMP test rejects if $Y_m > c$ and accepts if $Y_m \leqslant c$, where $Y_m = \Sigma_{i=1}^m X_i$. Here, $P_\theta(Y_m = c) = 0$ for all c, and $a = -\infty$, $b = \infty$. Hence the UMP curtailed test is the same as the fixed-sample test.

EXAMPLE 3. Suppose that the X_i are independent Bernoulli variables with mean θ. To test $\theta = \theta_0 (0 < \theta_0 < 1)$ against $\theta > \theta_0$, the fixed-sample UMP test rejects if $Y_m > c$ and accepts if $Y_m \leqslant c$ for some $0 \leqslant c \leqslant m$, where $Y_m = \Sigma_{i=1}^m X_i$. Here $P_\theta(Y_m = c) = 0$ if c is not an integer, and $a = 0$, $b = 1$. Hence the UMP curtailed test rejects if $Y_n > c$ and accepts if $Y_n \leqslant c - m + n$ for any $n \leqslant m$.

EXAMPLE 4. Suppose that the X_i are independent Poisson variables with mean θ. To test $\theta = \theta_0 (\theta_0 > 0)$ against $\theta > \theta_0$, the fixed-sample UMP test rejects if $Y_m > c$ accepts if $Y_m \leqslant c$ for some $c \geqslant 0$, where $Y_m = \Sigma_{i=1}^m X_i$. Here $P_\theta(Y_m = c) = 0$ if c is not an integer, and $a = 0$, $b = \infty$. Hence the UMP curtailed test rejects if $Y_n > c$ for any $n \leqslant m$ and accepts if $Y_m \leqslant c$.

5. Asymptotic relative sample size of UMP curtailed tests. Consider the UMP curtailed test, defined in Theorem 4.2, for testing $\theta = \theta_0$ against $\theta > \theta_0$ in the family (4.2). For present purposes, it will be more appropriate to denote N^* by N_m^*. We shall call

$$e_m(\theta) = E_\theta(N_m^*)/m \tag{5.1}$$

the relative sample size of the UMP curtailed test with respect to the fixed-sample UMP test. Clearly, $e_m(\theta) \leqslant 1$ for all $m \geqslant 1$ and θ, and $e_m(\theta)$ represents the fraction of the m available observations that one would require, on the average, under the curtailed procedure.

For a sequence of fixed-sample UMP tests and the corresponding UMP curtailed tests, the *asymptotic relative sample size at level* α is $\lim_{m\to\infty} e_m(\theta)$ when $\lim_{m\to\infty} P_{\theta_0}(D_m = 1) = \alpha$. We shall now compute this asymptotic relative sample size. Let

$$\mu_\theta = E_\theta(C(X_i)), \qquad \sigma_\theta^2 = \mathrm{Var}_\theta(C(X_i)), \tag{5.2}$$

and denote P_{θ_0} by P, μ_{θ_0} by μ, and σ_{θ_0} by σ. It is well known that μ_θ and σ_θ exist for all θ, and that μ_θ is an increasing function of θ (see [4], page 25, [5], pages 51, 58).

THEOREM 5.1. *The asymptotic relative sample size of the UMP curtailed test is given by*

$$(5.3)\qquad \lim_{m\to\infty} e_m(\theta) = (\mu - a)/(\mu_\theta - a) \quad \text{if } \theta \geqslant \theta_0$$
$$= (b-\mu)/(b-\mu_\theta) \quad \text{if } \theta \leqslant \theta_0,$$

for all levels $\alpha, 0 < \alpha < 1$, *where a and b are defined in* (4.4). *If* $a = -\infty$ *or* $b = \infty$, *the ratio in* (5.3) *is interpreted as* 1. *Moreover, as* $m \to \infty$, N_m^*/m *converges to* $\lim e_m(\theta)$ *in probability under* P_θ *for all* θ.

PROOF. Let $c_m = m\mu + z_\alpha m^{\frac{1}{2}}\sigma$ for $0 < \alpha < 1$, where z_α is defined by $(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{z_\alpha} \exp(-\frac{1}{2}x^2)\,dx = 1 - \alpha$. Then $P(Y_m > c_m) \to \alpha$ as $m \to \infty$, where Y_m is defined in (4.3). For any $r, 0 < r < 1$, the stopping rule in Theorem 4.2 yields

$$(5.4)\quad P_\theta(N_m^*/m > r) = P_\theta\big(c_m - (m - [rm])b < Y_{[rm]} < c_m - (m-[rm])a\big)$$
$$= P_\theta\big(u_m < (Y_{[rm]} - [rm]\mu_\theta)\sigma_\theta^{-1}([rm])^{-\frac{1}{2}} < v_m\big),$$

where

$$u_m = \left\{m(\mu - b) - [rm](\mu_\theta - b) + z_\alpha \sigma m^{\frac{1}{2}}\right\}\sigma_\theta^{-1}([rm])^{-\frac{1}{2}},$$
$$v_m = \left\{m(\mu - a) - [rm](\mu_\theta - a) + z_\alpha \sigma m^{\frac{1}{2}}\right\}\sigma_\theta^{-1}([rm])^{-\frac{1}{2}},$$

and $[rm]$ denotes the largest integer $\leqslant rm$. Suppose first that $\theta = \theta_0$ and $-\infty < a < b < \infty$. Then $a < \mu = \mu_\theta < b$ and, as $m \to \infty$, $u_m \to -\infty$ and $v_m \to \infty$ for all $0 < r < 1$. It follows from (5.4) and the central limit theorem that, as $m \to \infty$, $P(N_m^*/m > r) \to 1$ for all $r < 1$. Since $P(N_m^*/m \leqslant 1) = 1$, N_m^*/m converges to 1 in probability under P. Moreover, since $N_m^*/m \leqslant 1$, we also conclude that $e_m(\theta_0) \to 1$ as $m \to \infty$. Suppose next that $\theta > \theta_0$ and $-\infty < a < b < \infty$. Then $a < \mu < \mu_\theta < b$, so that $0 < \mu - a < \mu_\theta - a$ and $0 < b - \mu_\theta < b - \mu$. If one chooses $r < (\mu - a)(\mu_\theta - a)^{-1}$, it is easily verified that $u_m \to -\infty$ and $v_m \to \infty$ as $m \to \infty$, and therefore $P_\theta(N_m^*/m > r) \to 1$. If one chooses $r > (\mu - a)(\mu_\theta - a)^{-1}$, then $v_m \to -\infty$ and therefore $P_\theta(N_m^*/m > r) \to 0$. The assertions of the theorem now follow. The proof for the case $\theta < \theta_0$ and $-\infty < a < b < \infty$ is similar. If $a = -\infty (b = \infty)$, the stopping rule in Theorem 4.2 implies that $v_m(u_m)$ in (5.4) should be replaced by $\infty(-\infty)$. Then using the same technique as above for cases $\theta = \theta_0$, $\theta > \theta_0$ and $\theta < \theta_0$ separately we conclude that the present theorem holds with proper interpretation when $a = -\infty$ or $b = \infty$. To complete the proof, suppose that c_m' is another sequence satisfying $P(Y_m > c_m') \to \alpha$ as $m \to \infty$. Then, for any $\epsilon > 0$ and sufficiently large m, we have $m\mu + z_{\alpha+\epsilon} m^{\frac{1}{2}}\sigma \leqslant c_m' \leqslant m\mu + z_{\alpha-\epsilon} m^{\frac{1}{2}}\sigma$. Since the terms of order $m^{\frac{1}{2}}$ play no role in our proof, these sequences also lead to the assertions of the theorem. □

As applications of Theorem 5.1, consider the four examples in the preceding section. In Example 1, $C(X_i) = X_i^2$, $\mu_\theta = \theta^2$, and the asymptotic relative sample size of the UMP curtailed test is 1 if $0 < \theta \leqslant \theta_0$ and $(\theta_0/\theta)^2$ if $\theta \geqslant \theta_0$. In Example 2, $C(X_i) = X_i$, $\mu_\theta = \theta$, and the asymptotic relative sample size is 1 for all θ, as it ought to be since the two tests are identical. In Example 3, $C(X_i) = X_i$, $\mu_\theta = \theta$, and the asymptotic relative sample size is $(1 - \theta_0)/(1 - \theta)$ if $0 < \theta \leqslant \theta_0$ and θ_0/θ if $\theta_0 \leqslant \theta < 1$. In Example 4, $C(X_i) = X_i$, $\mu_\theta = \theta$, and the asymptotic relative sample size is 1 if $0 < \theta \leqslant \theta_0$ and θ_0/θ if $\theta \geqslant \theta_0$. The first and last examples show that, when θ is much larger than θ_0, the UMP curtailed test will achieve a considerable saving over the fixed-sample UMP test. The third example shows that this can be true also when θ is much smaller than θ_0.

It can be seen that in the notion of asymptotic relative sample size α is fixed between 0 and 1, but $P_\theta(D_m = 0) \to 0$ for any $0 > \theta_0$ as $m \to \infty$. This is related to the notion of efficiency proposed by Hodges and Lehmann ([6]) to compare two fixed-sample tests. A second way to compare the UMP curtailed test with the fixed-sample UMP test is to hold both α and $\beta = P_\theta(D_m = 0)$ fixed between 0 and 1 $(0 < \alpha < 1 - \beta < 1)$, and then compare $E_\theta(N_m^*)$ with $m = m(\theta)$ as $\theta \to \theta_0$. We define the *local asymptotic relative sample size* of the UMP curtailed test with respect to the fixed-sample UMP test as $\lim_{\theta \to \theta_0} e_{m(\theta)}(\theta)$, where $e_m(\theta)$ is given by (5.1) and $m(\theta)$ is the sample size required by the fixed-sample test to achieve the error probabilities α and β. This notion is closely related to the well-known Pitman-efficiency of fixed-sample tests. Our final result shows that there is no advantage in using a curtailed test in the sense of local asymptotic relative sample size.

THEOREM 5.2. *The local asymptotic relative sample size of the UMP curtailed test is one for all* $0 < \alpha < 1 - \beta < 1$.

PROOF. We shall use the notations of Theorem 5.1. Observe first that $m(\theta) = (z_\alpha \sigma - z_{1-\beta}\sigma_\theta)^2(\mu_\theta - \mu)^{-2}$ asymptotically, so that $m(\theta) \to \infty$ as $\theta \to \theta_0$ for any $0 < \alpha < 1 - \beta < 1$ (the condition $\alpha < 1 - \beta$ implies $\theta > \theta_0$). For any $\theta > \theta_0$ and $r < 1$,

$$P_\theta\big(N^*_{m(\theta)}/m(\theta) > r\big)$$
$$= P_\theta\Big(u_{m(\theta)} < \big(Y_{[rm(\theta)]} - [rm(\theta)]\mu_\theta\big)\sigma_\theta^{-1}([rm(\theta)])^{-\frac{1}{2}} < v_{m(\theta)}\Big).$$

Given any $\delta > 0$, one can choose θ sufficiently close to θ_0 such that $(\mu - a) - r(\mu_\theta - a) > \delta$. But $m(\theta) \to \infty$ as $\theta \to \theta_0$. Hence $u_{m(\theta)} \to -\infty$, $v_{m(\theta)} \to \infty$, and $P_\theta(N^*_{m(\theta)}/m(\theta) > r) \to 1$ for all $r < 1$, as $\theta \to \theta_0$. Since $N^*_{m(\theta)}/m(\theta) \leqslant 1$, we find $\lim_{\theta \to \theta_0} E_\theta(N^*_{m(\theta)}/m(\theta) = 1$. $\square$

Note that, if $\alpha \geqslant 1 - \beta$, we must have $\theta \leqslant \theta_0$, $m(\theta) = 1$, and the local asymptotic efficiency is trivially one.

APPENDIX

PROOF OF THEOREM 4.1. Let $C = [\lambda_n S_{n,m} \leqslant k]$, and denote by $\bar{C}$ the complement of C. Then

$$\int_C P(\lambda_m > k \mid \mathfrak{S}_n)\, dP = P(\lambda_m > k, \lambda_n S_{n,m} \leqslant k) = 0.$$

Hence $P(\lambda_m > k | \mathcal{E}_n) = 0$ on C. Let $D \in \mathcal{E}_n$ be a subset of $\bar{C}$ with $P(D) > 0$. Then $P(D, \lambda_n > k/S_{n,m}) > 0$ and therefore $P(D, \lambda_n \geqslant k(S_{n,m} - \epsilon)^{-1}) > 0$ for some $\epsilon > 0$. Now $\prod_{i=n+1}^{m} h_i$ is independent of $\mathcal{E}_n$, and $P(\prod_{i=n+1}^{m} h_i > S_{n,m} - \epsilon) > 0$ for all $\epsilon > 0$. Hence

$$P\big(D, \lambda_n \geqslant k(S_{n,m} - \epsilon)^{-1}, \textstyle\prod_{i=n+1}^{m} h_i > S_{n,m} - \epsilon\big) > 0,$$

which implies $P(D, \lambda_m > k) > 0$. Since $\int_D P(\lambda_m \geqslant k | \mathcal{E}_n) \, dP > 0$ for all D in $\mathcal{E}_n$ which are included in $\bar{C}$, it follows that $P(\lambda_m \geqslant k | \mathcal{E}_n) > 0$ a.s. P on $\bar{C}$. We have thus shown that

$$[P(\lambda_m > k | \mathcal{E}_n) = 0] = [\lambda_n \leqslant k/S_{n,m}] \text{ a.s. } P,$$

and similarly one gets

$$[P(\lambda_m < k | \mathcal{E}_n) = 0] = [\lambda_n \geqslant k/I_{n,m}] \text{ a.s. } P.$$

It follows from Theorem 2.2 that the curtailed version of (N, D) with $N \equiv m$ stops the first time $A_n = [\lambda_n \leqslant k/S_{n,m}]$ or $B_n = [\lambda_n \geqslant k/I_{n,m}]$ occurs, and $D^* = 0$ in the first case and $D^* = 1$ in the second case. This is identical to the rule of the generalized sequential probability ratio test stated in the present theorem. The last part of the theorem follows from the fact that $I_{n,m} = \text{ess}_P \inf h_{n+1} I_{n+1,m} \leqslant I_{n+1,m}$ and $S_{n,m} = \text{ess}_P \sup h_{n+1} S_{n+1,m} \geqslant S_{n+1,m}$. □

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