The Annals of Statistics
1977, Vol. 5, No. 3, 541–543

AN ESTIMATOR FOR THE SPECTRAL DENSITY OF A STATIONARY TIME SEQUENCE

BY A. C. BUTCHER

The City University, London

An estimator for the spectral density of a stationary time sequence is introduced. The form of the estimator is motivated by the analogy with the summability theory of Fourier series. A theorem relating to rate of consistency is proved. The effect of a mean correction is considered.

Let $\{X_n, -\infty < n < \infty\}$ be a real random sequence having the properties that $EX_n = 0$ all n, $R(n) = E(X_{m+n}X_m)$ is independent of m, and $\sum |R(n)|$ converges. Then the spectral density function (Doob (1953), page 476) is the continuous function $f(\lambda)$ given by the uniformly convergent series

$$f(\lambda) = \sum_{-\infty}^{\infty} R(n) \exp(-2\pi in\lambda) . \tag{1}$$

We have (Parzen (1957), equations 2.2, 2.3),

$$E(X_iX_jX_kX_l) = R(i-j)R(k-l) + R(i-k)R(j-l) + R(i-l)R(j-k) + Q ,$$

where Q is the fourth order cumulant. It is supposed that X_n is fourth order stationary, i.e., $Q = Q(j-i, k-i, l-i)$ and that $\sum\sum\sum |Q(l, m, n)|$ converges.

Let $\{a_n\}$ be a nonincreasing sequence of real numbers such that $(a_n)^{1/n}$ tends to unity and suppose that $\sum a_n x^n$ tends to infinity as x tends to $1-0$. Define

$$\phi_n(x) = \sum_{r=n}^{\infty} a_r x^r , \qquad \phi(x) = \phi_1(x) . \tag{2}$$

A family of estimators for $f(\lambda)$ is defined by

$$U_n = 2 \sum_{r=0}^{n-1} \{\phi_{r+1}(x)/\phi(x)\} R_n(r)\varepsilon_r \cos 2\pi r\lambda , \tag{3}$$

where $\varepsilon_0 = \frac{1}{2}$, $\varepsilon_r = 1$ for $r \geqq 1$, and

$$R_n(r) = (1/n) \sum_{i=1}^{n-r} X_i X_{i+r} . \tag{4}$$

THEOREM. (a) *If* $\sum_{n=1}^{\infty} |R(n)|(a_1 + a_2 + \cdots + a_n)$ *converges, then* $E\{U_n - f(\lambda)\} = O\{1/\phi(x)\} + O(k/n) + O\{1/(a_1 + a_2 + \cdots + a_k)\}$, *where* $k < n$, *as* $k, n \to \infty$, $x \to 1-0$, *uniformly with respect to* λ,

(b) Var $U_n = O\{n^{-1}(1-x)^{-1}\}$, *as* $n \to \infty$, $x \to 1-0$, *uniformly with respect to* λ.

PROOF. *For* (a) we have

$$f(\lambda) - EU_n = 2 \sum_{r=1}^{n-1} R(r)\{(n-r)/n\}\{1 - \phi_{r+1}(x)/\phi(x)\} \cos 2\pi r\lambda + 2 \sum_{r=1}^{n-1} R(r)(r/n) \cos 2\pi r\lambda + 2 \sum_{r=n}^{\infty} R(r) \cos 2\pi r\lambda$$

Received August 1974; revised August 1976.

AMS 1970 *subject classification.* Primary 62M15.

Key words and phrases. Stationary time series, rate of consistency, spectral density.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Statistics.* www.jstor.org

using (1) and (4), consequently

$$|f(\lambda) - EU_n| \leqq 2 \sum_{r=1}^{n-1} |R(r)|(a_1 x + a_2 x^2 + \cdots + a_r x^r)/\phi(x) + 2(k/n) \sum_{r=1}^{k-1} |R(r)| + 2 \sum_{r=k}^{\infty} |R(r)|$$

for $k < n$ using (2). But

$$\sum_{r=k}^{\infty} |R(r)| \leqq (a_1 + a_2 + \cdots + a_k)^{-1} \sum_{r=k}^{\infty} |R(r)|(a_1 + a_2 + \cdots + a_r) ,$$

and the result follows.

For (b) we have

$$\begin{aligned}\operatorname{Var} U_n &= 4 \sum_{r=0}^{n-1} \sum_{s=0}^{n-1} \varepsilon_r \varepsilon_s \{\phi_{r+1}(x)/\phi(x)\}\{\phi_{s+1}(x)/\phi(x)\} \operatorname{Cov}\{R_n(r)R_n(s)\} \cos 2\pi r\lambda \cos 2\pi s\lambda \\ &\leqq 4 \sum_{r=0}^{n-1} \sum_{s=0}^{n-1} x^{r+s} |\operatorname{Cov}\{R_n(r), R_n(s)\}|\end{aligned}$$

using (2). From (4)

$$\begin{aligned}&|\operatorname{Cov}\{R_n(r), R_n(s)\}| \\ &\quad\leqq (1/n) \sum_{i=-n}^{n} |R(i)||R(i + s - r)| + (1/n) \sum_{i=-n}^{n} |R(i + s)||R(i - r)| \\ &\qquad + (1/n) \sum_{i=-n}^{n} |Q(r, i, i + s)| .\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{Var} U_n &\leqq (4/n) \sum_{r=0}^{\infty} x^r \sum_{i=-n}^{n} |R(i)| \sum_{s=0}^{n-1} |R(i + s - r)| \\ &\quad + (4/n) \sum_{r=0}^{\infty} x^r \sum_{i=-n}^{n} |R(i - r)| \sum_{s=0}^{n-1} |R(i + s)| \\ &\quad + (4/n) \sum_{s=0}^{\infty} x^s \sum_{i=-n}^{n} \sum_{r=0}^{n-1} |Q(r, i, i + s)| \\ &\leqq 8\{\sum|R|\}^2/\{n(1 - x)\} + 4(\sum\sum\sum |Q|)/\{n(1 - x)\} .\end{aligned}$$

EXAMPLE. Let $x = 1 - n^{-s}$, then from (b), $\operatorname{Var} U_n = O(n^{s-1})$ and we require $0 < s < 1$. With Parzen's condition (Parzen (1957), page 339, equation 5.7d) $a_1 + a_2 + \cdots + a_n = n^q$ and then $\phi(x) = (1 - x) \sum_{r=1}^{\infty} r^q x^r$. We have $1/\phi(x) < l^{-q} x^{-l}$, $l = 1, 2, \cdots$ therefore if k and l have the order of n^s, $E\{U_n - f(\lambda)\}^2 = O(n^{s-1}) + O(n^{-2qs})$, supposing $q < 1$. Choosing $s = 1/(1 + 2q)$, we have that $n^{2q/(1+2q)} E\{U_n - f(\lambda)\}^2$ remains bounded, in agreement with Parzen's result (Parzen (1957), page 339, Theorems 5A, 5B).

With regard to the mean correction, let $Y_n = M + X_n$ where M is a constant and X_n has the properties given above. Then the theorem remains valid with the addition of a term $O\{n^{-1}(1 - x)^{-1}\}$ in (a), when $R_n(r)$ is replaced by

$$(1/n) \sum_{i=1}^{n-r} (Y_i - M_n)(Y_{i+r} - M_n)$$

where

$$M_n = (Y_1 + Y_2 + \cdots + Y_n)/n .$$

After some algebra it may be seen that the numerical value of the terms introduced into the expression for $f(\lambda) - EU_n$ cannot exceed $6 \sum |R| n^{-1}(1 - x)^{-1}$. Also it may be seen that the additional terms introduced into $\operatorname{Cov}\{R_n(r), R_n(s)\}$ are at most $O(1/n^2)$, consequently the additional terms in $\operatorname{Var} U_n$ are at most $O(n^{-2}(1 - x)^{-2})$.

REFERENCES

DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.

PARZEN, E. (1957). On consistent estimates of the spectrum of a stationary time series. *Ann. Math. Statist.* **28** 329–348.

DEPARTMENT OF MATHEMATICS
THE CITY UNIVERSITY
ST. JOHN STREET
LONDON EC1V 4PB
ENGLAND