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# EXACT BAHADUR EFFICIENCIES FOR TESTS OF THE MULTIVARIATE LINEAR HYPOTHESIS[1]

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The notion of Bahadur efficiency is used to compare multivariate linear hypothesis tests based on six criteria: (1) Roy's largest root, (2) the likelihood ratio test, (3) the Lawley-Hotelling trace, (4) Pillai's trace, (5) Wilks' $U$, and (6) Olson's statistic. Bahadur exact slope is computed for each statistic as a function of noncentrality parameters using results for probabilities of large deviations. The likelihood ratio test is shown to be asymptotically optimal in the sense that its slope attains the optimal information value, and the remaining tests are shown not to be asymptotically optimal. Inequalities are derived for the slopes showing order of preference.

**1. Introduction and summary.** We consider a hypothesis testing problem of the multivariate linear model (see, e.g., Anderson, 1958; Roy, 1957; Roy, Gnanadesikan and Srivastava, 1971). Let $\{X_i\}$ be a sequence of independent random vectors, $X_i \sim N_p(BZ_i, \Sigma)$. That is, $X_i$ is a $p$-variate normal variable with covariance matrix $\Sigma$ and expectation $EX_i = BZ_i$, $i = 1, 2, \cdots$. The vectors $Z_i$, each with $q$-components, are known, and the $p \times q$ matrix $B$ and the $p \times p$ matrix $\Sigma$ which is assumed to be positive definite (p.d.) are unknown. The null hypothesis is $H_0 : B_1 = 0_{p\times q_1}$ and the alternative is $H_1 : B_1 \neq 0_{p\times q_1}$, where $B = (B_1, B_2)$ such that $B_1$ has $q_1$ columns and $B_2$ has $q_2$ columns ($q_1 + q_2 = q$). A typical example of this multivariate linear model is the MANOVA problem. Some invariant procedures for testing $H_0$ are (see, e.g., Anderson, 1958, pages 222–223; Roy, Gnanadesikan and Srivastava, 1971, page 73): (1) the likelihood ratio criterion; (2) Roy's largest root; (3) the Lawley-Hotelling trace; (4) Pillai's trace (Pillai, 1955); (5) Wilks' $U$ (Wilks, 1932); and (6) Olson's criterion (Olson, 1974). These criteria are denoted by $W$, $R$, $T$, $V$, $U$, and $S$, respectively; their expressions as functions of some sample matrix are given in Section 2 below.

The monotonicity property of the power functions of tests (1) through (4) has been studied by Das Gupta, Anderson and Mudholkar (1964), Eaton and Perlman (1974), and Srivastava (1964). It follows from Theorem 1 of Schwartz (1967) (see also Kiefer and Schwartz, 1965) that $W$, $R$, $T$, and $V$ are admissible for finite samples. An associate editor has conjectured that $U$ and $S$ are inadmissible using

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Theorem 2 of Schwartz (1967). Further, it is known from numerical work of Fujikoshi (1970), Hart and Money (1976), Ito (1962), Lee (1971), Mikhail (1965), Pillai and Jayachandran (1967), Pillai and Sudjana (1975), Roy, Gnanadesikan and Srivastava (1971), Schatzoff (1966), and perhaps others, that for fixed sample sizes there is no UMP invariant test for the multivariate linear hypothesis problem. For large sample comparison, Gnanadesikan et al (1965) mentioned comparisons of asymptotic efficiencies of tests $W$, $R$ and $T$ using Bahadur's (1960) approximate measure (but to the author's knowledge, no published results are available). Gleser (1966) compared approximate asymptotic efficiencies of the LR test and Hotelling's $T^2$ test in the context of the modified $T^2$ problem discussed by Rao (1946). But as indicated by Bahadur (1967), knowledge of the exact measure is preferred when available. The main purpose of this paper is to compare criteria (1) through (6) using the notion of exact Bahadur efficiency suggested by Bahadur (1967).

The exact Bahadur efficiency between two test criteria is defined as the ratio of the exact slopes of the two sequences of statistics associated with the two criteria. The computation of exact slopes, which is also known as the theory of large deviation probabilities, has been discussed by authors including Bahadur (1971; and references therein), Bahadur and Raghavachari (1972), Book (1975), Gleser (1964), Hwang and Klotz (1975), Killeen, Hettmansperger and Sievers (1972), and Sievers (1976). Recently, Koziol (1978), Sievers (1975), and Steinebach (1976) discussed probabilities of large deviations for multivariate test statistics. However, application of known theories to the multivariate linear hypothesis problem is not straightforward, since the test statistics that we consider are functions of correlated variables with complicated distributions.

In this paper, the exact slope associated with each of the six test criteria mentioned previously is obtained as a function of noncentrality parameters for any fixed alternative and dimension. Inequalities for these slopes are derived showing order of preference. It is noted that among the six criteria considered, only the likelihood ratio criterion is asymptotically optimal in the sense that its exact slope attains the optimal information value for all alternatives. This provides an example to the general results of Bahadur (1967, 1971) and Bahadur and Raghavachari (1972) that under certain regularity conditions, likelihood ratios have optimal slopes. However, readers should notice that the results of this paper are asymptotic ones, and do not necessarily imply anything about what happens in finite samples. Therefore, the uniform asymptotic optimality of the likelihood ratio test should not be interpreted as an unqualified endorsement of the LR test in finite sample situations.

**2. Further notation and assumptions.** For large sample theory, we assume that the sequence $\{Z_i\}$ satisfies the following:

ASSUMPTION 1. All entries of $Z_i$ $(i = 1, 2, \cdots)$ are uniformly bounded.

ASSUMPTION 2. For each $n \geqslant p + q$, the $q \times n$ matrix

$$Z(n) = (Z_1, \cdots, Z_n)$$

is of rank $q$.

ASSUMPTION 3. There is a $q \times q$ p. d. matrix $M$ satisfying

$$\lim_{n\to\infty} n^{-1}Z(n)Z'(n) = M. \tag{2.1}$$

Assumption 1 can be satisfied whenever experimental designs are performed in a bounded domain. Assumption 2 implies that $Z(n)Z'(n)$ is positive definite for $n \geqslant p + q$ (see, e.g., Graybill, 1961, Theorem 1.24), and hence it has an inverse. In the context of the MANOVA problem with $q$ groups, Assumption 3 is equivalent to saying that $n_i/n \to \gamma_i$, $0 < \gamma_i < 1$, $\Sigma_{i=1}^q \gamma_i = 1$, where $n_i$ is the sample size of the $i$th group and $n = \Sigma_{i=1}^q n_i$.

The parameter space under the general model is

$$\Omega = \{(B_1, B_2, \Sigma) : B_1(p \times q_1), B_2(p \times q_2), \Sigma(p \times p) \text{ p.d.}\},$$

and under the null hypothesis $H_0$, it is

$$\Omega_0 = \{(0, B_{20}, \Sigma_0) : 0(p \times q_1), B_{20}(p \times q_2), \Sigma_0(p \times p) \text{ p.d.}\}.$$

Clearly $\Omega_0 \subset \Omega$ and $\Omega_1 = \Omega - \Omega_0$ is nonempty. Let

$$X(n) = (X_1, \cdots, X_n),$$

then based on $X(n)$, $n \geqslant p + q$, the MLE of $B$ and $\Sigma$ over $\Omega$ are given (see, e.g., Anderson, 1958, page 181) by

$$\hat{B}(n) = X(n)Z'(n)[Z(n)Z'(n)]^{-1}, \tag{2.2}$$

$$\hat{\Sigma}(n) = n^{-1}[X(n) - BZ(n)][X(n) - BZ(n)]', \tag{2.3}$$

respectively. Corresponding to the partition of $B$, we partition

$$\hat{B}(n) = (\hat{B}_1(n), \hat{B}_2(n)) \tag{2.4}$$

such that $\hat{B}_1(n)$ has $q_1$ columns;

$$Z(n)Z'(n) = \begin{pmatrix} A_{11}(n) & A_{12}(n) \\ A_{21}(n) & A_{22}(n) \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{2.5}$$

such that $A_{11}(n)$, $M_{11}$ are $q_1 \times q_1$ matrices, and $A_{22}(n)$, $M_{22}$ are $q_2 \times q_2$ matrices. Further, we define

$$A_{11\cdot2}(n) = A_{11}(n) - A_{12}(n)A_{22}^{-1}(n)A_{21}(n), \tag{2.6}$$

$$M_{11\cdot2} = M_{11} - M_{12}M_{22}^{-1}M_{21}. \tag{2.7}$$

The existence of the inverses $A_{22}^{-1}$ and $M_{22}^{-1}$ follow from Assumptions 2 and 3. Further, $A_{11\cdot2}(n)$ and $M_{11\cdot2}$ are positive definite and by (2.1)

$$\lim_{n\to\infty} n^{-1}A_{11\cdot2}(n) = M_{11\cdot2}. \tag{2.8}$$

Throughout this paper we use the following notation.

$$
\begin{aligned}
H(n) &= \hat{B}_1(n)A_{11\cdot 2}(n)\hat{B}_1'(n);\\
G(n) &= n\hat{\Sigma}(n);\\
K &= B_1 M_{11\cdot 2} B_1';\\
\Theta &= (B, \Sigma) = (B_1, B_2, \Sigma);\\
s &= \min(p, q_1);\\
v_i &= \text{the } i\text{th largest nonzero characteristic root of the}\\
&\qquad \text{random matrix } H(n)G^{-1}(n),\ 0 < v_s \leqslant \cdots \leqslant v_1 < \infty;\\
w_i &= v_i/(1+v_i),\ i = 1, \cdots, s;\\
\lambda_i &= \text{the } i\text{th largest characteristic root of } K\Sigma^{-1},\\
&\qquad 0 \leqslant \lambda_p \leqslant \cdots \leqslant \lambda_1 < \infty;\\
\theta_i &= \lambda_i/(1+\lambda_i),\ i = 1, 2, \cdots, p.
\end{aligned} \tag{2.9}
$$

We note that the $\lambda_i$ are also characteristic roots of the limiting matrix of noncentrality parameters $\Lambda = \frac{1}{2}\Sigma^{-\frac{1}{2}}K\Sigma^{-\frac{1}{2}}$ in the canonical form of the multivariate linear model (see, e.g., Anderson, 1958, Section 8.11 and Fujikoshi, 1970). In this paper, we call $K\Sigma^{-1}$ the matrix of noncentrality parameters.

The test statistics associated with the six criteria mentioned in Section 1 can be expressed in terms of $v_i$ and $w_i$ as follows:

The likelihood ratio, $W_n = \prod_{i=1}^{s}(1 + v_i)$;
Roy's largest root, $R_n = v_1$;
The Lawley-Hotelling trace, $T_n = \sum_{i=1}^{s} v_i$;
Pillai's statistic, $V_n = \sum_{i=1}^{s} w_i$;
Wilks' $U$, $U_n = \prod_{i=1}^{s} w_i$;
Olson's statistic, $S_n = \prod_{i=1}^{s} v_i$.

In each case the null hypothesis $H_0$ is rejected for large values. We suffix each statistic with $n$ to indicate that the associated statistic is based on a sample of size $n$. But suffixes on $v_i$ and $w_i$ are omitted for simplicity. Since when $s = 1$ all test criteria are reduced to the familiar $F$-test (see, e.g., Anderson, 1958, Theorem 8.5.3), in this paper we are mainly interested in cases where $s \geqslant 2$, i.e., we take $p \geqslant 2$ and $q_1 \geqslant 2$ throughout.

It is known (see, e.g., Anderson, 1958, Sections 13.2.3 and 13.2.4; Pillai, 1956) that under the null hypothesis $H_0$, the joint pdf of $w_i$ $(i = 1, \cdots, s)$ has the form

$$
c(s, m^*, n^*)\prod_{i=1}^{s} w_i^{m^*}(1 - w_i)^{n^*}\prod_{i<j}(w_i - w_j) \tag{2.10}
$$

for $0 < w_s \leqslant \cdots \leqslant w_1 < 1$, where

$$c(s, m^*, n^*) = \Pi^{\frac{1}{2}s}\Pi_{i=1}^{s}\Gamma\left(\frac{2m^* + 2n^* + s + 2 + i}{2}\right) \Big/ \left[\Gamma\left(\frac{2m^* + 1 + i}{2}\right)\Gamma\left(\frac{2n^* + 1 + i}{2}\right)\Gamma\left(\frac{i}{2}\right)\right],$$

$$m^* = \tfrac{1}{2}(|p - q_1| - 1) \tag{2.11}$$

$$n^* = \tfrac{1}{2}(n - p - q - 1).$$

In terms of $v_i$, (2.10) becomes

$$c(s, m^*, n^*)\Pi_{i=1}^{s} v_i^{m^*}(1 + v_i)^{-(n^* + m^* + s + 1)}\Pi_{i<j}(v_i - v_j) \tag{2.12}$$

for $0 < v_s \leqslant \cdots \leqslant v_1 < \infty$. Note that the value of $m^*$ can either be $-\frac{1}{2}$ or nonnegative. Since we are interested in large sample theory, we take $n^* \geqslant 0$ throughout the paper.

Further, we use $P_0[\cdot]$ to indicate the probability operator under the null hypothesis $H_0$. In view of (2.10) or (2.12), the operator $P_0[\cdot]$ is well defined.

**3. Preliminaries.** Using Stirling's approximation to the gamma function (e.g., Feller, 1968, page 66), it is easily seen that for any fixed number $r$

$$\lim_{u\to\infty} u^{-1}\ln[\Gamma(u)/\Gamma(u + r)] = 0. \tag{3.1}$$

Consequently, for fixed $s$ and $m^*$

$$\lim_{n\to\infty} n^{-1}\ln c(s, m^*, n^*) = 0. \tag{3.2}$$

LEMMA 3.1. *In the framework of the multivariate linear hypothesis, let* $\Theta = (B, \Sigma)$, *then*

$$\lim_{n\to\infty}\hat{B}(n) = B \qquad \text{a.e. } P_\Theta, \tag{3.3}$$

$$\lim_{n\to\infty}\hat{\Sigma}(n) = \Sigma \qquad \text{a.e. } P_\Theta. \tag{3.4}$$

*The convergence is element-wise.*

PROOF. (3.4) follows directly from Theorem 8.2.2 of Anderson (1958) and the strong law of large numbers (see, e.g., Rao, 1973, page 115). To show (3.3), we note

$$\hat{B}(n) - B = E(n)Z'(n)[Z(n)Z'(n)]^{-1}, \tag{3.5}$$

where $E(n) = (e_1, \cdots, e_n)$ such that $e_1, \cdots, e_n$ are i.i.d., $e_i \sim N_p(0, \Sigma)$. Define $D(n) = [Z(n)Z'(n)]^{-\frac{1}{2}}Z(n)$ and write

$$[Z(n)Z'(n)]^{-1}Z(n)E'(n) = n^{\frac{1}{2}}[Z(n)Z'(n)]^{-\frac{1}{2}}n^{-\frac{1}{2}}D(n)E'(n).$$

Applying Theorem 3 of Chow (1966), we obtain $n^{-\frac{1}{2}}D(n)E'(n) \to 0$ a.s. This and (2.1) imply that the expression in (3.5) converges to the zero matrix a.s. □

REMARK 3.1. Anderson and Taylor (1976) obtained result (3.3) for $p = 1$ under a condition similar to (2.1).

COROLLARY 3.1.1. *Let* $\Theta = (B, \Sigma)$, *then*

$$\lim_{n\to\infty} n^{-1}H(n) = K \qquad \text{a.s. } P_\Theta, \tag{3.6}$$

$$\lim_{n\to\infty} n^{-1}G(n) = \Sigma \qquad \text{a.s. } P_\Theta. \tag{3.7}$$

COROLLARY 3.1.2. *Let* $\Theta = (B, \Sigma)$, *then*

$$\begin{aligned}
&\lim_{n\to\infty} W_n = \Pi_{i=1}^{s}(1 + \lambda_i) \quad \text{as. } P_\Theta;\\
&\lim_{n\to\infty} R_n = \lambda_1 \quad \text{a.s. } P_\Theta;\\
&\lim_{n\to\infty} T_n = \Sigma_{i=1}^{s}\lambda_i \quad \text{a.s. } P_\Theta;\\
&\lim_{n\to\infty} V_n = \Sigma_{i=1}^{s}\theta_i \quad \text{a.s. } P_\Theta;\\
&\lim_{n\to\infty} U_n = \Pi_{i=1}^{s}\theta_i \quad \text{a.s. } P_\Theta;\\
&\lim_{n\to\infty} S_n = \Pi_{i=1}^{s}\lambda_i \quad \text{a.s. } P_\Theta.
\end{aligned} \tag{3.8}$$

The last corollary follows from Corollary 3.1.1 since as $n \to \infty$, $H(n)G^{-1}(n) \to K\Sigma^{-1}$ a.e. $P_\Theta$, and the $i$th largest characteristic root $H(n)G^{-1}(n)$ converges to that of $K\Sigma^{-1}$ a.e. $P_\Theta$. The following lemma is a consequence of Anderson (1958, page 223) and a result of Mitra (1970).

LEMMA 3.2. *Let* $\mathbf{a}$ *be any fixed nonzero p-vector, then under the null hypothesis* $H_0$, *the random variable*

$$y(n) = \mathbf{a}'G(n)\mathbf{a}/\mathbf{a}'\left[G(n) + H(n)\right]\mathbf{a}$$

*has beta* $[\frac{1}{2}(n - q), \frac{1}{2} q_1]$ *distribution independent of* $\mathbf{a}$.

We state below a particular case of Minkowski's inequality (see, e.g., Hardy, Littlewood and Polya, 1967, 2.13.8).

LEMMA 3.3. *If* $d_1, \cdots, d_m$ *are nonnegative numbers, then the following inequality holds*:

$$\Pi_{i=1}^{m}(1 + d_i)^{1/m} \geqslant 1 + (\Pi_{i=1}^{m} d_i)^{1/m}. \tag{3.9}$$

**4. The Kullback-Leibler information number.** Denote $x = (X_1, \ldots, X_n)$ with $X_i$ as defined in Section 1. For each $\Theta = (B_1, B_2, \Sigma) \in \Omega$ and $\Theta_0 = (0, B_{20}, \Sigma_0) \in \Omega_0$, we define (according to Bahadur and Raghavachari, 1972)

$$K_n(x; \Theta, \Theta_0) = n^{-1}\ln\{\Pi_{i=1}^{n}[f_i(X_i; \Theta)/f_i(X_i; \Theta_0)]\}, \tag{4.1}$$

where

$$f_i(X_i; \Theta) = (2\pi)^{-\frac{1}{2}}|\Sigma|^{-\frac{1}{2}}\exp\left[-\tfrac{1}{2}\mathrm{tr}(X_i - BZ_i)(X_i - BZ_i)'\Sigma^{-1}\right],$$

$$f_i(X_i; \Theta_0) = (2\pi)^{-\frac{1}{2}}|\Sigma_0|^{-\frac{1}{2}}\exp\left[-\tfrac{1}{2}\mathrm{tr}(X_i - B_{20}Z_i^{(2)})(X_i - B_{20}Z_i^{(2)})'\Sigma_0^{-1}\right],$$

$B = (B_1, B_2)$ and $Z_i' = (Z^{(1)\prime}{}_i, Z^{(2)\prime}{}_i)$ such that $Z_i^{(1)}$ has $q_1$ components and $Z_i^{(2)}$ has $q_2$ components. Then using Assumptions 1 and 3, and Kolmogorov's strong law of large numbers (see, e.g., Rao, 1973, page 114), it can be verified that

$$\lim_{n\to\infty} K_n(x; \Theta, \Theta_0) = I(\Theta, \Theta_0) \quad \text{a.s. } P_\Theta \tag{4.2}$$

and $0 \leqslant I(\Theta, \Theta_0) \leqslant \infty$, where

$$I(\Theta, \Theta_0) = -\tfrac{1}{2}\ln|\Sigma| - \tfrac{1}{2}p + \tfrac{1}{2}\ln|\Sigma_0| + \tfrac{1}{2}\mathrm{tr}\{[\Sigma + BMB' - B_1M_{12}B_{20}' - B_2M_{22}B_{20}' - B_{20}M_{21}B_1' - B_{20}M_{22}B_2' + B_{20}M_{22}B_{20}']\Sigma_0^{-1}\}.$$

Further, if $J(\Theta)$ denotes the inf of $I(\Theta, \Theta_0)$ over $\Theta_0 \in \Omega_0$, then

$$J(\Theta) = \tfrac{1}{2}\ln[\Pi_{i=1}^s(1 + \lambda_i)]. \tag{4.3}$$

Results (4.2) and (4.3) show that Corollary 3 of Bahadur and Raghavachari (1972) is applicable to the multivariate linear hypothesis problem. Hence we have the following.

THEOREM 4.1. *In the context of the multivariate linear hypothesis model defined in Sections* 1 *and* 2, *let* $t_n$ *be a test statistic based on* $x = (X_1, \cdots, X_n)$ *for testing* $H_0$ *against* $H_1$ *such that* $H_0$ *is rejected for large values of* $t_n$. *Let* $L_n(x) = \sup_{\Theta_0 \in \Omega_0} P_{\Theta_0}[t_n \geqslant x]$ *be the level attained by* $t_n$ *for* $x$. *Then for any fixed alternative* $\Theta = (B_1, B_2, \Sigma) \in \Omega_1$,

$$-\lim\inf_{n\to\infty} n^{-1}\ln L_n(x) \leqslant \tfrac{1}{2}\ln[\Pi_{i=1}^s(1 + \lambda_i)] \quad \text{a.s. } P_\Theta. \tag{4.4}$$

Inequality (4.4) implies that in the multivariate linear hypothesis model, the exact slope (if it exists) of any test sequence for any fixed $\Theta \in \Omega_1$ can not exceed $2J(\Theta)$. Accordingly, a test criterion is said to be asymptotically optimal if its associated slope equals the optimal information value $2J(\Theta)$ for every $\Theta \in \Omega_1$.

**5. Exact slopes for the six test sequences.** Using Bahadur's (1967, 1971) definition for exact slopes of a test sequence, we have the following results.

THEOREM 5.1. *In the framework of the multivariate linear hypothesis model, for each* $\Theta = (B_1, B_2, \Sigma) \in \Omega_1$, *the exact slopes of* $\{W_n\}$, $\{R_n\}$, $\{T_n\}$, $\{V_n\}$, $\{U_n\}$ *and* $\{S_n\}$ *exist and equal*

$$\begin{aligned}
C_W(\Theta) &= \ln[\Pi_{i=1}^s(1 + \lambda_i)] \\
C_R(\Theta) &= \ln(1 + \lambda_1), \\
C_T(\Theta) &= \ln(1 + \Sigma_{i=1}^s\lambda_i), \\
C_V(\Theta) &= \ln[1 - (\Sigma_{i=1}^s\theta_i)/s]^{-s}, \\
C_U(\Theta) &= \ln[1 - (\Pi_{i=1}^s\theta_i)^{1/s}]^{-s}, \\
C_S(\Theta) &= \ln[1 + (\Pi_{i=1}^s\lambda_i)^{1/s}]^s,
\end{aligned}$$

*respectively, where* $s$, $\lambda_i$, $\theta_i$ *are defined in Section* 2. *Consequently, the* LR *test is asymptotically optimal and the remaining ones are not.*

This theorem follows from Theorem 7.2 of Bahadur (1971) using Corollary 3.1.2 of this paper and Lemmas 5.1 through 5.6 below.

LEMMA 5.1. *Let* $\alpha_W(n, t) = P_0[W_n \geqslant t]$. *Then for each* $d > 1$,

$$\lim_{n\to\infty} n^{-1} \ln \alpha_W(n, d) = -\tfrac{1}{2}\ln d. \tag{5.1}$$

PROOF. Let $U(n) = W_n^{-1} = |G(n)|/|G(n) + H(n)|$. Consul (1969) shows that under the null hypothesis $H_0$, $U(n)$ has exact pdf given by

$$f(u) = k(n)u^{\frac{1}{2}(n-p-q-1)}g(u), \qquad 0 \leqslant u < 1,$$

where

$$k(n) = \Pi_{j=1}^{p}\Gamma\left[\tfrac{1}{2}(n+1-q_2-j)\right]/\Gamma\left[\tfrac{1}{2}(n+1-q-j)\right]$$

and

$$g(u) = G_{p,p}^{p,0}\left[u : \begin{matrix} \frac{1}{2}q_1, \frac{1}{2}(q_1+1), \cdots, \frac{1}{2}(q_1+p-1) \\ 0, \frac{1}{2}, 1, \cdots, \frac{1}{2}(p-1) \end{matrix}\right]$$

is the Meijer $G$-function (see e.g., Erdélyi et al., 1953), which is known to be continuous on the closed interval $[0, d^{-1}]$, $0 < d^{-1} < 1$, and independent of $n$. Therefore, for each $d > 1$,

$$\begin{aligned} \alpha_W(n, d) &= P_0[W_n \geqslant d] = P_0[U(n) \leqslant d^{-1}] \\ &= k(n)\textstyle\int_0^{d^{-1}} u^{\frac{1}{2}(n-p-q-1)}g(u)\,du \\ &\leqslant k(n)d^{-\frac{1}{2}(n-p-q-1)}\textstyle\int_0^{d^{-1}} g(u)\,du. \end{aligned} \tag{5.2}$$

By (3.1), $n^{-1}\ln k(n) \to 0$, as $n \to \infty$. This and the finiteness of the integral on the right-hand side of (5.2) imply

$$\limsup_{n\to\infty} n^{-1}\ln \alpha_W(n, d) \leqslant -\tfrac{1}{2}\ln d. \tag{5.3}$$

This inequality and results (3.8) and (4.3) show that the conditions required by Corollary 5 of Bahadur and Raghavachari (1972) are satisfied by the sequence $\{\frac{1}{2}\ln W_n\}$. Hence as a consequence of that corollary, (5.1) holds. □

LEMMA 5.2. *Let* $\alpha_R(n, t) = P_0[R_n \geqslant t]$. *Then for each* $t > 0$,

$$\lim_{n\to\infty} n^{-1}\ln \alpha_R(n, t) = -\tfrac{1}{2}\ln(1+t).$$

PROOF. For $t > 0$, define

$$D_1 = \{(w_1, \cdots, w_s) : 0 < w_s \leqslant \cdots \leqslant w_1 < 1, w \geqslant t/(1+t)\}.$$

Using (2.10) with $w_i = v_i/(1+v_i)$, we have

$$\begin{aligned} \alpha_R(n, t) &= P_0[v_1 \geqslant t] = P_0[w_1 \geqslant t/(1+t)] \\ &= c(s, m^*, n^*)\textstyle\int_{D_1} \cdots \int \Pi_{i=1}^{s} w_i^{m^*}(1-w_i)^{n^*}\Pi_{i<j}(w_i - w_j)\Pi_{i=1}^{s}dw_i \\ &\leqslant c(s, m^*, n^*)(1+t)^{-n^*}\textstyle\int \cdots \int_{D_1}\Pi_{i=1}^{s} w_i^{m^*}\Pi_{i=1}^{s}dw_i. \end{aligned} \tag{5.5}$$

Note that the multiple integral on the right hand side of (5.5) is bounded by $(m^* + 1)^{-s}$. This and (3.2) imply

$$\limsup_{n\to\infty} n^{-1}\ln \alpha_R(n, t) \leqslant -\tfrac{1}{2}\ln(1 + t). \tag{5.6}$$

On the other hand, by the definition of $v_1$,

$$v_1 = \sup_{\mathbf{x}\in R^p} \frac{\mathbf{x}'H(n)\mathbf{x}}{\mathbf{x}'G(n)\mathbf{x}} \geqslant \frac{\mathbf{a}'H(n)\mathbf{a}}{\mathbf{a}'G(n)\mathbf{a}}$$

for any nonzero $p$-vector $\mathbf{a}$, where $R^p$ represents the $p$-dimensional Euclidean space. Hence for $t > 0$,

$$\begin{aligned}\alpha_R(n, t) &\geqslant P_0\left[\frac{\mathbf{a}'H(n)\mathbf{a}}{\mathbf{a}'G(n)\mathbf{a}} \geqslant t\right]\\ &= P_0\left[\frac{\mathbf{a}'G(n)\mathbf{a}}{\mathbf{a}'[G(n) + H(n)]\mathbf{a}} \leqslant (1 + t)^{-1}\right]\\ &= \frac{\Gamma\left[\frac{1}{2}(n - q + q_1)\right]}{\Gamma\left[\frac{1}{2}(n - q)\right]\Gamma\left(\frac{1}{2}q_1\right)}\int_0^{(1+t)^{-1}} y^{\frac{1}{2}(n-q)-1}(1 - y)^{\frac{1}{2}q_1 - 1}\,dy.\end{aligned} \tag{5.7}$$

The last equality follows from Lemma 3.2. From (5.7) and recalling $q_1 \geqslant 2$,

$$\alpha_R(n, t) \geqslant k_1(n)(1 + t)^{-\frac{1}{2}(n-q)}, \tag{5.8}$$

where $k_1(n) = 2(n - q)^{-1}[t/(1 + t)]^{\frac{1}{2}q_1 - 1}\{\Gamma[\frac{1}{2}(n - q_2)]/\Gamma[\frac{1}{2}(n - q)]\Gamma(\frac{1}{2}q_1)\}$. Since (3.1) implies $n^{-1}\ln k_1(n) \to 0$ as $n \to \infty$, (5.8) gives

$$\liminf_{n\to\infty} n^{-1}\ln \alpha_R(n, t) \geqslant -\tfrac{1}{2}\ln(1 + t). \tag{5.9}$$

Combining (5.6) and (5.9) gives (5.4). □

LEMMA 5.3. *Let $\alpha_T(n, t) = P_0[T_n \geqslant t]$. Then for each $t > 0$,*

$$\lim_{n\to\infty} n^{-1}\ln \alpha_T(n, t) = -\tfrac{1}{2}\ln(1 + t). \tag{5.10}$$

PROOF. For each $t > 0$, (5.10) is established by applying (5.1) and (5.4) to the following inequalities:

$$P_0\left[\Pi_{i=1}^s(1 + v_i) \geqslant 1 + t\right] \geqslant P_0\left[\Sigma_{i=1}^s v_i \geqslant t\right] \geqslant P_0\left[v_1 \geqslant t\right]. \qquad \square$$

LEMMA 5.4. *Let $\alpha_V(n, t) = P_0[V_n \geqslant t]$. Then for each $t$, $0 < t < s$,*

$$\lim_{n\to\infty} n^{-1}\ln \alpha_V(n, t) = \tfrac{1}{2}\ln(1 - t/s)^s. \tag{5.11}$$

PROOF. Denote

$$D_2 = \{(w_1, \cdots, w_s) : 0 < w_s \leqslant \cdots \leqslant w_1 < 1,\ \Sigma_{i=1}^s w_i \geqslant t\}.$$

Using (2.10) and applying the arithmetic-geometric mean inequality to the product $\Pi_{i=1}^s(1 - w_i)$, we have

(5.12)

$$\alpha_V(n, t) \leqslant c(s, m^*, n^*)\int\cdots\int_{D_2}\Pi_{i=1}^s w_i^{m^*}\left[1 - (\Sigma_{i=1}^s w_i)/s\right]^{sn^*}\Pi_{i<j}(w_i - w_j)\Pi_{i=1}^s\,dw_i.$$

Consider the following set of symmetric transformations $F$ (see, e.g., Van der Waerden, 1949, page 78)

$$\begin{aligned} u_1 &= w_1 + \cdots + w_s \\ u_2 &= w_1 w_2 + \cdots + w_{s-1} w_s \\ &\vdots \\ u_s &= w_1 \cdots w_s \end{aligned} \tag{5.13}$$

and denote the image set of $D_2$ under $F$ by $D_u$. Then $F$ is one-to-one between $D_2$ and $D_u$, and

$$\Pi_{i=1}^{s} du_i = \Pi_{i<j}(w_i - w_j)\Pi_{i=1}^{s} dw_i. \tag{5.14}$$

(For $s = 2$, see, e.g., Hotelling, 1951.) Now (5.12) is equivalent to

$$\alpha_V(n, t) \leqslant c(s, m^*, n^*)\int \cdots \int_{D_u} u_s^{m^*}(1 - u_1/s)^{sn^*}\Pi_{i=1}^{s} du_i. \tag{5.15}$$

Since the coordinates of any vector $(u_1, \cdots, u_s)$ in $D_u$ satisfy

$$t < u_1 < s, \qquad 0 < u_i < b(i, s), \qquad i = 2, \cdots, s, \tag{5.16}$$

where $b(i, s) = s/[i!(s - i)!]$ is the binomial coefficient, (5.15) leads to

$$\alpha_V(n, t) \leqslant c(s, m^*, n^*)(1 - t/s)^{sn^*} Q_1, \tag{5.17}$$

where $Q_1 = \int \cdots \int_{D_u} u_s^{m^*}\Pi_{i=1}^{s} du_i \leqslant (m^* + 1)^{-1}(s - t)\Pi_{i=2}^{s} b(i, s)$. Using (3.2) and the finiteness of $Q_1$, (5.17) gives

$$\limsup_{n\to\infty} n^{-1} \ln \alpha_V(n, t) \leqslant \tfrac{1}{2} \ln(1 - t/s)^s. \tag{5.18}$$

On the other hand, since in $D_2$ $w_1 \geqslant w_i$ $(i = 2, \cdots, s)$, replacing $w_i$ $(i = 2, \cdots, s)$ by $w_1$ in the factor $(1 - w_i)^{n^*}$ of (2.10) gives

(5.19)

$$\alpha_V(n, t) \geqslant c(s, m^*, n^*)\int \cdots \int_{D_2}\left[\Pi_{i=1}^{s} w_i^{m^*}\right](1 - w_1)^{sn^*}\Pi_{i<j}(w_i - w_j)\Pi_{i=1}^{s} dw_i.$$

For each $h$ satisfying $0 < h < m_1$ with $m_1 = \min\{(s - t)/s, (s - 1)t/s\}$, define

$$D_h^{(1)} = \left\{(w_1, \cdots, w_s) : \frac{t}{s} - \frac{h}{s-1} < w_s \leqslant \cdots \leqslant w_2 < \frac{t}{s}, \frac{t}{s} + h < w_1 < 1\right\}.$$

Clearly, $D_h^{(1)}$ is nonempty and it is a subset of $D_2$. Putting $a = (t/s) - h/(s - 1)$ and replacing $D_2$ by $D_h^{(1)}$ in (5.19), we get

$$\begin{aligned} \alpha_V(n, t) &\geqslant c(s, m^*, n^*)\int_{t/s+h}^{1} w_1^{m^*}(1 - w_1)^{sn^*}\int_a^{t/s} \cdots \int_a^{w_{s-1}}\Pi_{i=1}^{s} w_i^{m^*} \\ &\quad \Pi_{i<j}(w_i - w_j)\, dw_s \cdots dw_1 \geqslant k_2(n)\left(1 - \frac{t}{s} - h\right)^{sn^*+1} Q_2, \end{aligned} \tag{5.20}$$

where $k_2(n) = c(s, m^*, n^*)\ \min\{1, ((t/s) + h)^{m^*}\}(sn^* + 1)^{-1}$, and $Q_2 = \int_a^{t/s} \cdots \int_a^{w_{s-1}}\Pi_{i=2}^{s} w_i^{m^*}((t/s) + h - w_i)\Pi_{i=2}^{s-1}\Pi_{j=i+1}^{s}(w_i - w_j)\, dw_s \cdots dw_2$. Since $n^{-1} \ln k_2(n) \to 0$ as $n \to \infty$, and $Q_2$ is positive and independent of $n$, (5.20) implies

$$\liminf_{n\to\infty} n^{-1} \ln \alpha_V(n, t) \geqslant \tfrac{1}{2} \ln\left(1 - \frac{t}{s} - h\right)^s. \tag{5.21}$$

Since (5.21) holds for all $h$ in the interval $E_1 = (0, m_1)$, it follows that

$$\liminf_{n\to\infty} n^{-1} \ln \alpha_V(n, t) \geq \sup_{h\in E_1} \tfrac{1}{2} \ln\left(1 - \frac{t}{s} - h\right)^s = \tfrac{1}{2} \ln\left(1 - \frac{t}{s}\right)^s.$$

The last inequality and (5.18) give (5.11). □

LEMMA 5.5. *Let* $\alpha_U(n, t) = P_0[U_n \geq t]$. *Then for each* $t$, $0 < t < 1$,

$$\lim_{n\to\infty} n^{-1} \ln \alpha_U(n, t) = \tfrac{1}{2} \ln(1 - t^{1/s})^s. \tag{5.22}$$

PROOF. By the arithmetic-geometric mean inequality,

$$\alpha_U(n, t) = P_0\left[\Pi_{i=1}^s w_i \geq t\right] \leq P_0\left[\Sigma_{i=1}^s w_i \geq st^{1/s}\right] = \alpha_V(n, st^{1/s}),$$

where $\alpha_V(n, \cdot)$ is defined in Lemma 5.4. With $t$ replaced by $st^{1/s}$ in (5.18), we get

$$\limsup_{n\to\infty} n^{-1} \ln \alpha_U(n, t) \leq \tfrac{1}{2} \ln(1 - t^{1/s})^s. \tag{5.23}$$

Let $D_3 = \{(w_1, \cdots, w_s) : 0 < w_s \leq \cdots \leq w_1 < 1, \Pi_{i=1}^s w_i > t\}$. By definition of $U_n$ and (2.10), one obtains

$$\alpha_U(n, t) \geq c(s, m^*, n^*) m_2 \int \cdots \int_{D_3} (1 - w_1)^{sn^*} \Pi_{i<j}(w_i - w_j) \Pi_{i=1}^s dw_i \tag{5.24}$$

where $m_2 = \min(1, t^{m^*})$. For each $h$ satisfying $t^{1/s} < h < 1$, define

$$D_h^{(2)} = \{(w_1, \cdots, w_s) : t^{1/s} \cdot h^{1/(s-1)} < w_s \leq \cdots \leq w_2 < t^{1/s},\quad t^{1/s}/h < w_1 < 1\},$$

which is nonempty and is a subset of $D_3$. As in (5.20), (5.24) leads to

$$\alpha_U(n, t) \geq k_3(n)(1 - t^{1/s}/h)^{sn^*} \cdot Q_3, \tag{5.25}$$

defining $a = t^{1/s} \cdot h^{1/(s-1)}$, $k_3(n) = c(s, m^*, n^*) m_2 (sn^* + 1)^{-1}$,

$$Q_3 = \int_a^{t^{1/s}} \cdots \int_a^{w_{s-1}} \Pi_{i=2}^s (t^{1/s}/h - w_i) \Pi_{i=2}^{s-1} \Pi_{j=i+1}^s (w_i - w_j)\, dw_s \cdots dw_2.$$

Since $Q_3$ is positive and independent of $n$, and since by (3.2) $n^{-1} \ln k_3(n) \to 0$ as $n \to \infty$, with $E_2 = (t^{1/s}, 1)$, (5.25) yields

$$\liminf_{n\to\infty} n^{-1} \ln \alpha_U(n, t) \geq \sup_{h\in E_2} \tfrac{1}{2} \ln(1 - t^{1/s}/h)^s = \tfrac{1}{2} \ln(1 - t^{1/s})^s. \tag{5.26}$$

Inequalities (5.23) and (5.26) establish (5.22). □

LEMMA 5.6. *Let* $\alpha_S(n, t) = P_0[S_n \geq t]$. *Then for each* $t > 0$,

$$\lim_{n\to\infty} n^{-1} \ln \alpha_S(n, t) = -\tfrac{1}{2} \ln(1 + t^{1/s})^s. \tag{5.27}$$

PROOF. By (3.9),

$$P_0\left[\Pi_{i=1}^s v_i \geq t\right] \leq P_0\left[\Pi_{i=1}^s (1 + v_i) \geq (1 + t^{1/s})^s\right].$$

Applying (5.3) to the above inequality gives

$$\limsup_{n\to\infty} n^{-1} \ln \alpha_S(n, t) \leq -\tfrac{1}{2} \ln(1 + t^{1/s})^s. \tag{5.28}$$

On the other hand, using (2.12) with $(1 + v_i)$ replaced by $(1 + v_1)$, we get

$$(5.29)\quad \alpha_S(n, t) \geqslant c(s, m^*, n^*)\int \cdots \int_{D_4}\left[\Pi_{i=1}^s v_i^{m^*}\right](1 + v_1)^{-s(n^*+m^*+s+1)} \times \Pi_{i<j}(v_i - v_j)\Pi_{i=1}^s\, dv_i$$

where the domain of integration is defined by

$$D_4 = \left\{(v_1, \cdots, v_s) : 0 < v_s \leqslant \cdots \leqslant v_1, \Pi_{i=1}^s v_i \geqslant t\right\}.$$

As in preceding lemmas, for each $h$ satisfying $0 < h < 1$, we define

$$D_h^{(3)} = \left\{(v_1, \cdots, v_s) : t^{1/s} \cdot h^{1/(s-1)} < v_s \leqslant \cdots \leqslant v_2 < t^{1/s}, t^{1/s}/h < v_1 < \infty\right\},$$

which is nonempty and $D_h^{(3)} \subset D_4$. Denoting $a = t^{1/s} \cdot h^{1/(s-1)}$, and replacing $D_4$ in (5.29) by $D_h^{(3)}$, we get

$$(5.30)\quad \alpha_S(n, t) \geqslant c(s, m^*, n^*)\int_{t^{1/s}/h}^{\infty}\int_a^{t^{1/s}} \cdots \int_a^{v_{s-1}}\left(\Pi_{i=1}^s v_i^{m^*}\right)(1 + v_1)^{-s(n^*+m^*+s+1)} \times \Pi_{i<j}(v_i - v_j)\, dv_s \cdots dv_1.$$

The cases $m^* > 0$ and $m^* = -\frac{1}{2}$ are discussed separately as follows.

CASE 1. $m^* > 0$. Then, in the domain of integration $D_h^{(3)}$, $\Pi_{i=1}^s v_i^{m^*} > t^{m^*}$. Hence from (5.30) we have

$$(5.31)\qquad \alpha_S(n, t) \geqslant k_4(n)(1 + t^{1/s}/h)^{-s(n^*+m^*+s+1)+1} \cdot Q_4,$$

where $k_4(n) = c(s, m^*, n^*)t^{m^*}[s(n^* + m^* + s + 1) - 1]^{-1}$,

$$Q_4 = \int_a^{t^{1/s}} \cdots \int_a^{v_{s-1}}\Pi_{i=2}^s(t^{1/s}/h - v_i)\Pi_{i=2}^{s-1}\Pi_{j=i+1}^s(v_i - v_j)\, dv_s \cdots dv_2.$$

CASE 2. $m^* = -\frac{1}{2}$. In this case $v_1^{-\frac{1}{2}} \geqslant (1 + v_1)^{-\frac{1}{2}}$, (5.30) gives

$$(5.32)\qquad \alpha_S(n, t) \geqslant k_5(n)(1 + t^{1/s}/h)^{-s\left(n^*+s+\frac{1}{2}\right)+\frac{1}{2}} \cdot Q_5,$$

where $k_5(n) = c(s, m^*, n^*)[s(n^* + s + \frac{1}{2}) - \frac{1}{2}]^{-1}$

$$Q_5 = \int_a^{t^{1/s}} \cdots \int_a^{v_{s-1}}\Pi_{i=2}^s v_i^{-\frac{1}{2}}(t^{1/s}/h - v_i)\Pi_{i=2}^{s-1}\Pi_{j=i+1}^s(v_i - v_j)\, dv_s \cdots dv_2.$$

Since $Q_4$ and $Q_5$ are positive and independent of $n$, and since $n^{-1}\ln k_4(n) \to 0$, $n^{-1}\ln k_5(n) \to 0$ as $n \to \infty$, (5.31) and (5.32) show that no matter the value of $m^*$,

$$\liminf_{n\to\infty} n^{-1}\ln \alpha_S(n, t) \geqslant \sup_{h\in(0,1)} -\tfrac{1}{2}\ln\left(1 + t^{1/s}/h\right)^s = -\tfrac{1}{2}\ln(1 + t^{1/s})^s.$$

This and (5.28) prove the lemma. □

**6. Orderings of the six slopes.** For any fixed alternative $\Theta \in \Omega_1$, let $C_W(\Theta)$, $C_R(\Theta)$, $C_T(\Theta)$, $C_U(\Theta)$, $C_V(\Theta)$, and $C_S(\Theta)$ be the slopes defined in Theorem 5.1. It can be verified that the following inequalities hold:

(i) $C_R(\Theta) \leqslant C_T(\Theta) \leqslant C_W(\Theta)$;
(ii) $C_U(\Theta) \leqslant C_V(\Theta) \leqslant C_W(\Theta)$;
(iii) $C_U(\Theta) \leqslant C_S(\Theta) \leqslant C_W(\Theta)$.

Definite orderings of the six slopes have been exhausted by inequalities (i)–(iii). Other than these orderings, each slope could be larger than others for some alternatives (Hsieh, 1976, Tables 1–15). In particular, if the noncentrality matrix $K\Sigma^{-1}$ has only one nonzero characteristic root, i.e., if $\lambda_1 \neq 0, \lambda_2 = \cdots = \lambda_p = 0$, then

(iv) $C_U(\lambda_1) = C_S(\lambda_1) < C_V(\lambda_1) < C_R(\lambda_1) = C_T(\lambda_1) = C_W(\lambda_1)$;

if the $p$ characteristic roots are equal, i.e., if $\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda$, then

(v) $C_R(\lambda) < C_T(\lambda) < C_U(\lambda) = C_S(\lambda) = C_V(\lambda) = C_W(\lambda)$.

Since, as indicated by Bahadur (1967), the most important property of an exact slope is its value in the immediate vicinity of the null hypothesis, we investigate the leading term of the Taylor expansion for each slope. One of the referees pointed out that it follows from Theorem 5.1, as $\lambda_1 \to 0$,

$$C_W(\Theta) = \Sigma_{i=1}^s \lambda_i + o(\lambda_1);$$
$$C_R(\Theta) = \lambda_1 + o(\lambda_1);$$
$$C_T(\Theta) = \Sigma_{i=1}^s \lambda_i + o(\lambda_1);$$
$$C_V(\Theta) = \Sigma_{i=1}^s \lambda_i + o(\lambda_1);$$
$$C_U(\Theta) = s(\Pi_{i=1}^s \lambda_i)^{1/s} + o(\lambda_1);$$
$$C_S(\Theta) = s(\Pi_{i=1}^s \lambda_i)^{1/s} + o(\lambda_1).$$

Therefore, in terms of exact local slopes, $W$, $T$, and $V$ are equivalent; $R$, $U$, and $S$ are inferior to these; and $U$ and $S$ are equivalent.

We note here that the approximate slope of the LR criterion for the modified $T^2$-problem obtained by Gleser (1966) essentially takes the same form as the exact slope of the LR criterion discussed in this paper; and the approximate slope of Hotelling's $T^2$-test for the same problem of Gleser takes the same form as the local slope of $T$ criterion of this paper. Further, when $q = 2$, $q_1 = 1$ (or $p = 1$), results of Theorem 5.1 reduce to those obtained by Bahadur (1971, Example 5.1), Killeen and Hettmansperger (1972) and Klotz (1967).

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