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AN UPPER BOUND OF RESOLUTION IN SYMMETRICAL FRACTIONAL FACTORIAL DESIGNS

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For the minimum weight in s-ary (n, p) linear codes (or for the maximum resolution in s^{n-p} designs), an upper bound has been obtained by Plotkin [4] where s is a prime power.

The purpose of this note is to obtain an improvement of the Plotkin's upper bound. Main result is as follows: when $p \geqq 2$, the maximum resolution $R_p(n, s)$ of any s^{n-p} design satisfies the following:

$$\begin{aligned} R_p(n, s) &= s^{p-1}q & &\text{if} \quad m = 0, 1 \\ &\leqq s^{p-1}q + [s^{p-2}(s-1)(m-1)/(s^{p-1}-1)] & & \\ & & &\text{if} \quad m = 2, 3, \cdots, s^{p-1} \\ &\leqq s^{p-1}q + [(s-1)m/s] & &\text{if} \quad m = s^{p-1}+1, \cdots, N-1, \end{aligned}$$

where $[x]$ is the greatest integer not exceeding x, $n = qN + m$ and $N = (s^p - 1)/(s - 1)$.

1. Introduction and summary. Consider s^n factorial designs with n factors each at s levels, where s is a prime power. A $1/s^p$ fraction of an s^n factorial design is called an s^{n-p} fractional factorial design, or briefly, an s^{n-p} design. An s^{n-p} design in which no t-factor or lower order interaction is aliased with another u-factor or lower order interaction is called an s^{n-p} design of resolution $t + u + 1$ [1]. It is desirable to obtain, in s^{n-p} designs, an s^{n-p} design having the maximum resolution. In the special case $p = 1$ it is easy to see that the maximum resolution of any s^{n-1} design is equal to n; for $p = 2$ it is well known [5] that the maximum resolution of any s^{n-2} design is equal to $[ns/(s + 1)]$ where $[x]$ is the greatest integer not exceeding x. But it is very difficult, in general, to obtain an s^{n-p} design having the maximum resolution.

Plotkin [4] obtained the following upper bound for resolution in s^{n-p} designs (or for the minimum distance in s-ary (n, p) linear codes):

In s^{n-p} designs, the resolution R cannot exceed $[s^{p-1}(s-1)n/(s^p-1)]$, i.e.,

$$R \leqq [s^{p-1}(s-1)n/(s^p-1)] . \tag{1.1}$$

The purpose of this paper is to obtain an improvement of the Plotkin's upper bound. (Plotkin's upper bound works for any symmetrical fractional factorial being regular or irregular. But in this paper, we consider the comparison of our upper bound and Plotkin's upper bound within the limits of the regular case.) The main result is given as Theorem 4.1.

In Sections 4 and 5, it is shown that (i) our upper bound, BF, is less than or

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equal to Plotkin's upper bound, BP, for any s^{n-p} design and (ii) there are examples such that $BF < BP$.

2. The resolution of the s^{n-p} design D_F. An s^{n-p} (fractional factorial) design D_F may be defined as the set of the s^{n-p} treatment combinations $\mathbf{x}'=(x_1, x_2, \cdots, x_n)$ such that x_i's are elements of the finite field $GF(s)$ and $F\mathbf{x} = \mathbf{c}$ over $GF(s)$ for some $p \times n$ matrix F whose elements are those of $GF(s)$ and whose rank is p over $GF(s)$.

It is well known [1, 3] that the resolution R_F of the s^{n-p} design D_F is given by

$$R_F = \min \{r(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \mathscr{P}(F')\} \tag{2.1}$$

where $\mathscr{P}(F')$ denotes the $(p-1)$-dimensional subspace (or the $(p-1)$-flat) of an $(n-1)$-dimensional projective space $PG(n-1, s)$ over $GF(s)$ generated by the linear closure of column vectors of the matrix F' and $r(\boldsymbol{\xi})$ denotes the number of nonzero elements in the n components of the n-vector $\boldsymbol{\xi}$. The maximum resolution R of s^{n-p} designs can be expressed as

$$R = \max_F R_F \, . \tag{2.2}$$

3. The modular representation of the s^{n-p} design D_F. In order to obtain an upper bound for the resolution R of s^{n-p} designs, we shall use the concept of the modular representation which was introduced by Slepian [6] in connection with problems in coding theory and independently by Burton and Connor [2] for the case of factorial designs.

Let α be a primitive element of $GF(s^p)$. Then every nonzero element of $GF(s^p)$ may be represented as a power of α, say α^i $(0 \leqq i \leqq s^p - 2)$, and every point of $PG(p-1, s)$ may be represented as (α^j) $(0 \leqq j < N)$ where $N = (s^p - 1)/(s-1)$. Two points (α^{i_1}) and (α^{i_2}) represent the same point when and only when $i_1 \equiv i_2 \bmod N$. Since every nonzero element α^i $(i = 0, 1, 2, \cdots, s^p - 2)$ of $GF(s^p)$ can also be represented uniquely by a polynomial, $x_0 + x_1\alpha + \cdots + x_{p-1}\alpha^{p-1}$, in α, of degree at most $p-1$, with coefficients from $GF(s)$. It is clear that every point of $PG(p-1, s)$ has also such a representation. In the following, we shall denote by $\boldsymbol{\alpha}_i'$ $(i = 0, 1, \cdots, N-1)$ the vector $(x_{i0}, x_{i1}, \cdots, x_{ip-1})$ such that

$$(\alpha^i) = x_{i0} + x_{i1}\alpha + \cdots + x_{ip-1}\alpha^{p-1} \, . \tag{3.1}$$

The vectors $\boldsymbol{\alpha}_i$ $(i = 0, 1, \cdots, N-1)$ may be regarded as the N points of $PG(p-1, s)$ and every nonzero p-vector $\mathbf{f}_i$ of F can be taken as a points of $PG(p-1, s)$.

DEFINITION 3.1. The modular representation of the s^{n-p} design D_F is defined as the vector $(z, M_0, M_1, \cdots, M_{N-1})$ where z and M_i denote the number of the zero column vectors $\mathbf{0}$ in F and the number of the column vectors in F corresponding to the point (α^i), respectively. We call the vector $\mathbf{M}' = (M_0, M_1, \cdots, M_{N-1})$ the modular vector of the s^{n-p} design D_F.

The maximum resolution R for s^{n-p} designs can also be represented by

$$R = \max_F \min_{\boldsymbol{\xi} \in \mathscr{P}(F')} r(\boldsymbol{\xi}) \tag{3.2}$$

where $\mathscr{P}(F') = \{\boldsymbol{\xi}_i \mid \boldsymbol{\xi}_i = F'\boldsymbol{\alpha}_i, i = 0, 1, \cdots, N-1\}$.

4. A new upper bound for the resolution R of s^{n-p} designs. For an improvement of the Plotkin's upper bound, we have the following theorem.

THEOREM 4.1. *In the case $p \geqq 2$, the maximum resolution $R_p(n, s)$ of s^{n-p} designs is given by*

$$
\begin{aligned}
R_p(n, s) &= s^{p-1}q && \text{if } m = 0, 1 \\
&\leqq s^{p-1}q + [s^{p-2}(s-1)(m-1)/(s^{p-1}-1)] && \\
& && \text{if } m = 2, 3, \cdots, s^{p-1} \\
&\leqq s^{p-1}q + [(s-1)m/s] && \text{if } m = s^{p-1}+1, \cdots, N-1,
\end{aligned}
\tag{4.1}
$$

where $n = qN + m$ and $N = (s^p - 1)/(s - 1)$.

In order to prove the above theorem, we prepare the following two lemmas given by Robillard [5].

LEMMA 4.1. *Between the vector $\mathbf{r}' = (r(\boldsymbol{\xi}_0), r(\boldsymbol{\xi}_1), \cdots, r(\boldsymbol{\xi}_{N-1}))$, or, briefly, $\mathbf{r}' = (r_0, r_1, \cdots, r_{N-1})$, and the modular representation $(z, \mathbf{M}')$ of the s^{n-p} design D_F where $\boldsymbol{\xi}_i = F'\boldsymbol{\alpha}_i$ and $\mathbf{M}' = (M_0, M_1, \cdots, M_{N-1})$, there is the following relation:*

$$\mathbf{r} = A\mathbf{M} \tag{4.2}$$

where $z + \sum_{i=0}^{N-1} M_i = n$ and A is the $N \times N$ matrix whose elements a_{ij} are given by

$$
\begin{aligned}
a_{ij} &= 0 && \text{if the } j\text{th point } (\alpha^j) \text{ in } PG(p-1, s) \text{ lies on} \\
& && \text{the } (p-2)\text{-flat } V_{p-2}(i) \\
&= 1 && \text{otherwise,}
\end{aligned}
\tag{4.3}
$$

where $V_{p-2}(i) = \{\boldsymbol{\xi} \mid \boldsymbol{\alpha}_i'\boldsymbol{\xi} = 0, \boldsymbol{\xi} \in PG(p-1, s)\}$.

LEMMA 4.2. *For any integer l, the values of nonnegative integers z and m_i $(i = 1, 2, \cdots, l)$ which maximizes the minimum of l integers m_i $(i = 1, 2, \cdots, l)$ subject to the condition $z + \sum_{i=1}^{l} m_i = n$ are given by*

$$
\begin{aligned}
&z = 0 && && z = 0 \\
&m_i = q \quad (i = 1, 2, \cdots, l) && \text{or} && m_i = q + 1 \quad (i = 1, 2, \cdots, r) \\
& && && m_j = q \quad (j = r+1, \cdots, l)
\end{aligned}
\tag{4.4}
$$

according as $n = ql$ or $n = ql + r$ $(1 \leqq r \leqq l - 1)$ for some nonnegative integers q and r.

From Lemma 4.2, it is easy to see that the maximum value of the minimum of l nonnegative integers m_i $(i = 1, \cdots, l)$ subject to the condition $z + \sum_{i=1}^{l} m_i = n$, $z \geqq 0$, is given by $[n/l]$ for any positive integers l and n.

The following concept plays an important role in proving Theorem 4.1.

A symmetric (v, k, λ)-configuration is defined as an arrangement of v elements $x_1, x_2, \cdots, x_v$ into v sets $B_1, B_2, \cdots, B_v$ such that (i) each set contains exactly k distinct elements and (ii) each pair of elements occurs together in exactly λ sets. We define the incidence matrix of the symmetric (v, k, λ)-configuration to be the matrix $A = [a_{ij}]$ $(i, j = 1, 2, \cdots, v)$ where $a_{ij} = 1$ or 0 according as the ith element x_i belongs to the jth set B_j or not. Then it is easy to see that each element occurs in exactly k distinct sets.

PROOF OF THEOREM 4.1. It is well known [5] that the matrix A in Lemma 4.1 is the incidence matrix of a symmetric (v, k, λ)-configuration with $v = N$, $k = s^{p-1}$ and $\lambda = s^{p-2}(s-1)$. From the definition of a symmetric (v, k, λ)-configuration, (4.2) and $\sum_{i=0}^{N-1} M_i = n$, we have

$$\sum_{i=0}^{N-1} r_i = \sum_{j=0}^{N-1} M_j \sum_{i=0}^{N-1} a_{ij} = kn \tag{4.5}$$

$$\sum_{i=0}^{N-1} r_i a_{il} = \sum_{j=0}^{N-1} M_j \sum_{i=0}^{N-1} a_{ij} a_{il} = (k-\lambda)M_l + \lambda n \tag{4.6}$$

for $l = 0, 1, \cdots, N-1$. Let us denote by $i(0, l), i(1, l), \cdots, i(k-1, l)$ the k integers i such that $a_{il} = 1$ for l and by $i(k, l), i(k+1, l), \cdots, i(N-1, l)$ the $N-k$ integers i such that $a_{il} = 0$ for l. Then from (4.5) and (4.6), we obtain the following conditions for the vector $\mathbf{r}' = (r_0, r_1, \cdots, r_{N-1})$.

$$\mathscr{C}(M_l, n): \begin{cases} \sum_{j=0}^{k-1} r_{i(j,l)} = (k-\lambda)M_l + \lambda n \\ \sum_{j=k}^{N-1} r_{i(j,l)} = (k-\lambda)(n-M_l) \qquad l = 0, 1, \cdots, N-1 . \end{cases} \tag{4.7}$$

We define the following $N+3$ sets of vectors $\mathbf{M}' = (M_0, M_1, \cdots, M_{N-1})$ and $\mathbf{r}' = (r_0, r_1, \cdots, r_{N-1})$ respectively.

$$\mathscr{M} = \{\mathbf{M} \mid \mathbf{M}' = (M_0, M_1, \cdots, M_{N-1}),\ \sum_{i=0}^{N-1} M_i = n,\ M_i \geqq 0\}$$
$$\mathscr{R} = \{\mathbf{r} \mid \mathbf{r}' = (r_0, r_1, \cdots, r_{N-1}),\ \sum_{i=0}^{N-1} r_i = kn,\ r_i \geqq 0\}$$
$$\mathscr{R}(A, n) = \{\mathbf{r} \mid \mathbf{r} = A\mathbf{M},\ \mathbf{M} \in \mathscr{M}\}$$
$$\mathscr{R}(M_l, n) = \{\mathbf{r} \mid \mathscr{C}(M_l, n)\} \qquad l = 0, 1, \cdots, N-1 .$$

Then from the definition of above sets, we have

$$\mathscr{R}(A, n) \subset \bigcup_{\mathscr{M}} \bigcap_{l=0}^{N-1} \mathscr{R}(M_l, n) \subset \mathscr{R} . \tag{4.8}$$

From (3.2), (4.8) and Lemma 4.2, the resolution R for s^{n-p} designs satisfies the following inequality.

$$\begin{aligned} R &= \max_{\mathscr{R}(A,n)} \min \{r_0, r_1, \cdots, r_{N-1}\} \\ &\leqq \max_{\mathscr{M}} \max_{\bigcap_{l=0}^{N-1} \mathscr{R}(M_l,n)} \min \{r_0, r_1, \cdots, r_{N-1}\} \\ &\leqq \max_{\mathscr{M}} \min_{l=0}^{N-1} \max_{\mathscr{R}(M_l,n)} \min \{\min_{j=0}^{k-1} r_{i(j,l)}, \min_{j=k}^{N-1} r_{i(j,l)}\} . \end{aligned}$$

Thus we have

$$\begin{aligned} R &\leqq \max_{\mathscr{M}} \min_{l=0}^{N-1} \min \{\max_{\mathscr{R}(M_l,n)} \min_{j=0}^{k-1} r_{i(j,l)}, \max_{\mathscr{R}(M_l,n)} \min_{j=k}^{N-1} r_{i(j,l)}\} \\ &= \max_{\mathscr{M}} \min_{l=0}^{N-1} \min \{[((k-\lambda)M_l + \lambda n)/k], [(k-\lambda)(n-M_l)/(N-k)]\} . \end{aligned}$$

Then it follows that

$$\begin{aligned} R \leqq \max_{\mathscr{M}} \min_{l=0}^{N-1} \min \{&[((s-1)n + M_l)/s], \\ &[s^{p-2}(s-1)(n-M_l)/(s^{p-1}-1)]\} . \end{aligned} \tag{4.9}$$

Let $n = qN + m$ $(0 \leqq m \leqq N-1,\ q \geqq 0)$, then we have

$$(4.10)\qquad \begin{aligned} &((s-1)n + M_l)/s \leqq s^{p-2}(s-1)(n-M_l)/(s^{p-1}-1) \quad \text{if} \quad M_l \leqq q \\ &((s-1)n + M_l)/s > s^{p-2}(s-1)(n-M_l)/(s^{p-1}-1) \quad \text{if} \quad M_l \geqq q+1\,. \end{aligned}$$

As the inequalities

$$(4.11)\qquad \begin{aligned} &((s-1)n + q)/s \geqq s^{p-2}(s-1)(n-q-1)/(s^{p-1}-1) \quad \text{if} \quad m \leqq s^{p-1} \\ &((s-1)n + q)/s < s^{p-2}(s-1)(n-q-1)/(s^{p-1}-1) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad m \geqq s^{p-1}+1 \end{aligned}$$

hold, then from (4.9), (4.10) and (4.11), we have

$$(4.12)\qquad \begin{aligned} R &\leqq [s^{p-2}(s-1)(n-q-1)/(s^{p-1}-1)] \\ &= s^{p-1}q + [s^{p-2}(s-1)(m-1)/(s^{p-1}-1)] \quad \text{if} \quad 2 \leqq m \leqq s^{p-1} \\ &\leqq s^{p-1}q + [(s-1)m/s] \quad \text{if} \quad m \geqq s^{p-1}+1\,. \end{aligned}$$

It follows that the second and third equations in Theorem 4.1 hold. When $m = 0$, from (4.9) and (4.10), and when $m = 1$, from (4.9) and (4.11), we have $R \leqq s^{p-1}q$ respectively. However we select qN or $qN+1$ points in $PG(p-1, s)$, corresponding to the column vectors in F, each point repeatedly q or $q+1$ times, i.e., $\mathbf{M}' = (q, q, \cdots, q)$ or $\mathbf{M}' = (q, \cdots, q, q+1, q, \cdots, q)$ when $n = qN$ or $n = qN+1$ respectively, so that the resolution R for s^{n-p} design is $s^{p-1}q$. It follows that the first equation in the theorem holds.

This completes the proof of Theorem 4.1.

From (4.9), it follows that the following corollary holds.

COROLLARY 4.1. *In an s^{n-p} design, let M be the maximum value of the N components of modular vector $\mathbf{M}' = (M_0, M_1, \cdots, M_{N-1})$. Then the resolution R for s^{n-p} design is less than or equal to* $[s^{p-2}(s-1)(n-M)/(s^{p-1}-1)]$, i.e.,

$$(4.13)\qquad R \leqq [s^{p-2}(s-1)(n-M)/(s^{p-1}-1)]\,.$$

Our upper bound BF is derived under conditions (4.7), and Plotkin's upper bound BP under condition (4.5). However any set of r_i which satisfy (4.7) automatically satisfy (4.5), hence $BF \leqq BP$.

5. Examples of the maximum resolution for s^{n-p} designs. In this section, it is shown that there are examples such that our upper bound is less than Plotkin's upper bound.

EXAMPLE 1. Consider the case $s = 2$, $p = 3$ and $n = 9$. In this case, it follows from (1.1) and (4.1) that $BP = 5$ and $BF = 4$. This shows that $BF < BP$ and $R \leqq 4$. On the other hand, there exists at least one 2^{9-3} design with resolution $R = 4$. For example, $D_F = \{\mathbf{x}' \mid F\mathbf{x} = \mathbf{c}\}$ is a 2^{9-3} design with $R = 4$ where

$$F = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Therefore, we have $BF = R = 4$.

There are many examples such that $BF < BP$. We list such examples in Tables 1, 2 and 3.

TABLE 1
3^{n-3} *designs*

n	3	6	15	16	19
R	1	3	9	10	12
BF	1	3	9	10	12
BP	2	4	10	11	13

TABLE 2
4^{n-3} *designs*

n	7	8	12	23	24	25	28	29	33
R	4	5	8	16	17	18	20	21	24
BF	4	5	8	16	17	18	20	21	24
BP	5	6	9	17	18	19	21	22	25

TABLE 3
2^{n-4} *designs*

n	4	6	17	19	21
R	1	2	8	9	10
BF	1	2	8	9	10
BP	2	3	9	10	11

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