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EIGENFUNCTIONS OF EXPECTED VALUE OPERATORS IN THE WISHART DISTRIBUTION

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Let $(X_{1,m}, X_{2,m}, \cdots X_{k,m})$, $1 \leqslant m \leqslant n$, be a sample of size n from the k dimensional normal distribution with mean vector μ and covariance matrix Σ. Let $V = (v_{ij})$, $1 \leqslant i, j \leqslant k$, denote the symmetric scatter matrix where $v_{ij} = \Sigma_m(X_{i,m} - \mu_i)(X_{j,m} - \mu_j)$. The problem posed is to characterize the eigenfunctions of the expectation operators of the Wishart distribution, i.e., those scalar valued functions, $f(V)$, such that $E(f(V)) = \lambda_{n,k}f(\Sigma)$. If f is an eigenfunction then (a) for nonsingular T, $f(T'VT)$ is an eigenfunction and (b) for integral p, $|V|^{p/2}f(V)$ is an eigenfunction. For $k \leqslant 2$, a complete solution of the problem is given. For $k = 1$ the functions $f(v) = cv^{\alpha}$ are the only eigenfunctions. For $k = 2$, a function f is an eigenfunction if and only if (i) f is homogeneous and (ii) $4\frac{\partial^2 f}{\partial v_{22}\partial v_{11}} - \frac{\partial^2 f}{\partial^2 v_{21}} = C|V|^{-1}f$. A representation of eigenfunctions is given in terms of sums of associated Legendre functions. Relationships between eigenfunctions and harmonic functions are indicated. Any homogeneous polynomial is proved to be a linear combination of polynomial eigenfunctions.

1. Introduction and summary. Let $V = (v_{ij})$ follow the Wishart distribution $W(k, n, \Sigma)$. Properties of this distribution, as well as a comprehensive list of references, are given in Johnson and Kotz [4].

In this paper we pose and partially solve the following problem: Characterize those (scalar-valued) functions, $f(V)$, in the matrix variable V with the property

$$E(f(V)) = \lambda_{n,k}f(\Sigma). \tag{1}$$

In other words find all functions $f(V)$ whose expected value is "evaluation at Σ" multiplied by a constant depending only on k, the dimension of the sample, and on n, the sample size. We call this property EP, the "evaluation property" or "expectation property." We can write equation (1) more precisely as

$$E_{n,k}(f(V)) = \int f(V)K(V, \Sigma, k, n)\,dV = \lambda_{n,k}f(\Sigma) \tag{2}$$

where $K(V, \Sigma, k, n) = C_{n,k}|\Sigma|^{-n/2}|V|^{\frac{n-1-k}{2}}e^{-\frac{1}{2}\mathrm{tr}(V\Sigma^{-1})}$ is the density function of V, $C_{n,k} = 1/2^{\frac{kn}{2}}\pi^{k(k-1)/4}\prod_{j=1}^{k}\Gamma\left(\frac{n-j+1}{2}\right)$, and the range of integration is the space of all $k \times k$ symmetric positive definite matrices, $V > 0$. Equation (2) represents, for each fixed k, an infinite number of integral equations ($k \leqslant n$) with kernel K. The function $f(V)$ is then a common eigenfunction (thus EP can also stand for "eigenfunction property") of all the integral operators defined by equation (2) and $\lambda_{n,k}$ are the corresponding eigenvalues of $f(V)$. For example $f(V) = 1$

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is an eigenfunction with $\lambda_{n,k} = 1$, as is the error sum of squares of multivariate regression theory. Other simple examples of EP functions are $f_{ij}(V) = v_{ij}$, $f(V) = |V|^r$, and $f^{ij}(V) = v^{ij}$ where $(v^{ij}) = V^{-1}$. As a final, more recondite example, we note that equation (1) of Constantine [1] states that James' "zonal polynomial" [5] $C_\kappa(S)$ is an EP function.

The Wishart distribution is the distribution of the scatter matrix and, accordingly, its chief interest is for integral n (i.e., sample size). However, it also has been extended (Eaton [3]) to the case when n is complex; we denote this distribution by $W(k, \nu, \Sigma)$. (The following statement will be proved for $k \leqslant 2$: f is EP with respect to $W(k, n, \Sigma)$ implies f is EP with respect to $W(k, \nu, \Sigma)$).

Frequently, for a given f, $E_{n,k}(f(V))$ may fail to exist unless n is sufficiently large; for example $E_{n,2}(v^{11}) = \dfrac{\sigma^{11}}{n-3}$ provided $n \geqslant 4$. We shall still refer to the function $f(v) = v^{11}$ as an EP function. Throughout this paper, the expressions $E_{n,k}\,(f(V))$ are tacitly assumed to exist with $n \geqslant n_0(k, f)$.

As a final preparatory observation, we note that an EP function can be viewed as a multidimensional Laplace transform. In particular, partial differentiation under the integral is permissible.

This paper presents a complete solution to the problem when $k \leqslant 2$. For arbitrary k we prove two theorems about the construction of new EP functions from a given EP function. As an application of these theorems we give a simple proof that $f^{ij}(V) = v^{ij}$ are EP functions. In Section 3 all EP functions and their eigenvalues are obtained for $k = 1$. Section 4 shows, for $k = 2$, that an EP function is homogeneous and satisfies a certain second order partial differential equation. This second order partial differential equation is shown after a change of coordinates, to be the classical potential equation. The generation of all EP homogeneous polynomials, for $k = 2$, is the subject of Section 5. Finally, Section 6 shows the equivalence of four different statements defining EP functions and concludes with two consequences of the preceding results: (a) $f(V^{-1})$ is EP if f is EP; and (b) the distribution of certain EP functions is a product of powers of independent gamma variates.

2. General theorems. We begin by noting some general theorems concerning EP functions. Let T be a nonsingular $k \times k$ matrix. Denote the transpose of T by T'.

THEOREM 1. *If $f(V)$ is an EP function with eigenvalues $\lambda_{n,k}$, T a nonsingular $k \times k$ matrix, then $f_T(V) = f(T'VT)$ is also an EP function with the same eigenvalues $\lambda_{n,k}$.*

THEOREM 2. *If $f(V)$ is an EP function with eigenvalues $\lambda_{n,k}$, and α is an integer or half an integer, then $|V|^\alpha f(V)$ is an EP function with eigenvalues*

$$\lambda_{n+2\alpha,k}\frac{C_{n,k}}{C_{n+2,k}} \qquad n \geqslant k - 2\alpha.$$

At the end of this paper, we shall prove this theorem for arbitrary α, when $k = 2$. When $k = 1$, one can prove this theorem for arbitrary α using the results of Section 3.

We note that when $\alpha = -1$, the eigenvalues of $|V|^{-1}f(V)$ are $\lambda_{n-2,k}\dfrac{(n-k-2)!}{(n-2)!}$. As an illustration of these two theorems, we prove $E_{n,k}(v^{ij}) = \sigma^{ij}/(n-k-1)$, (Seal [3]). For, $f(V) = v^{11}$ is the quotient of two determinants. Using 18.2.33 of Wilks [9] and Theorem 2 with $\alpha = -1$, one finds $E(v^{11}) = \sigma^{11}/(n - k - 1)$. Since $f(T'VT) = s'V^{-1}s$ (where s is the first row vector of T^{-1}), it follows from Theorem 1 that for any vector s, $s'V^{-1}s$ is an EP function with the same eigenvalues as v^{11}. The result now follows.

3. Eigenfunctions and eigenvalues for $k = 1$. We now begin a systematic study of the eigenfunctions of the Wishart operators $E_{n,k}$. We start with the case $k = 1$ which, of course, is the familiar (scaled) chi-square distribution. The problem here is to find all functions $f(v)$ such that for $n \geqslant 1$

(3) $$E_n(f(v)) = C_n\sigma^{-n/2}\int_0^\infty v^{n/2-1}e^{-\frac{1}{2}v/\sigma}f(v)\,dv = \lambda_n f(\sigma).$$

Here we denote $E_{n,1}$ by $E_n, \lambda_{n,1}$ by λ_n, σ_{11} by σ and $C_{n,1}$ by $C_n = 1/(2^{n/2}\Gamma(n/2))$.

THEOREM 3. *The functions $f(v) = Cv^\alpha(\operatorname{Re}\alpha > -\frac{1}{2})$ are the only eigenfunctions of the expected value operators E_n of the (scaled) chi-square distributions $n \geqslant 1$. Their associated eigenvalues are*

$$\lambda_n = \frac{2^\alpha\Gamma(\alpha + n/2)}{\Gamma(n/2)}.$$

PROOF. Multiply equation (3) by $\sigma^{n/2}$ and differentiate with respect to σ. Comparing the resulting equation with (3) with n replaced by $n + 2$ yields

$$2\frac{C_{n+2}}{C_n}\lambda_n\sigma^{-n/2+1}(\sigma^{n/2}f(\sigma))' = \lambda_{n+2}f(\sigma)$$

which simplifies to

$$\frac{2}{n}\frac{\lambda_n}{\lambda_{n+2} - \lambda_n}\sigma f'(\sigma) = f(\sigma).$$

Now

$$\frac{n}{2}\left(\frac{\lambda_{n+2}}{\lambda_n} - 1\right) = \alpha,$$

independent of n, implying

(4) $$f(\sigma) = C\sigma^\alpha$$

To prove sufficiency, i.e., that equation (4) actually is an EP function and also to calculate λ_n, substitute $f(v) = v^\alpha$ into equation (3). We note that α can also be complex in which case

$$f(v) = v^a(\cos b \ln v + i(\sin b \ln v)) \qquad \alpha = a + ib$$

$$\lambda_n = 2^a \frac{(\cos b \ln 2 + i(\sin b \ln 2))\Gamma(\alpha + n/2)}{\Gamma(n/2)}$$

Since $E_n(v^\alpha)$ exists only when $\operatorname{Re}\alpha + \frac{n}{2} - 1 > -1$ i.e., when $\operatorname{Re}\alpha > -n/2$, we require $\operatorname{Re}\alpha > -\frac{1}{2}$ if $f(v)$ is to be an EP function for all the expectation operators $\{E_n\}$ $n \geqslant 1$. This completes the proof of Theorem 3.

In conclusion, we note that the sufficiency proof establishes that the eigenfunctions Cv^α are also EP functions for all the expectation operators corresponding to $W(1, \nu, \sigma)$.

4. Some necessary conditions for EP functions ($k = 2$). For the case $k = 2$, the problem is to find all functions $f(V)$ of the symmetric positive definite matrix variable V,

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}$$

such that

$$E_n(f(V)) = C_n|\Sigma|^{-n/2}\int |V|^{\frac{n-3}{2}} e^{-\frac{1}{2}\operatorname{tr}(V\Sigma^{-1})} f(V)\, dV = \lambda_n f(\Sigma) \tag{5}$$

Here, again we denote $E_{n,2}$ by E_n, $\lambda_{n,2}$ by λ_n and $C_{n,2}$ by $C_n = 1/4\pi\Gamma(n-1)$. We note the following two computational lemmas.

LEMMA 1. *Let D be the operator $4\dfrac{\partial^2}{\partial v^{22}\,\partial v^{11}} - \dfrac{\partial^2}{\partial (v^{12})^2}$. Then*

$$D(FG) = GDF + FDG + 2\left(2\frac{\partial F}{\partial v^{11}}\frac{\partial G}{\partial v^{22}} + 2\frac{\partial G}{\partial v^{11}}\frac{\partial F}{\partial v^{22}} - \frac{\partial F}{\partial v^{12}}\frac{\partial G}{\partial v^{12}}\right)$$

where $F = F(v^{11}, v^{12}, v^{22})$ and $G = G(v^{11}, v^{12}, v^{22})$.

LEMMA 2. (a) *The partial derivatives of $|V|^\alpha$ with respect to v^{ij} are given by the expressions*

$$\frac{\partial |V|^\alpha}{\partial v^{11}} = -\alpha v^{22}|V|^{\alpha+1},$$

$$\frac{\partial |V|^\alpha}{\partial v^{22}} = -\alpha v^{11}|V|^{\alpha+1},$$

$$\frac{\partial |V|^\alpha}{\partial v^{12}} = 2\alpha v^{12}|V|^{\alpha+1}.$$

(b) $D(|V|^\alpha) = (2\alpha)(2\alpha - 1)|V|^{\alpha+1}$ *where D is defined in Lemma 1.*

THEOREM 4. *Let D be the operator defined in Lemma 1 and let $f(V)$ be an EP function with eigenvalues λ_n. Then f satisfies the partial differential equations*

$$D(|V|^{n/2}f(V)) = \frac{\lambda_{n+2}}{\lambda_n} n(n-1)|V|^{(n+2)/2} f(V) \qquad n \geqslant n_0.$$

PROOF. Multiply equation (5) by $|\Sigma|^{n/2}$ and operate with D (with respect to σ^{ij}) on the left and right hand sides of equation (5). We get

$$C_n \int |V|^{\frac{n-3}{2}} D e^{-\frac{1}{2}\mathrm{tr}(V\Sigma^{-1})} f(V)\, dV = \lambda_n D(|\Sigma|^{n/2} f(\Sigma)). \tag{6}$$

Now $De^{-\frac{1}{2}\mathrm{tr}(V\Sigma^{-1})} = |V| e^{-\frac{1}{2}\mathrm{tr}(V\Sigma^{-1})}$ implies that the left hand side of equation (6) is

$$C_n \int |V|^{\frac{n+2-3}{2}} e^{-\frac{1}{2}\mathrm{tr}(V\Sigma^{-1})} f(V)\, dV.$$

Comparing this expression to equation (5) in which n is replaced by $n + 2$ we obtain

$$\lambda_n D(|\Sigma|^{n/2} f(\Sigma)) = \frac{C_n}{C_{n+2}} |\Sigma|^{(n+2)/2} \lambda_{n+2} f(\Sigma),$$

from which the result follows. □

A homogeneous function of the three variables v_{11}, v_{12}, v_{22} is a function satisfying the equation $f(tV) = t^d f(V)$ for all $t > 0$, where d is the degree of homogeneity. By viewing f as a function of v^{11}, v^{12}, v^{22}, f is seen to be homogeneous of degree $-d$.

As a key consequence of Theorem 4, we prove three necessary conditions, two that an EP function must satisfy and the other that its eigenvalues must satisfy.

THEOREM 5. *Let $f(V)$ be an* EP *function and $\{\lambda_n\}$ its associated eigenvalues. Then*

(a) *an* EP *function is homogeneous*, i.e., $f(tV) = t^d f(V)$. *Here, d, the degree of homogeneity of f (with respect to V) can be complex.*

(b) $Df = \lambda |V| f$ *for some constant* λ

(c) *The eigenvalues of an* EP *function satisfy the relation*

$$\frac{\lambda_{n+2}}{\lambda_n}(n)(n-1) = n(n-1) + 2nd + \lambda.$$

PROOF. From Theorem 4,

$$\frac{\lambda_{n+2}}{\lambda_n}(n)(n-1)|V|^{\frac{n+2}{2}} f = D|V|^{\frac{n}{2}} f \qquad n > n_0(f) \tag{7}$$

By Lemma 1, the right side of this equation is

$$fD|V|^{n/2} + |V|^{n/2} Df + 2\left(2\frac{\partial f}{\partial v^{11}}\frac{\partial |V|^{n/2}}{\partial v^{22}} + 2\frac{\partial f}{\partial v^{22}}\frac{\partial |V|^{n/2}}{\partial v^{11}} - \frac{\partial f}{\partial v^{12}}\frac{\partial |V|^{\frac{n}{2}}}{\partial v^{12}}\right) \tag{8}$$

which, using Lemma 2 simplifies to

$$n(n-1)|V|^{(n+2)/2} f + |V|^{n/2} Df - 2n|V|^{n/2+1}\left(v^{11}\frac{\partial f}{\partial v^{11}} + v^{12}\frac{\partial f}{\partial v^{12}} + v^{22}\frac{\partial f}{\partial v^{22}}\right).$$

Equating this to the left side of (7) we get

$$\frac{\lambda_{n+2}}{\lambda_n} n(n-1) f(V) = n(n-1) f(V) + |V|^{-1} Df$$
$$- 2n\left(v^{11}\frac{\partial f}{\partial v^{11}} + v^{12}\frac{\partial f}{\partial v^{12}} + v^{22}\frac{\partial f}{\partial v^{22}}\right).$$

For a fixed V, this is a quadratic polynomial equation in n. Since the polynomials $\{1, n, n(n-1)\}$ are linearly independent we then must have

$$\lambda f(V) = |V|^{-1} Df$$

and

$$-df(V) = v^{11}\frac{\partial f}{\partial v^{11}} + v^{12}\frac{\partial f}{\partial v^{12}} + v^{22}\frac{\partial f}{\partial v^{22}}$$

for constants λ and d. The first of these equations establishes (b). The last equation is Euler's differential expression and shows that $f(V)$ is homogeneous (w.r.t. v^{ij}) with degree of homogeneity $-d$; equivalently, $f(V)$ is homogeneous (w.r.t. v_{ij}) with degree of homogeneity d, proving (a). Finally, with these constants d and λ, one clearly has (c).

5. Polynomial EP functions. In this section we characterize, when $k = 2$, the polynomial EP functions. According to the results of Section 4, we can restrict ourselves to homogeneous EP polynomials.

THEOREM 6. *Consider the subspace H_{ts} of homogeneous polynomials of degree d spanned by*

$$\{|V|^t p_{r,s}(V)\} \quad -s \leqslant r \leqslant s$$

where s and t are nonnegative integers such that $2t + s = d$ and $p_{r,s}(V)$ are the coefficients of α^{s+r} in the expansion of $(v_{11}\alpha^2 + 2v_{12}\alpha + v_{22})^s$. Then

(a) *f is an* EP *polynomial of degree d if and only if $f \in H_{ts}$ for some s and t.*

(b) *The space of homogeneous polynomials of degree d is a direct sum of the* "EP" *subspaces $\{H_{ts}\}$ where s and t run through all nonnegative integers s and t satisfying $2t + s = d$.*

(c) *The eigenvalues of $|V|^t p_{r,s}(V)$ are $\lambda_n = \Pi_{j=1}^{s}(2s + 2t + n - 2j)\Pi_{k=1}^{2t}(2t + n - 1 - k)$.*

PROOF. We know that $f(V) = v_{11}^s$ is EP when s is any nonnegative integer. Also, since

$$T = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}$$

is a nonsingular 2×2 matrix, the function $f_T(V) = f(T'VT) = (v_{11}\alpha^2 + 2v_{12}\alpha + v_{22})^s$ is also, by Theorem 1, an EP function. Now define $2s + 1$ polynomials $p_{r,s}(V)$ by means of the equation

$$(v_{11}\alpha^2 + 2v_{12}\alpha + v_{22})^s = \alpha^s \Sigma_{r=-s}^{s} p_{r,s}(V)\alpha^r.$$

Arguing as at the end of Section 2, we deduce that $\{p_{r,s}(V)\}$ $-s \leqslant r \leqslant s$ are $2s + 1$ EP polynomials of degree s, and conclude that if t and s are nonnegative integers such that $2t + s = d$, then

$$\{|V|^t p_{r,s}(V)\}, -s \leqslant r \leqslant s$$

are $2s + 1$ homogeneous EP polynomials of degree d. The total number of such polynomials is $(d + 1)(d + 2)/2$ which is also just the number of linearly independent homogeneous polynomials (in three variables) of degree d.

The linear independence of these EP homogeneous polynomials may be proved by identifying them with the spherical harmonics defined, for example, by equation (120), page 540 of Courant-Hilbert [2]. In particular, make the formal change of variables

$$\begin{aligned} v_{11} &= -(x + iy) \\ v_{12} &= -z \\ v_{22} &= x - iy. \end{aligned}$$

Then $|V| = v_{11}v_{22} - v_{12}^2 = -(x^2 + y^2 + z^2)$ and so, according to (121) and (122) of Courant, our EP functions $|V|^t p_{l,s}(V)$ are equal to a constant multiple of the harmonic polynomial $r^{2t}P_s^l(\cos\theta)e^{il\phi}$ $-s \leqslant l \leqslant s$ where P_s^l is the associated Legendre function and r, θ, ϕ are the spherical coordinates of x, y, z. Since, as is proved in Courant's discussion, the functions $r^{2t}P_s^l(\cos\theta)e^{il\phi}$ are linearly independent, the same is true for the polynomials $|V|^t P_{l,s}(V)$. This concludes the proof of (a) and (b).

The connection between EP functions and harmonic functions, observed above, will reappear in the sequel (Theorem 9) although in an apparently different form. For the present, we note that $p_{r,s}(V)$ satisfies the equation

$$\Delta p_{r,s}(V) = 0,$$

where Δ is the partial differential operator

$$\Delta = 4\frac{\partial^2}{\partial v_{22}\,\partial v_{11}} - \frac{\partial^2}{\partial v_{12}^2}.$$

(This equation is easily verified by applying the operator to the function $(v_{11}\alpha^2 + 2v_{12}\alpha + v_{22})^s$, the generating function of $p_{r,s}(V)$). The operator Δ can be viewed as an analogue of the classical Laplacian operator. The connection between the operators Δ and D will be noted in Section 6.

To prove (c), we observe that, for fixed s, the eigenvalues of $p_{r,s}(V)$ are, by Theorem 1 and by equating coefficients of powers of α, the same as the eigenvalues of v_{11}^s. The eigenvalues of $|V|^t p_{r,s}(V)$ are obtained using Theorem 3 and Theorem 2; they are given by the expression

$$\lambda_n = \frac{2^s\Gamma(s + t + n/2)\Gamma(n + 2t - 1)}{\Gamma(t + n/2)\Gamma(n - 1)}.$$

Using the relation $\Gamma(x + 1) = x\Gamma(x)$, and multiplying though by 2^s yields

$$\lambda_n = \Pi_{j=1}^{s}(2s + 2t + n - 2j)\Pi_{k=1}^{2t}(2t + n - 1 - k). \qquad \square$$

6. Equivalent necessary and sufficient conditions for EP functions. In this section we conclude the study of eigenfunctions for the case $k = 2$. Our goal is to establish a certain set of equivalent conditions, each necessary and sufficient for a function to be an EP function. More specifically, we prove the equivalence of the following statements.

I. f is an EP function with eigenvalues $\{\lambda_n\}$.

(9) II. f satisfies $D(|V|^{n/2}f) = \dfrac{\lambda_{n+2}}{\lambda_n} n(n-1)|V|^{(n+2)/2}f$.

(10) III. a) f is homogeneous of degree d.

(11) b) $Df = \lambda|V|f$.

IV. f has the series representation given in Theorem 8.

Equation (11), which involves differentiation with respect to v^{ij}, can be expressed in an equivalent form, using the operator Δ where

$$\Delta = 4\frac{\partial^2}{\partial v_{22}\,\partial v_{11}} - \frac{\partial^2}{\partial v_{12}^2}; \tag{12}$$

namely, (computation omitted) for any homogeneous function, of degree d, the equation

$$\Delta f = (\lambda + 2d)|V|^{-1}f \tag{13}$$

is equivalent to equation (11). Thus IIIb. can be replaced by the equivalent condition given in equation (13).

Theorem 4 of the previous section establishes that I implies II; the proof of Theorem 5 establishes that II implies III. Theorem 7 proves the series representation of f, i.e., III implies IV. The implication IV implies I is established by Theorem 8 which shows that any f given by equation (16) is an EP function. This section concludes with miscellaneous results and remarks.

The results of this section may be considered a generalization of those in Section 5. There we proved that any polynomial EP function must belong to exactly one of the "EP subspaces," H_{ts}. To arrive at the analogue of H_{ts} subspaces we proceed as follows. From IIIa and IIIb two numbers, λ and d are associated with any EP function. λ is determined by the eigenvalues of the EP function and d is the degree of homogeneity. Let s be determined by equation (15) and t by the equation, $s + 2t = d$. Let the subspace H_{ts} consist of all functions having the representation given in Theorem 8 or, equivalently, satisfying IIIa and IIIb (in this last formulation the parameters λ and d play the role of s and t). Then: f is EP iff $f \in H_{ts}$ for some (unique) s and t.

At this point, we have reduced the problem of solving the set of partial differential equations (9), to the solution of the single equation (11). We now further reduce equation (11) to a homogeneous partial differential equation.

LEMMA 3. *Let $f(V)$, homogeneous of degree d satisfy the equation*

$$Df = \lambda|V|f.$$

Define $g(V)$ by the equation $f(V) = |V|^t g(V)$, where t satisfies

$$t^2 + \left(\tfrac{1}{2} - d\right)t + \lambda/4 = 0.$$

Then $g(V)$ satisfies the homogeneous equation

$$Dg = 0.$$

PROOF. Apply Lemma 1 to the product FG where $F = |V|^t$, $G = g(V)$ yielding

$$D(f) = D(|V|^t g) = gD(|V|^t) + |V|^t Dg$$
$$+ 2\left(2\frac{\partial|V|^t}{\partial v^{11}}\frac{\partial g}{\partial v^{22}} + 2\frac{\partial|V|^t}{\partial v^{22}}\frac{\partial g}{\partial v^{11}} - \frac{\partial|V|^t}{\partial v^{12}}\frac{\partial g}{\partial v^{12}}\right).$$

Evaluating terms by Lemma 2, and using the fact that $g(V)$ is homogeneous of degree, say s, with respect to v_{ij} and therefore of degree $-s$ with respect to v^{ij}, where $s = d - 2t$, we obtain the equation,

$$Df = (2t)(2t-1)g|V|^{t+1} + |V|^t Dg - 4t|V|^{t+1}(2t-d)g.$$

Further, equation (11) indicates that

$$Df = \lambda|V|^{t+1}g.$$

If we now choose t so that

$$(2t)(2t-1) - 4t(2t-d) = \lambda$$

we obtain $Dg = 0$, i.e., the homogeneous case of equation (11). The quadratic equation which t must satisfy simplifies to

$$t^2 + \left(\tfrac{1}{2} - d\right)t + \lambda/4 = 0. \tag{14}$$

This concludes the proof of Lemma 3.

Written in terms of s, the degree of $g(V)$ with respect to v_{ij}, this quadratic equation is, (since $s = d - 2t$)

$$s^2 - s + \lambda - d(d-1) = 0 \tag{15}$$

The solution of this homogeneous partial differential equation may be obtained by classical methods, and is outlined in the proof of Theorem 7. Further, we shall show that the two different choices of t, i.e., the two solutions of equation (14), will give rise to the same representation of f.

THEOREM 7. *Let D be the partial differential operator defined in Lemma 1. Let f be homogeneous of degree d with respect to v_{ij}. Then the solution of the equation $Df = \lambda|V|f$ is given by a sum of linearly independent functions*

$$f(v_{11}, v_{12}, v_{22}) = |V|^{\frac{d}{2}}\left[\Sigma_{-\infty}^{\infty} A_m P_{-s}^m\left(\frac{v_{11}+v_{22}}{2|V|^{\frac{1}{2}}}\right)\left[\frac{v_{11}-v_{22}-2iv_{12}}{\left((v_{11}-v_{22})^2+4v_{12}^2\right)^{\frac{1}{2}}}\right]^m\right]$$

where A_m are constants and P^m_{-s} is the associated Legendre function of the first kind. Here s is any root of the quadratic $s^2 - s + \lambda - d(d-1) = 0$.

PROOF. Applying Lemma 3, we search for g, the solution of the homogeneous equation.

Put

$$\begin{aligned} v_{11} &= r^{-1}(\cosh\theta - \sinh\theta\cos\phi) \\ v_{12} &= -r^{-1}\sinh\theta\sin\phi \\ v_{22} &= r^{-1}(\cosh\theta + \sinh\theta\cos\phi) \\ \text{where } r^2 &= |V|^{-1}, 0 \leqslant r < \infty, 0 \leqslant \theta < \infty, 0 \leqslant \phi < 2\pi. \end{aligned}$$

The differential equation for g becomes

$$\frac{\partial}{\partial r}\left(-r^2\sinh\theta g_r\right) + \frac{\partial}{\partial\theta}(\sinh\theta g_\theta) + \frac{\partial}{\partial\phi}\frac{(g_\phi)}{\sinh\theta} = 0$$

This is a potential equation for the sphere with $\sinh\theta$ in place of $\sin\theta$. Using the homogeneity of $g(V)$ we have

$$g(r,\phi,\theta) = r^{-s}\eta(\theta,\phi)$$

where $\eta(\theta,\phi)$ satisfies the p.d.e.

$$s(1-s)\sinh\eta(\theta,\phi) + (\sinh\theta\eta_\theta(\theta,\phi))_\theta + \frac{n_{\phi\phi}(\theta,\phi)}{\sinh\theta} = 0.$$

By using the separation of variables technique, and recalling that $g(V)$ must be well behaved, one finds that

$$\eta(\theta,\phi) = \Sigma_{-\infty}^{\infty} A_m P^m_{-s}(\cosh\theta)e^{im\phi}.$$

To obtain the solution to the nonhomogeneous equation $Df = \lambda|V|f$, recall $f(V) = |V|^t g(V)$. Thus we have, using the v_{ij} variables,

$$f(V) = |V|^{d/2}\Sigma_m A_m P^m_{-s}\left(\frac{v_{11}+v_{22}}{2|V|^{\frac{1}{2}}}\right)\left[\frac{v_{11}-v_{22}-2iv_{12}}{\left((v_{11}-v_{22})^2+4v_{12}^2\right)^{\frac{1}{2}}}\right]^m \tag{16}$$

Note that $f(V)$ is essentially unique in equation (16) in the sense that both solutions t of the quadratic equation (14) will give rise to equation (16). This follows since the two roots of equation (15), which is the lower index of the associated Legendre function, sum to $+1$ and $P^m_{-s} = P^m_{-s^*}$ if $s + s^* = 1$. This completes the proof of Theorem 7, establishing that III implies IV.

$f(V)$ may be expressed in terms of the invariants of V. Let $(\xi, 1)$ be an eigenvector of V. Then (omitting the computations) we have

$$f(V) = |V|^{d/2}\Sigma A_m P^m_{-s}\left(\frac{1}{2}\frac{\operatorname{tr} V}{|V|^{\frac{1}{2}}}\right)\left(\frac{\xi - i}{\xi + i}\right)^{2m}. \tag{17}$$

Before proving the next theorem, we state and prove a lemma which concerns in a sense the "prototype" of EP function for $k = 2$.

LEMMA 4. *For* $\mathrm{Re}(b) > -n_0/2$ *and* $\mathrm{Re}(a+b) > -n_0/2$, $E_n(v_{11}^a|V|^b)$ *exists for* $n > n_0$, *and* $f(V) = v_{11}^a|V|^b$ *is an* EP *function, with associated eigenvalues given by*

$$\lambda_n = 2^a \frac{\Gamma(a+b+n/2)\Gamma(n+2b-1)}{\Gamma(b+n/2)\Gamma(n-1)}.$$

PROOF. Since $E_n(v_{11}^a|V|^b) = \dfrac{C_n}{C_{n+2b}} E_{n+2b}(v_{11}^a)$, the result follows from Theorem 3.

We now conclude with the proof that IV implies I, establishing the equivalence of the four statements.

THEOREM 8. *For any* $\{A_m\}$, *let* $f(V)$ *be given by the equation*

$$f(V) = |V|^{d/2}\Sigma_m A_m P_{-s}^m\left(\frac{v_{11}+v_{22}}{2|V|^{\frac{1}{2}}}\right)\left[\frac{v_{11}-v_{22}-2iv_{12}}{\left((v_{11}-v_{22})^2+4v_{12}^2\right)^{\frac{1}{2}}}\right]^m$$

where $\mathrm{Re}\dfrac{d-s+1}{2} > -n_0/2$ *and* $\mathrm{Re}\dfrac{d+s-1}{2} \geqslant -n_0/2$, *so that, by Lemma* 4, $E(v_{11}^{s-1}|V|^{\frac{d-s+1}{2}})$ *exists. Then*: $f(V)$ *is an* EP *function.*

PROOF. Consider the identity

$$|V|^{d/2}\Sigma B_m P_{-s}^m\left(\frac{v_{11}+v_{22}}{2|V|^{\frac{1}{2}}}\right)\left[\frac{v_{11}-v_{22}+2iv_{12}}{\left((v_{11}-v_{22})^2+4v_{12}^2\right)^{\frac{1}{2}}}\right]^m e^{2im\omega}$$
$$= |V|^{\frac{d-s+1}{2}}\left(v_{11}\cos^2\omega + 2v_{12}\cos\omega\sin\omega + v_{22}\sin^2\omega\right)^{s-1}.$$

By the hypotheses of Theorem 8, Lemma 4, and the transformation theorem (Theorem 1) the left-hand side of this last equation is an EP function for each ω. Arguing as in Section 3, we deduce that the coefficients

$$\left\{|V|^{d/2}P_{-s}^m\left(\frac{v_{11}+v_{22}}{2|V|^{\frac{1}{2}}}\right)\left[\frac{v_{11}-v_{11}+2iv_{12}}{\left((v_{11}-v_{22})^2+4v_{12}\right)^{\frac{1}{2}}}\right]^m\right\}$$

are also EP functions. This concludes the proof of the theorem and establishes the implication IV $\rightarrow$ I. Thus, all four statements I, II, III, IV have been shown to be equivalent.

The restriction α an integer or half integer in Theorem 2, which stated that an EP function multiplied by $|V|^\alpha$ results in an EP function, may now be removed for $k=2$. This extension of Theorem 2 follows readily from Theorem 8 since $|V|^{d/2}$ is a factor in the representation of f.

The connection between EP functions of the variables v_{11}, v_{12}, v_{22} and harmonic functions of three variables x,y,z has been noted in the present and preceding sections. The following theorem is an analogue of a theorem in Courant-Hilbert [2], page 515.

THEOREM 9. *If $f(V)$ is an* EP *function, then so is $f(V^{-1})$.*

PROOF. Since

$$V^{-1} = |V|^{-1}\begin{pmatrix} v_{22} & -v_{12} \\ -v_{21} & v_{11} \end{pmatrix} = |V|^{-1}T'VT, \qquad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we then have, since $f(V)$ is homogeneous of degree d

$$f(V^{-1}) = |V|^{-d}f_T(V).$$

The assertion of the theorem now follows from Theorem 1 and Theorem 2 (extended above to arbitrary α).

James' zonal polynomials $C_{(k_1k_2)}(V)$ [6] are also related to harmonic polynomials. For $k = 2$, $C_{(k_1k_2)}(V) = \text{const}\, x|V|^{d/2}P_{k_1-k_2}\left(\frac{\operatorname{tr} V}{2|V|^{\frac{1}{2}}}\right)$ where $k_1 + k_2 = d$. (This also follows from (17).) Indeed, for $k = 2$, one can easily show that EP functions depending only on the eigenvalues of V, are eigenfunctions of the Laplace-Beltrami operator.

The concluding theorem gives the distribution of the prototype EP function $v_{11}^a|V|^b$.

THEOREM 10. *$v_{11}^a|V|^b$ is distributed as $\sigma_{11}^a|\Sigma|^b 2^{a+2b}Z_1{}^bZ_2{}^{a+b}$ where Z_1 and Z_2 are independent gamma variates, $Z_1 \sim G\left(\frac{n-1}{2}\right)$ and $Z_2 \sim G\left(\frac{n}{2}\right)$.*

PROOF. Immediate, using Wilks' moment method [8].

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